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The Extension of Isometry between Unit Spheres of Normed Space E and l^1

ZHAN Hua Ying
(College of Science, Tianjin University of Technology, Tianjin 300384, China)
(E-mail: zhanhuaying@gmail.com)

Abstract The main result of this paper is to prove Fang and Wang's result by another method: Let E be any normed linear space and $V_0 : S(E) \to S(l^1)$ be a surjective isometry. Then V_0 can be linearly isometrically extended to E.

Keywords isometry; surjective; linearly isometric extension.

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1. Introduction

We recall that a mapping T from a subset M of a normed space E to a subset of another normed space F is called an isometry if $\|Tx - Ty\| = \|x - y\|$ for all $x,\ y \in M$. In 1972, Mankiewicz proved in [1] that an isometry mapping an open connected subset of a normed space E onto an open subset of another normed space F can be extended to be an affine isometry from E onto F. In 1987, Tingley raised in [2] the following problem:

Problem Let E and F be normed spaces with unit spheres $S(E)$ and $S(F)$. Assume that $V_0 : S(E) \to S(F)$ is an onto isometry. Does there exist a linear or affine isometry $V : E \to F$ such that $V|_{S(E)} = V_0$?

Tingley just obtained the following result: If E and F are finite dimensional Banach spaces and $V_0 : S(E) \to S(F)$ is an onto isometry, then $V_0(-x) = -V_0(x)$ for any $x \in S(E)$. That is, V_0 preserves anti-polar points. In the past decade, Ding and his group kept on working on this topic and had obtained a number of significant results[3,4]. Most of these works just concerned the surjective isometries between spaces of the same type. Ding discussed first in [5] the extension of isometries between unit spheres of different type spaces. In [6], Fang and Wang gave an affirmative answer to Tingley's problem for the case that $F = l^1$. In this paper, we will give another method to prove Fang and Wang's result. Our notation and terminology are standard.

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In the complex spaces, it is evident that the answer to Tingley's problem is negative. An obvious counterexample is that $E = F = \mathbf{C}$ (complex plane) and $V_0(x) = \bar{x}$. Hence, we just need to study the problem in the real spaces.

2. Main result

We note that for any $x = (x_i)$, $y = (y_i) \in S(l^1)$, $\|x - y\| = 2$ if and only if x_i and y_i have different signs for each $i \in \operatorname{supp}(x) \bigcap \operatorname{supp}(y)$.

First, we prove Fang and Wang's result for the case that both of the spaces are 2-dimensional. $l^1_{(2)}$ stands for $\mathbf{R}^2$ with l^1-norm, that is, $\|(\alpha_1, \alpha_2)\| = |\alpha_1| + |\alpha_2|$. We need the following lemma.

Lemma 1 *Let $E = l^1_{(2)}$, $d_1 = (\frac{1}{2}, \frac{1}{2})$, $d_2 = (-\frac{1}{2}, \frac{1}{2})$. Then for any $x, y \in S(E)$, $\|x \pm d_1\| = \|y \pm d_1\|$ and $\|x \pm d_2\| = \|y \pm d_2\|$ imply that $x = y$.*

Proof Let $x = (\alpha_1, \alpha_2)$, $y = (\beta_1, \beta_2) \in S(E)$. Suppose that $0 \le \alpha_1,\ \alpha_2 \le 1$. Then $\|x + d_1\| = 2 = \|y + d_1\|$, which implies that $0 \le \beta_1,\ \beta_2 \le 1$. Since $\|x - d_1\| = \|y - d_1\|$, if $x \neq y$, then $d_1 = \frac{x+y}{2}$. Hence, from $\|x + d_2\| = \|y + d_2\|$, we can get a contradiction. In fact, from $d_1 = \frac{x+y}{2}$ we know that $\beta_1 = 1 - \alpha_1$, $\beta_2 = 1 - \alpha_2$. Hence,

$$\begin{aligned} \|x - d_2\| &= \|(\alpha_1 + \frac{1}{2}, \alpha_2 - \frac{1}{2})\| = |\alpha_1 + \frac{1}{2}| + |\alpha_2 - \frac{1}{2}| \\ &= \|y - d_2\| = \|(1 - \alpha_1 + \frac{1}{2}, 1 - \alpha_2 - \frac{1}{2})\| = |\frac{3}{2} - \alpha_1| + |\frac{1}{2} - \alpha_2|. \end{aligned}$$

Since $0 \le \alpha_1 \le 1$, we know that $\frac{3}{2} - \alpha_1 = \alpha_1 + \frac{1}{2}$. It follows that $\alpha_1 = \frac{1}{2}$. Hence, $\alpha_2 = \beta_1 = \beta_2 = \frac{1}{2}$. That is, $x = y$.

