Composition Operators from $\alpha$-Bloch Spaces into $Q_K$ Type Spaces

YU Yan Yan$^1$, LIU Yong Min$^2$

(1. School of Mathematics and Physics Science, Xuzhou Institute of Technology, Jiangsu 221008, China; 2. Department of Mathematics, Xuzhou Normal University, Jiangsu 221116, China)

(E-mail: minliu@xznu.edu.cn)

Abstract Suppose $\phi$ is an analytic map of the unit disk $D$ into itself, $X$ is a Banach space of analytic functions on $D$. Define the composition operator $C_\phi: C_\phi f = f \circ \phi$, for all $f \in X$. In this paper, the boundedness and compactness of the composition operators from $\alpha$-Bloch spaces into $Q_K(p,q)$ and $Q_K,0(p,q)$ spaces are discussed, where $0 < \alpha < \infty$.

Keywords Composition operator; analytic function; $\mathcal{B}_\alpha$ space; $K$-Carleson measure; compact $K$-Carleson measure.

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1. Introduction

First, we introduce some basic notations, which are used in this paper. The unit disk in the finite complex plane $\mathbb{C}$ will be denoted by $D$. $H(D)$ will denote the space of all analytic functions on $D$, $B(D)$ will denote the subset of $H(D)$ consisting of these $f \in H(D)$ for which $|f(z)| < 1$, $dA$ will denote the Lebesgue measure on $D$, normalized so that $A(D) = 1$. For $a \in D$, $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ is the Möbius transformation of $D$ to itself, $g(z,a) = \log |\frac{1-\overline{a}z}{1-a\overline{z}}|$ is the Green function of $D$ with singularity at $a$. Every analytic self-map $\phi$ of the unit disk $D$ induces through composition a linear composition operator $C_\phi: C_\phi f = f \circ \phi$ from $X$ to itself. $\mathbb{N}^+$ is the natural numbers set. We say the function $f \in \mathcal{B}_\alpha$, if $f \in H(D)$ and

$$\|f\|_\alpha = |f(0)| + \|f\|_{\mathcal{B}_\alpha} < \infty,$$

where

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)|.$$

We say the function $f \in \mathcal{B}_{0,\alpha}$, if $f \in H(D)$ and

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$
The space $B^\alpha$ is a Banach space under the norm $\| \cdot \|_\alpha$, $B_0^\alpha$ is the closed subset of $B^\alpha$. When $\alpha = 1$, we get the Bloch space and the little Bloch space. In recent years a special class of M"obius invariant function spaces, the so-called $Q_K(p, q)$ spaces, has attracted a lot of attention. One important property of $Q_K(p, q)$ spaces is the inclusionship with $\alpha$-Bloch spaces $B^\alpha$. It was shown in [1] that $Q_K(p, q) \subset B^{\alpha + 2}$. Furthermore, $Q_K(p, q) = B^{\alpha + 2}$ if and only if

$$
\int_0^1 (1 - r^2)^{\alpha - 2} (\log \frac{1}{r}) \, dr < \infty.
$$

We recall some facts about $Q_K(p, q)$ spaces. We let $K : [0, \infty) \to [0, \infty)$ be a right-continuous and nondecreasing function. For $0 < p < \infty$, $Q_K(p, q)$ is two times differentiable on $(0, \infty)$.

Then we say $f$ analytic in $D$ belongs to the space $Q_K(p, q)$, if

$$
\|f\|_{K,p,q} = \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z) < \infty.
$$

Let $\|f\| = |f(0)| + \|f\|_{K,p,q}$. $Q_K(p, q)$ is a Banach space under the norm $\| \cdot \|$ when $p \geq 1$. If $f \in H(D)$ and

$$
\lim_{|s| \to 0} \int_D |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z) = 0,
$$