So, $x = y$ if $0 \le \alpha_1,\ \alpha_2 \le 1$. Similarly, we can get the same result for other cases.

Proposition 2 *Let E be a 2-dimensional normed space. Then any isometry $V_0 : S(E) \to S(l^1_{(2)})$ can be linearly extended to an isometry on E.*

Proof Since E and $l^1_{(2)}$ are both 2-dimensional, following Tingley's result, V_0 preserves anti-polar points. Let $d_1 = (\frac{1}{2}, \frac{1}{2})$, $d_2 = (-\frac{1}{2}, \frac{1}{2})$ and $e_1 = V_0^{-1} d_1$, $e_2 = V_0^{-1} d_2$. Obviously, e_1 and e_2 are linearly independent. Moreover, $\|e_1 + e_2\| = \|d_1 + d_2\| = \|d_1 - d_2\| = \|e_1 - e_2\| = 1$. Put $A \triangleq \{x \in S(E) : V_0 x(1),\ V_0 x(2) \ge 0\}$. We will show that A is a convex subset.

Obviously, $e_1 \in A$. Fix any $x_1,\ x_2 \in A$. By the definition of A and that of the norm, $\|V_0 x_1 + d_1\| = 2$. Hence, $\|\frac{x_1 + e_1}{2}\| = 1$. By the Hahn-Banach theorem, there is $x_1^* \in S(E^*)$ such that $x_1^*(\frac{x_1+e_1}{2}) = \|\frac{x_1+e_1}{2}\| = 1$. Consequently, $x_1^*(x_1) = x_1^*(e_1) = 1$, which implies that

$$2 = x_1^*(\frac{x_1 + e_1}{2} + e_1) \le \|\frac{x_1 + e_1}{2} + e_1\| \le 2.$$

Then,

$$\|V_0(\frac{x_1 + e_1}{2}) + d_1\| = \|\frac{x_1 + e_1}{2} + e_1\| = 2.$$

That means $V_0(\frac{x_1+e_1}{2})(i) \ge 0$ $(i = 1, 2)$, which implies that

$$\|\frac{x_1 + e_1}{2} + x_2\| = \|V_0(\frac{x_1 + e_1}{2}) + V_0 x_2\| = 2.$$

Similarly, there is $x_2^* \in S(E^*)$ such that $x_2^*(x_2) = x_2^*(x_1) = x_2^*(e_1) = 1$. Then

$$2 = x_2^*(\frac{x_1 + x_2}{2} + e_1) \le \|\frac{x_1 + x_2}{2} + e_1\| \le 2,$$

which means that

$$\|V_0(\frac{x_1 + x_2}{2}) + d_1\| = \|\frac{x_1 + x_2}{2} + e_1\| = 2.$$

Hence, $\frac{x_1+x_2}{2} \in A$. Since V_0 is continuous and x_1 and x_2 are arbitrarily chosen, A is convex.

Similarly, $B \triangleq \{x \in S(E) : V_0x(1) \le 0,\ V_0x(2) \ge 0\}$ is also a convex set. Moreover, $x_0 \triangleq V_0^{-1}(d_1 + d_2) \in A \bigcap B$. It is straightforward to have that $x_0 = e_1 + e_2$.