then we say $f \in Q_{K,0}(p, q)$. If $p = q + 2$, the space $Q_K(p, q)$ is a M"obius-invariant, i.e., $\|f \circ \varphi_a\|_{K,p,q} = \|f\|_{K,p,q}$ for all $a \in D$. $Q_{K,0}(p, q)$ is a closed subset of $Q_K(p, q)$. Three special cases are worth mentioning. When $p = 2$, $q = 0$, $Q_K(p, q) = Q_K^{[2]}$; When $K(t) = t^s$, $Q_K(p, q) = F(p, q, s)$, $Q_{K,0}(p, q) = F_0(p, q, s)$; When $p = 2$, $q = 0$, $K(t) = t^\alpha$, $Q_K(p, q) = Q_\alpha$, $Q_{K,0}(p, q) = Q_{\alpha,0}$. The space $Q_K(p, q)$ is trivial, if $Q_K(p, q)$ contains only constant functions. If the integral

$$
\int_0^1 (1 - r^2)^{q \alpha} K(\log \frac{1}{r}) \, dr
$$

is divergent, then $Q_K(p, q)$ is trivial. It is clear that the function-theoretic properties of $Q_K(p, q)$ depend on the structure of $K$. So, as in [7], from now on we take it for granted that the above weight function $K$ always satisfies the following conditions:

(a) $K : [0, \infty) \to [0, \infty)$ is nondecreasing;
(b) $K$ is two times differentiable on $(0, 1)$;
(c) The above integral is convergent;
(d) $K(1) > 0$, $t \geq 1$;
(e) $K(2t) \simeq K(t)$;
(f) $\int_0^1 \varphi_K(s) \, ds < \infty$, where $\varphi_K(s) = \sup_{0 \leq t \leq 1} \frac{K(st)}{K(t)}$ $(0 < s < \infty)$;
(g) $I = \sup_{a \in D} \int_D \frac{(1 - |z|^2)^{p - 2}}{|1 - az|^p} K(\log \frac{1}{|z|}) \, dA(z) < \infty$.

It is a well-known consequence of Littlewood’s subordination principle[8] that the formula $C_\phi f = f \circ \phi$ defines a bounded linear operator on the classical Hardy and Bergman spaces. That is, $C_\phi : H^p \to H^p$ and $C_\phi : A^p \to A^p$ are bounded operators. There has been done much research on the relations between the function theoretic properties of $\phi$ and the topological properties of the operator $C_\phi$ in different circumstances. Lou[9] discussed the boundedness and the compactness of the composition operators from $B^\alpha$ to $B^\beta$ when $0 < \alpha < \infty, 0 < \beta < \infty$;
Zhang\cite{10} discussed the boundedness and the compactness of the composition operators and the weighted composition operators from $B^p$ to $B^q$ when $p, q \geq 0$; Zhang and Xiao\cite{11} gave the boundedness and the compactness of the composition operators from $\mu$-Bloch to $\nu$-Bloch on the unit ball when $\mu$ and $\nu$ are the normal functions on $[0,1)$.

Recently, Wulan and Zhou\cite{1} gave many results on $Q_K(p, q)$ spaces; Wulan and Wu\cite{4,12} discussed the boundedness and compactness of the composition operators from $\mu$-Bloch to $\nu$-Bloch on the unit ball when $\mu$ and $\nu$ are the normal functions on $[0,1)$; Kotilainen\cite{14} discussed the boundedness and compactness of the composition operators from $B^\alpha$ to $Q_K(p, q)$; Yu and Liu\cite{15} discussed the boundedness of the composition operators from $B^\alpha$ to $Q_K(p, q)$ and composition operators from hyperbolic $\alpha$-Bloch spaces into hyperbolic $Q_K$ type spaces. We want to characterize here by means of $K$-Carleson measures and compact $K$-Carleson measures the boundedness and compactness of $C_\phi$ from $B^\alpha$ spaces into $Q_K(p, q)$ and $Q_K(p, q)$ spaces. Throughout this paper, given a subarc $I \subset \partial D$, the boundary of $D$, we denote by $S(I)$ the Carleson box based on $I$

$$S(I) = \{ r\zeta \in D : 1 - |I| < r < 1, \zeta \in I \}.$$ 

If $|I| \geq 1$, then we set $S(I) = D$. For $0 < p < \infty$, we say that a positive measure $\mu$ on $D$ is a $p$-Carleson measure if

$$\|\mu\|_p = \sup_{I \subset \partial D} \frac{\mu(S(I))}{|I|^p} < \infty,$$

where the supremum is taken over all subarcs $I$ of $\partial D$. If the right hand fractions tend to zero as $|I| \to 0$, then $\mu$ is said to be a compact $p$-Carleson measure. Note that the 1-Carleson measures are the classical Carleson measures. In a similar way, a positive measure $\mu$ on $D$ is said to be a $K$-Carleson measure if

$$\|\mu\|_K = \sup_{I \subset \partial D} \mu_K(S(I)) < \infty,$$

where the supremum is taken over all subarcs $I$ of $\partial D$, and

$$\mu_K(S(I)) = \int_{S(I)} K \left( \frac{1 - |z|}{|I|} \right) d\mu(z).$$

Also, $\mu$ is said to be a compact $K$-Carleson measure if $\|\mu\|_K < \infty$ and

$$\lim_{|I| \to 0} \mu_K(S(I)) = 0.$$ 

Clearly, if $K(t) = t^p$, then $\mu$ is a $K$-Carleson measure if and only if the measure $(1 - |z|^2)^p d\mu(z)$ is a $p$-Carleson measure.

We use the notation $a \simeq b$ to denote the comparability of the quantities $a$ and $b$; i.e., the existence of two positive constants $C_1$ and $C_2$ satisfying $C_1a \leq b \leq C_2a$. For convenience, we will always use the letter $C$ to denote a positive constant, which may change from one equation to the next. The constants usually depend on $a$ and other fixed parameters.
2. Preliminaries

The following result (part (i) proved in [7]) characterizes $K$-Carleson measures in conformally invariant terms.

**Lemma 1**\textsuperscript{[7,16,17]} Suppose $K$ satisfies (f). Then

(i) $\mu$ is a $K$-Carleson measure if and only if

$$\sup_{a \in D} \int_D K(1 - |\varphi_a(z)|^2) \, d\mu(z) < \infty; \quad (2.1)$$

(ii) $\mu$ is a compact $K$-Carleson measure if and only if (2.1) holds and

$$\lim_{|a| \to 0} \int_D K(1 - |\varphi_a(z)|^2) \, d\mu(z) = 0.$$

**Lemma 2**\textsuperscript{[18]} Let $K$ satisfy (f), (g), $0 < p < \infty$ and $-1 < q < \infty$. Suppose $n$ is a positive integer. Then $f \in Q_K(p,q)$ if and only if

$$|f^{(n)}(z)|^p (1 - |z|^2)^{nq-p+q} \, dA(z)$$

is a $K$-Carleson measure.

**Remark** When $n = 1$, the result holds for $-2 < q < \infty$\textsuperscript{[1]}.

By Lemmas 1 and 2, we have the following

**Lemma 3** Suppose $p > 0$, $-2 < q < \infty$. Then the following statements are equivalent:

1. $f \in Q_K(p,q)$;
2. $\sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) \, dA(z) < \infty$;
3. $|f'(z)|^p (1 - |z|^2)^q \, dA(z)$ is a $K$-Carleson measure.

The following lemma is a generalization of the result in [5].

**Lemma 4** Suppose $p > 0$, $-2 < q < \infty$ and $\mu$ is a $K$-Carleson measure. Then the following statements are equivalent:

1. $f \in Q_{K,0}(p,q)$;
2. $\lim_{|a| \to 0} \int_D |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) \, dA(z) = 0$;
3. $|f'(z)|^p (1 - |z|^2)^q \, dA(z)$ is a compact $K$-Carleson measure.