In fact, e_1 and e_2 are linearly independent, hence, $\{e_1,\ e_2\}$ is a basis of E. Say, $x_0 = \alpha_1 e_1 + \alpha_2 e_2$. If $x_0 \ne e_1 + e_2$, following $\|x_0\| = \|e_1\| = \|e_2\| = \|x_0 - e_1\| = \|x_0 - e_2\| = 1$ and $\|x_0 + e_1\| = \|x_0 + e_2\| = 2$, we may claim that

$$\begin{aligned} x_0 \notin &\{\lambda e_1 : \lambda \in \mathbf{R}\} \bigcup \{\lambda e_2 : \lambda \in \mathbf{R}\} \bigcup \{\alpha e_1 + \alpha e_2 : \alpha \in \mathbf{R}, |\alpha| \ne 1\} \bigcup \\ &\{e_1 + \lambda e_2 : \lambda \in \mathbf{R}, \lambda \ne 1\} \bigcup \{\lambda e_1 + e_2 : \lambda \in \mathbf{R}, \lambda \ne 1\}. \end{aligned}$$

If $0 < \alpha_1 < \alpha_2$, then $[x_0, e_1] \triangleq \{\lambda x_0 + (1-\lambda)e_1 : \lambda \in [0, 1]\}$ intersects with the line that joints θ and $e_1 + e_2$ at some point x_1. Obviously, $x_1 = \lambda(e_1 + e_2)$ for some $\lambda \ne 1$. Hence, $\|x_1\| \ne 1$. On the other hand, following the convexity of A, $[x_0, e_1] \subset A \subset S(E)$. It means that $x_1 \in S(E)$. It is impossible.

If $0 < \alpha_2 < \alpha_1$, it may also lead a contradiction similarly.

If $\alpha_1 < \alpha_2 < 0$, then

$$\|x_0 - e_1\| = \|(\alpha_1 - 1)e_1 + \alpha_2 e_2\| \ge |1 - \alpha_1| - |\alpha_2| = 1 - \alpha_1 + \alpha_2 > 1$$

leads a contradiction.

Similarly, it is impossible for $\alpha_2 < \alpha_1 < 0$.

Hence, $x_0 = e_1 + e_2$.

Since $e_1 + \lambda e_2 \in [e_1, e_1 + e_2]$ for any $0 \le \lambda \le 1$ and $e_1,\ e_1 + e_2 \in A$, then $\|e_1 + \lambda e_2\| = 1 = \|d_1 + \lambda d_2\|$. Similarly, $\|\lambda e_1 + e_2\| = 1 = \|\lambda d_1 + d_2\|$ for any $0 \le \lambda \le 1$. Then for any $\lambda > 1$, $\|e_1 + \lambda e_2\| = \lambda\|\frac{1}{\lambda}e_1 + e_2\| = \lambda = \lambda\|\frac{1}{\lambda}d_1 + d_2\| = \|d_1 + \lambda d_2\|$. Following Tingley's result, $V_0(-e_i) = -d_i$, $i = 1, 2$. Hence, $\|e_1 + \lambda e_2\| = \|d_1 + \lambda d_2\|$ for any $\lambda \in \mathbf{R}$.

Now, for any $x \in S(E)$, say $x = \lambda_1 e_1 + \lambda_2 e_2$. It is immediate from the above that

$$\begin{aligned} \|V_0x \pm d_1\| &= \|x \pm e_1\| = \|(\lambda_1 \pm 1)e_1 + \lambda_2 e_2\| \\ &= \|(\lambda_1 \pm 1)d_1 + \lambda_2 d_2\| = \|(\lambda_1 d_1 + \lambda_2 d_2) \pm d_1\|. \end{aligned}$$

Similarly, $\|V_0x \pm d_2\| = \|(\lambda_1 d_1 + \lambda_2 d_2) \pm d_2\|$. Following Lemma 1, $V_0(\lambda_1 e_1 + \lambda_2 e_2) = \lambda_1 d_1 + \lambda_2 d_2$. That is, V_0 is linear on $S(E)$. Then it is easy to show that V_0 has a linearly isometric extension on the whole space E.

For infinite dimensional case, we have the following result.