**Proof** The (ii) of Lemma 1 implies that the equivalence of (2) and (3). (1) $\Rightarrow$ (2) follows from the following inequalities

$$\int_D |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) \, dA(z) \leq C \int_D |f'(z)|^p (1 - |z|^2)^q K(g(z,a)) \, dA(z).$$

To prove (2) $\Rightarrow$ (1), we set $\Delta(a, \frac{1}{4}) = \{z \in D : |\varphi_a(z)| \leq \frac{1}{4}\}$ and $D_{\frac{1}{4}} = D - \Delta(a, \frac{1}{4})$, respectively. Since $g(z,a) \leq 8(1 - |\varphi_a(z)|^2)$, $|\varphi_a(z)| \geq \frac{1}{4}$, we obtain that

$$\int_{D - \Delta(a, \frac{1}{4})} |f'(z)|^p (1 - |z|^2)^q K(g(z,a)) \, dA(z)$$
\[
\begin{align*}
&\leq C \int_{D-\Delta(a,\frac{1}{4})} |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) \, dA(z) \\
&\leq C \int_{D} |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) \, dA(z). \\
&\quad (2.2)
\end{align*}
\]

The Möbius-invariance of measure \((1 - |z|^2)^{-2} \, dA(z)\) implies that
\[
\int_{\Delta(a,\frac{1}{2})} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z)
\leq \sup_{z \in \Delta(a,\frac{1}{2})} \left\{ |f'(z)|^p (1 - |z|^2)^{q+2} \right\} \int_{\Delta(a,\frac{1}{2})} (1 - |z|^2)^{-2} K(g(z, a)) \, dA(z)
\leq \sup_{z \in \Delta(a,\frac{1}{2})} \left\{ |f'(z)|^p (1 - |z|^2)^{q+2} \right\} \int_{\Delta(0,\frac{1}{2})} (1 - |z|^2)^{-2} K(g(z, 0)) \, dA(z)
\leq \sup_{z \in \Delta(a,\frac{1}{2})} \left\{ |f'(z)|^p (1 - |z|^2)^{q+2} \right\} \int_0^1 r(1 - r^2)^{-2} K(\log \frac{1}{r}) \, dr
\leq C \int_{D} |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) \, dA(z).
\]

So
\[
\int_{D} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z) \leq C \int_{D} |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) \, dA(z). \quad (2.3)
\]

We get the desired condition, which completes the proof of Lemma 4.

**Lemma 5**\(^{[19,19]}\) Suppose \(0 < \alpha < \infty\). Then there exist \(f, g \in \mathcal{B}^\alpha\), such that
\[
|f'(z)| + |g'(z)| \geq \frac{1}{(1 - |z|)^\alpha} \geq \frac{1}{(1 - |z|^2)^\alpha}
\]
for all \(z \in D\).

**Lemma 6**\(^{[14,15]}\) Suppose \(0 < p, \alpha < \infty, -2 < q < \infty, \phi \in B(D)\). Then the following statements are equivalent:

1. \(C_\phi : \mathcal{B}^\alpha \to Q_K(p, q)\) is bounded;
2. \(C_\phi : \mathcal{B}^\alpha_0 \to Q_K(p, q)\) is bounded;
3. \(\sup_{a \in D} \int_D \frac{|\phi(z)|^p}{(1 - |\phi(z)|^2)^{q\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z) < \infty\).

**Lemma 7**\(^{[14]}\) Suppose \(0 < \alpha < \infty, p \geq 1, -2 < q < \infty, \phi \in B(D)\). Then the following statements are equivalent:

1. \(C_\phi : \mathcal{B}^\alpha \to Q_K(p, q)\) is compact;
2. \(C_\phi : \mathcal{B}^\alpha_0 \to Q_K(p, q)\) is compact;
3. \(\phi \in Q_K(p, q)\) and
\[
\lim_{r \to 1} \sup_{a \in D} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{q\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z) = 0.
\]
3. Composition operators from the $\alpha$-Bloch spaces into $Q_K(p, q)$ and $Q_K,0(p, q)$

In this section we are ready to prove the following results.

**Theorem 1** Suppose $0 < p, \alpha < \infty$, $-2 < q < \infty$, $\phi \in B(D)$. Then $C_\phi : B^\alpha \to Q_K(p, q)$ is bounded if and only if
\[
\frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(g(z, a)) dA(z) \leq \infty.
\]

**Proof** Necessity. Using Lemma 6, we have
\[
\sup_{a \in D} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty.
\]
By Lemma 1, it suffices to prove that
\[
\sup_{a \in D} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) < \infty.
\]
Since $K$ is nondecreasing and $(1 - t^2) \leq 2\log \frac{1}{t}$, for $0 < t \leq 1$, we have $(1 - |\varphi_a(z)|^2) \leq 2\log \frac{1}{|\varphi_a(z)|^2} = 2g(z, a)$, for $z, a \in D$. Therefore,
\[
\sup_{a \in D} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) \\
\leq \sup_{a \in D} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(2g(z, a)) dA(z) \\
\leq C \sup_{a \in D} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty.
\]
Sufficiency. Assume that \[
\frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q dA(z) \leq \infty
\]
Then
\[
\sup_{a \in D} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) < \infty.
\]
We obtain that for all $f \in B^\alpha$,
\[
\sup_{a \in D} \int_D \frac{|(C_\phi f)'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) \\
= \sup_{a \in D} \int_D |f'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) \\
\leq \|f\|^p \sup_{a \in D} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\rho\alpha}} (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) < \infty.
\]
By Lemma 3, $C_\phi f \in Q_K(p, q)$. Thus $C_\phi : B^\alpha \to Q_K(p, q)$ is bounded. The proof is completed.$\square$

**Theorem 2** Suppose $0 < p, \alpha < \infty$, $-2 < q < \infty$, $\phi \in B(D)$. Then the following statements are equivalent:

1. $C_\phi : B^\alpha \to Q_K,0(p, q)$ is bounded;

(2) $\phi \in Q_{K,0}(p,q)$ and for all $r \in (0,1)$

$$\sup_{a \in D} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z,a)) \, dA(z) < \infty.$$ 

**Proof** (1) $\Rightarrow$ (2). Suppose $C_\phi : \mathcal{B}_0^\infty \to Q_{K,0}(p,q)$ is bounded. We get $C_\phi : \mathcal{B}_0^\infty \to Q_K(p,q)$ is bounded. By Lemma 6, we get for all $r \in (0,1),

$$\sup_{a \in D} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z,a)) \, dA(z) \leq \sup_{a \in D} \int_D (1 - |\phi(z)|^2)^\alpha (1 - |z|^2)^q K(g(z,a)) \, dA(z) < \infty.$$ 

Take $f(z) = z \in \mathcal{B}_0^\infty$. Since $C_\phi : \mathcal{B}_0^\infty \to Q_{K,0}(p,q)$ is bounded, $C_\phi f = \varphi \in Q_{K,0}(p,q)$.

(2) $\Rightarrow$ (1). To prove that $C_\phi : \mathcal{B}_0^\infty \to Q_{K,0}(p,q)$ is bounded, by the closed graph theorem we only need to show that for all $f \in \mathcal{B}_0^\infty$, then $C_\phi f \in Q_{K,0}(p,q)$. For all $\varepsilon > 0$, since $f \in \mathcal{B}_0^\infty$, there exists $r \in (0,1)$, such that $|f(w)|^p(1 - |w|^2)^p \alpha < \varepsilon$, for all $|w| > r$. For $z \in \{z : |\phi(z)| > r\}$, then $|f'(\phi(z))|^p(1 - |\phi(z)|^2)^p \alpha < \varepsilon$. For the above $r$, by the condition, there exists $C > 0$, such that for all $a \in D$

$$\int_{|\phi(z)| > r} |(f \circ \phi)'(z)|^p(1 - |z|^2)^q K(g(z,a)) \, dA(z) = \int_{|\phi(z)| > r} |f'(\phi(z))|^p(1 - |\phi(z)|^2)^p \alpha|\phi'(z)|^p (1 - |z|^2)^q K(g(z,a)) \, dA(z) \leq \varepsilon \sup_{a \in D} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z,a)) \, dA(z) \leq C \varepsilon.$$ 