Lemma 3 *Suppose that V_0 is an isometry from $S(E)$ into $S(l^1)$, and that $\{\pm e_i\} \in V_0(S(E))$, where $\{e_i\}$ is the unit basis of l^1. Let $x_i = V_0^{-1}e_i$. Then $V_0(-x_i) = -e_i$.*

Proof Fix $i_0 \in \mathbf{N}$. Let $\mathcal{A}^+ = \{x \in S(E) : V_0x(i_0) > 0\}$, where $V_0x(i_0)$ is the i_0th coordinate of V_0x. We classify $\mathcal{A}^+$ in the following way: x, y are in the same class A if and only if $V_0x(j) \cdot V_0y(j) \geq 0$ for any $j \in \mathbf{N}$. Let $\mathcal{A}' = S(E) \backslash \mathcal{A}^+$. We may classify it by the same way.

We note that if $V_0x(j) \neq 0$, then $V_0x(j) \cdot V_0(-x)(j) \leq 0$, since $\|V_0x - V_0(-x)\| = 2$. Hence, for any $j \in \mathbf{N}$,

$$V_0x(j) \cdot V_0y(j) > 0 \text{ implies that } V_0(-x)(j) \cdot V_0(-y)(j) \geq 0.$$

If $x \in \mathcal{A}^+$ and $V_0x(j) = 0$ for some $j \in \mathbf{N}$, then there are classes A and B such that

$$V_0y(j) \geq 0,\ V_0z(j) \leq 0,\ V_0y(k) \cdot V_0z(k) \geq 0,$$

for all $y \in A$, $z \in B$, $k \in \mathbf{N}$ ($k \neq j$), and that $x \in A \bigcap B$.

Consider such A and B. If y_1, $y_2 \in A$ with $V_0y_1(j) \cdot V_0y_2(j) > 0$, then

$$V_0(-y_1)(j)V_0(-y_2)(j) \geq 0.$$

Thus, $-y_1$ and $-y_2$ are in the same class of $\mathcal{A}'$, denote it by A'. Similarly, if $z_1,\ z_2 \in B$ with $V_0z_1(j) \cdot V_0z_2(j) > 0$, then $-z_1$ and $-z_2$ are in the same class of $\mathcal{A}'$, denote it by B'. On the other hand, since $V_0x(j) = 0$, $V_0(-x)(j)$ may be 0 or ≥ 0 or ≤ 0. If $V_0(-x)(j) < 0$ ($V_0(-x)(j) > 0$, respectively), then x is regarded as in A and $-x$ is regarded as in A' ($x \in B$ and $-x \in B'$, respectively).

Hence, we say that $-A \subset A'$, which means that $-\bigcap_{A \subset \mathcal{A}^+} A \subset \bigcap_{A' \subset \mathcal{A}'} A'$. Since $\bigcap_{A \subset \mathcal{A}^+} A = \{x_i\}$ and $\bigcap_{A' \subset \mathcal{A}'} A' = V_0^{-1}(-e_i)$, we know that $V_0(-x_i) = -e_i$.

Lemma 4 *For any* $x = \{\alpha_i\},\ y = \{\beta_i\} \in S(l^1)$ *and any* $j \in \mathbf{N}$, *if* $\|x \pm e_j\| = \|y \pm e_j\|$, *then* $\alpha_j = \beta_j$.

Proof In fact,

$$\begin{aligned}\|x \pm e_j\| &= \|\sum_{i=1}^{\infty} \alpha_i e_i \pm e_j\| = \sum_{i \neq j} |\alpha_i| + |1 \pm \alpha_j| = 1 - |\alpha_j| + |1 \pm \alpha_j| \\ &= \|y \pm e_j\| = \|\sum_{i=1}^{\infty} \beta_i e_i \pm e_j\| = \sum_{i \neq j} |\beta_i| + |1 \pm \beta_j| = 1 - |\beta_j| + |1 \pm \beta_j|.\end{aligned}$$

Hence, $|1 \pm \alpha_j| - |\alpha_j| = |1 \pm \beta_j| - |\beta_j|$. Obviously, $\alpha_j = 0$ if and only if $\beta_j = 0$. Then we just need to show the case that $\alpha_j \cdot \beta_j \neq 0$. If $\alpha_j \cdot \beta_j < 0$, say $\alpha_j < 0 < \beta_j$, then

$$1 + \alpha_j - |\alpha_j| = 1 + \alpha_j + \alpha_j < 1 = 1 + \beta_j - \beta_j = 1 + \beta_j - |\beta_j|$$

leads a contradiction. It is similar for $\beta_j < 0 < \alpha_j$. If α_j, $\beta_j > 0$, then

$$1 - \alpha_j - |\alpha_j| = 1 - 2\alpha_j = 1 - \beta_j - |\beta_j| = 1 - 2\beta_j$$

implies that $\alpha_j = \beta_j$. Similarly, we can get the same result if α_j, $\beta_j < 0$.