Also, $\phi \in Q_{K,0}(p,q)$ implies that

$$\lim_{|a| \to 1} \int_{|\phi(z)| \leq r} |(f \circ \phi)'(z)|^p(1 - |z|^2)^q K(g(z,a)) \, dA(z) = \lim_{|a| \to 1} \int_{|\phi(z)| \leq r} |f'(\phi(z))|^p(1 - |\phi(z)|^2)^p \alpha|\phi'(z)|^p (1 - |z|^2)^q K(g(z,a)) \, dA(z) \leq \frac{\|f\|_p^p}{(1 - r^2)^{p\alpha}} \lim_{|a| \to 1} \int_{|\phi(z)| \leq r} |\phi'(z)|^p(1 - |z|^2)^q K(g(z,a)) \, dA(z) \leq \frac{\|f\|_p^p}{(1 - r^2)^{p\alpha}} \lim_{|a| \to 1} \int_D |\phi'(z)|^p(1 - |z|^2)^q K(g(z,a)) \, dA(z) = 0.$$ 

Combining all the above, we get

$$\lim_{|a| \to 1} \int_D |(f \circ \phi)'(z)|^p(1 - |z|^2)^q K(g(z,a)) \, dA(z) = \lim_{|a| \to 1} \left( \int_{|\phi(z)| > r} + \int_{|\phi(z)| \leq r} |(f \circ \phi)'(z)|^p(1 - |z|^2)^q K(g(z,a)) \, dA(z) \right) = 0.$$
Therefore \( C_\phi f \in Q_{K,0}(p, q) \), i.e., \( C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p, q) \) is bounded. The proof is completed. \( \square \)

**Theorem 3** Suppose \( 0 < \alpha < \infty \), \( p \geq 1 \), \(-2 < q < \infty\), \( \phi \in B(D) \). Then the following statements are equivalent:

1. \( C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p, q) \) is bounded;
2. \( C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p, q) \) is compact;
3. \( \lim_{|a| \to 1} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z) = 0; \)
4. \( \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z) \) is a compact \( K \)-Carleson measure;
5. \( C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p, q) \) is compact;
6. \( \phi \in Q_{K,0}(p, q) \) and

\[
\limsup_{r \to 1} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z) = 0. \tag{3.1}
\]

**Proof** (2) \( \Rightarrow \) (1). This implication is obvious.

(1) \( \Rightarrow \) (3). Suppose (1) is satisfied. By Lemma 5, there exist \( f_1, f_2 \in \mathcal{B}^\alpha \), such that

\[
|f_1'(z)| + |f_2'(z)| \geq \frac{1}{(1 - |z|)^\alpha} \geq \frac{1}{(1 - |z|^2)^\alpha}
\]

for all \( z \in D \). Since \( C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p, q) \) is bounded, we get \( C_\phi f_1 \in Q_{K,0}(p, q) \) and \( C_\phi f_2 \in Q_{K,0}(p, q) \). Thus

\[
\lim_{|a| \to 1} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z)
\]

\[
\leq C \lim_{|a| \to 1} \int_D ((f_1 \circ \phi)'(z)|^p + |(f_2 \circ \phi)'(z)|^p)(1 - |z|^2)^q K(g(z, a)) \, dA(z)
\]

\[
= 0.
\]

(3) \( \Rightarrow \) (4). Assume (3) holds. Then we have that

\[
\int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(1 - |\phi(z)|)^2 \, dA(z)
\]

\[
\leq C \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z)
\]

\[
\to 0 \quad \text{as} \quad |a| \to 1.
\]

By Lemma 1, \( \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q \, dA(z) \) is a compact \( K \)-Carleson measure.

(4) \( \Rightarrow \) (1). (4) gives that

\[
\lim_{|a| \to 1} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(1 - |\phi(z)|)^2) \, dA(z) = 0. \tag{3.2}
\]

For all \( f \in \mathcal{B}^\alpha \), we have that

\[
\int_D |(C_\phi f)'(z)|^p (1 - |z|^2)^q K(1 - |\phi(z)|^2) \, dA(z)
\]

\[
\leq C\|f\|^p \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(1 - |\phi(z)|^2) \, dA(z)
\]

\[
\to 0 \quad \text{as} \quad |a| \to 1.
\]
By Lemma 4, $C_\phi f \in Q_{K,0}(p, q)$, so $C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p, q)$ is bounded.

(4) $\Rightarrow$ (2). Let $\{f_n\} \subset \mathcal{B}^\alpha$, $\|f_n\|_\alpha \leq 1$ and $\{f_n\}$ converge on compact subsets of $D$ to 0 uniformly. Next, we will prove

$$\|C_\phi f_n\| \to 0 \text{ as } n \to \infty.$$ 

Firstly, $f_n(\phi(0)) \to 0$ as $n \to \infty$. Also, by (2.3) and (3.2), for all $\varepsilon > 0$, there exists $\delta : 0 < \delta < 1$ such that if $|a| > \delta$, then for all $n \in N^+$

$$\sup_{|a| > \delta} \int_D |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^\delta K(g(z, a)) \, dA(z)$$

$$\leq C \sup_{|a| > \delta} \int_D |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^\delta K(1 - |\phi_a(z)|^2) \, dA(z)$$

$$\leq C\|f_n\|_p^p \sup_{|a| > \delta} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)\rho_a} (1 - |z|^2)^\delta K(1 - |\phi_a(z)|^2) \, dA(z)$$

$$< \|f_n\|_p^p \varepsilon.$$

Since $\{f_n\}$ converges on a compact subset $rD = \{z : |z| \leq r\}$ of $D$ to 0 uniformly, for $0 < r < 1$, there exists $N > 0$, such that if $n \geq N$, then $|f_n'(\phi(z))|^p < \varepsilon$, for all $z \in rD$. We then get

$$\int_{rD} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^\delta K(1 - |\phi_a(z)|^2) \, dA(z)$$

$$< \varepsilon \int_{rD} |\phi'(z)|^p (1 - |z|^2)^\delta K(1 - |\phi_a(z)|^2) \, dA(z).$$

Thus

$$\sup_{|a| \leq \delta} \int_{rD} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^\delta K(1 - |\phi_a(z)|^2) \, dA(z)$$

$$< \varepsilon \|\phi\|_{Q_{K,p,q}}^p \quad \forall r \in (0, 1), \ n \geq N.$$ 

Since (4)$\Rightarrow$(1), we have $f_n \circ \phi \in Q_{K,0}(p, q) \subset Q_K(p, q)$. For all $a$, $|a| \leq \delta$, we have

$$\int_D |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^\delta K(g(z, a)) \, dA(z) \leq C < \infty.$$ 

For $0 < t < 1$, set $I_t(a) = \int_{D-tD} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^\delta K(g(z, a)) \, dA(z)$. We can choose $t(a) \in (0, 1)$, such that

$$I_t(a)(a) = \int_{D-t(a)D} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^\delta K(g(z, a)) \, dA(z) < \varepsilon.$$ 

Since $I_t(a)$ is a continuous function of $a^{[13]}$, there is a neighborhood $U(a)$ of $a$, for all $b \in U(a)$, such that

$$I_t(a)(b) = \int_{D-t(a)D} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^\delta K(g(z, b)) \, dA(z) < \varepsilon.$$ 

Thus, using the compactness of $\{a : |a| \leq \delta\}$, there exist $a_i \in D$, $i = 1, 2, \ldots, N_1$, such that $\{a : |a| \leq \delta\} \subset \cup_{i=1}^{N_1} U(a_i)$. We take $t_0 = \max_{1 \leq i \leq N_1} t(a_i)$, then for all $a$, $|a| \leq \delta$,

$$\int_{D-t_0D} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^\delta K(g(z, a)) \, dA(z) < \varepsilon.$$
Thus, using (3.3), we have

$$\sup_{|a| \leq \delta} \int_{D} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$= \sup_{|a| \leq \delta} \left[ \int_{D - t_0 D} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z) + \int_{t_0 D} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z) \right]$$

$$\leq \sup_{|a| \leq \delta} \int_{D - t_0 D} |(f_n \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z) + \sup_{|a| \leq \delta} \int_{t_0 D} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$< (1 + \|\phi\|_{Q_{K,p,q}}^p) \varepsilon.$$ 

Combining all the above, we get that for all $\varepsilon > 0$, there exists $N > 0$, for $n > N$,

$$\sup_{a \in D} \int_{D} |f_n'(\phi(z))|^p |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$\leq (2 + \|\phi\|_{Q_{K,p,q}}^p) \varepsilon < C \varepsilon.$$

Thus $\|C_\phi f_n\| = \|f_n(\phi(0))\| + \|C_\phi f_n\|_{K,p,q} \to 0$ as $n \to \infty$, i.e., $C_\phi : B^\alpha \to Q_{K,0}(p, q)$ is compact.