Next, we show the main result.

Theorem 5 *Let* E *be any normed linear space and* $V_0 : S(E) \to S(l^1)$ *be a surjective isometry. Then* V_0 *can be linearly isometrically extended to* E.

Proof Fix $n \in \mathbf{N}$. For any subset Δ_1 of $\{1, \dots, n\}$, let $\Delta_2 = \{1, \dots, n\} \backslash \Delta_1$. We denote $B_{\Delta_1} = \{x \in S(E) : V_0 x(i) \le 0,\ V_0 x(j) \ge 0 \text{ for any } i \in \Delta_1,\ j \in \Delta_2\}$, and $d_{\Delta_1} = \frac{1}{n}(\sum_{i \in \Delta_1} e_i - \sum_{j \in \Delta_2} e_j)$. Then $d_{\Delta_1}(i) \ge 0,\ d_{\Delta_1}(j) \le 0$ for any $i \in \Delta_1,\ j \in \Delta_2$. Since V_0 is surjective, there is $z \in S(E)$ such that $V_0 z = d_{\Delta_1}$.

Following the same steps shown in the proof of Proposition 2, we know that B_{Δ_1} is convex.

Now, for any real scalar α_i $(1 \le i \le n)$ with $\alpha = \sum_{i=1}^n |\alpha_i| \ne 0$. Let $\Delta_1 = \{1 \le i \le n : \alpha_i \ge 0\}$, $\Delta_2 = \{1, \dots, n\} \backslash \Delta_1$, and $x_i = V_0^{-1} e_i$. Following Lemma 3, $V_0(-x_i) = -e_i$. Moreover, $x_i \in B_{\Delta_2},\ -x_j \in B_{\Delta_2}$ for any $i \in \Delta_1,\ j \in \Delta_2$. Since B_{Δ_2} is convex,

$$\sum_{i \in \Delta_1} \frac{\alpha_i}{\alpha} x_i + \sum_{j \in \Delta_2} \frac{-\alpha_j}{\alpha}(-x_j) \in B_{\Delta_2}.$$

That is, $\sum_{i=1}^n \alpha_i x_i \in \alpha \cdot B_{\Delta_2}$. It follows that

$$\|\sum_{i=1}^n \alpha_i x_i\| = \alpha = \sum_{i=1}^n |\alpha_i| = \|\sum_{i=1}^n \alpha_i e_i\|.$$

Now, for any $\sum_{i=1}^n \alpha_i x_i \in S(E)$, say $V_0(\sum_{i=1}^n \alpha_i x_i) = \sum_{i=1}^\infty \beta_i e_i$, then for all $1 \le j \le n$,

$$\|\sum_{i=1}^\infty \beta_i e_i \pm e_j\| = \|V_0(\sum_{i=1}^n \alpha_i x_i) \pm V_0 x_j\| = \|\sum_{i=1}^n \alpha_i x_i \pm x_j\| = \|\sum_{i=1}^n \alpha_i e_i \pm e_j\|.$$

Following Lemma 4, $\beta_j = \alpha_j$ for any $1 \le j \le n$, and $\beta_j = 0$ for any $j > n$. That is, $V_0(\sum_{i=1}^n \alpha_i x_i) = \sum_{i=1}^n \alpha_i e_i$.

It is easy to check that V_0 can be extended to $S(E) \bigcup [x_i : 1 \le i \le n]$. Denote the extension by V_n, where V_n is a linear isometry on $[x_i : 1 \le i \le n]$.

Now, for any $x \in S(E)$, say $V_0 x = \sum_{i=1}^\infty \alpha_i e_i$. Let $\beta_n = \sum_{i=1}^n |\alpha_i|$. Then $\beta_n = \|\sum_{i=1}^n \alpha_i e_i\| = \|\sum_{i=1}^n \alpha_i x_i\|$, and $\beta_n \to 1$ $(n \to \infty)$. Since

$$\begin{aligned}
\|x - \sum_{i=1}^n \alpha_i x_i\| &\le \|x - \sum_{i=1}^n \frac{\alpha_i}{\beta_n} x_i\| + \|\sum_{i=1}^n \frac{\alpha_i}{\beta_n} x_i - \sum_{i=1}^n \alpha_i x_i\| \\
&= \|V_0 x - V_0(\sum_{i=1}^n \frac{\alpha_i}{\beta_n} x_i)\| + \|V_n(\sum_{i=1}^n \frac{\alpha_i}{\beta_n} x_i) - V_n(\sum_{i=1}^n \alpha_i x_i)\| \\
&= \|\sum_{i=1}^\infty \alpha_i e_i - V_n(\sum_{i=1}^n \frac{\alpha_i}{\beta_n} x_i)\| + \|\frac{1}{\beta_n}\sum_{i=1}^n \alpha_i e_i - \sum_{i=1}^n \alpha_i e_i\| \\
&= \|\sum_{i=1}^\infty \alpha_i e_i - \frac{1}{\beta_n}\sum_{i=1}^n \alpha_i e_i\| + \frac{1-\beta_n}{\beta_n}\|\sum_{i=1}^n \alpha_i e_i\| \\
&\le \frac{2(1-\beta_n)}{\beta_n}\|\sum_{i=1}^n \alpha_i e_i\| + \|\sum_{i=n+1}^\infty \alpha_i e_i\| \\
&= 2(1-\beta_n) + \|\sum_{i=n+1}^\infty \alpha_i e_i\|,
\end{aligned}$$

which is convergent to 0, we have $x = \sum_{i=1}^\infty \alpha_i x_i$. That is, $E = [x_i : i \in \mathbf{N}]$.

Now, we can define the desired isometry. For any $x = \sum_{i=1}^\infty \alpha_i x_i$, let $Vx = \sum_{i=1}^\infty \alpha_i e_i$. V is well defined since, for any $m > n$,

$$V_m(\sum_{i=1}^{n}\alpha_i x_i) = \sum_{i=1}^{n}\alpha_i e_i,$$

and

$$\begin{aligned}\|\sum_{i=1}^{n}\alpha_i e_i - \sum_{i=1}^{m}\alpha_i e_i\| &= \|V_m(\sum_{i=1}^{n}\alpha_i x_i) - V_m(\sum_{i=1}^{m}\alpha_i x_i)\| \\ &= \|V_m(\sum_{i=n+1}^{m}\alpha_i x_i)\| \\ &= \|\sum_{i=n+1}^{m}\alpha_i x_i\| \to 0\ (m, n \to 0).\end{aligned}$$

Obviously, V is a linear isometry. Moreover, we show that V is an extension of V_0. For any $x = \sum_{i=1}^{\infty}\alpha_i x_i \in S(E)$, let $\beta_n = \|\sum_{i=1}^{n}\alpha_i x_i\|$. Then $\beta_n \to 1$ $(n \to \infty)$. Hence,

$$\begin{aligned}Vx &= \sum_{i=1}^{\infty}\alpha_i e_i = \lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i e_i \\ &= \lim_{n\to\infty} V_n(\sum_{i=1}^{n}\alpha_i x_i) = \lim_{n\to\infty}\beta_n \cdot V_n(\sum_{i=1}^{n}\frac{\alpha_i}{\beta_n}x_i) \\ &= \lim_{n\to\infty}\beta_n \cdot V_0(\sum_{i=1}^{n}\frac{\alpha_i}{\beta_n}x_i) = \lim_{n\to\infty} V_0(\sum_{i=1}^{n}\frac{\alpha_i}{\beta_n}x_i) \\ &= V_0(\sum_{i=1}^{\infty}\alpha_i x_i) = V_0 x.\end{aligned}$$

The proof is completed. □

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