(6) $\Rightarrow$ (1). Suppose (6) is satisfied. For all $\varepsilon > 0$, there exists $\delta : 0 < \delta < 1$, such that if $\delta < r < 1$, then for $a \in D$

$$\int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$\leq \sup_{a \in D} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$< \varepsilon.$$ 

Therefore,

$$\int_{|\phi(z)| > r} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$= \int_{|\phi(z)| > r} |f'(\phi(z))|^p (1 - |\phi(z)|^2)^{p\alpha} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$\leq \|f\|_p^p \sup_{a \in D} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$\leq C \varepsilon.$$ 

On the other hand, 

$$\int_{|\phi(z)| \leq r} |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$= \int_{|\phi(z)| \leq r} |f'(\phi(z))|^p (1 - |\phi(z)|^2)^{p\alpha} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$

$$\leq \frac{\|f\|_p^p}{(1 - r^2)^{p\alpha}} \int_{|\phi(z)| \leq r} |\phi'(z)|^p (1 - |z|^2)^q K(g(z, a)) \, dA(z)$$
Corollary 4 holds, and the proof is completed.

Combining all the above, we get

\[ \lim_{|a| \to 1} \int_D |(f \circ \phi)'(z)|^p (1 - |z|^2)^q K(\alpha, z, a) \, dA(z) = 0. \]

Therefore \( C_{\phi} f \in Q_{K,0}(p, q) \), i.e., \( C_{\phi} : B^\alpha \to Q_{K,0}(p, q) \) is bounded.

(2)⇒(5). It is easy.

(5)⇒(6). Since the identical mapping belongs to \( B^\alpha_0 \), \( \phi \in Q_{K,0}(p, q) \). Since \( C_{\phi} : B^\alpha_0 \to Q_{K,0}(p, q) \) is compact, \( C_{\phi} : B^\alpha \to Q_K(p, q) \) is compact. By Lemma 7, it is easy to see that (3.1) holds, and the proof is completed.

**Corollary 4**\(^{[20]}\) Let \( 0 < p < \infty \), and \( \phi \) be an analytic self-map of \( D \). Then the following statements are equivalent:

(1) \( C_{\phi} : B \to Q_p \) is bounded;

(2) \( \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^p} (1 - |z|^2)^q \, dA(z) \) is a bounded \( p \)-Carleson measure.

**Corollary 5**\(^{[20]}\) Let \( 0 < p < \infty \), and \( \phi \) be an analytic self-map of \( D \). Then the following statements are equivalent:

(1) \( C_{\phi} : B \to Q_{p,0} \) is bounded;

(2) \( C_{\phi} : B \to Q_{p,0} \) is compact;

(3) \( \lim_{|a| \to 1} \int_D \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^p} g^p(z, a) \, dA(z) = 0; \)

(4) \( \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^p} (1 - |z|^2)^q \, dA(z) \) is a compact \( p \)-Carleson measure.

**Corollary 6**\(^{[21]}\) Let \( 0 < p, s < \infty \), \( -2 < q < \infty \), and \( \phi \) be an analytic self-map of \( D \). Then the following statements are equivalent:

(1) \( C_{\phi} : B^\alpha \to F(p, q, s) \) is bounded;

(2) \( \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^p} (1 - |z|^2)^q g^s(z, a) \, dA(z) \) is a bounded \( s \)-Carleson measure.

**Corollary 7**\(^{[21]}\) Let \( 0 < s < \infty \), \( p \geq 1 \), \( -2 < q < \infty \), and \( \phi \) be an analytic self-map of \( D \). Then the following statements are equivalent:

(1) \( C_{\phi} : B^\alpha \to F_0(p, q, s) \) is bounded;

(2) \( C_{\phi} : B^\alpha \to F_0(p, q, s) \) is compact;

(3) \( \lim_{|a| \to 1} \int_D \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^p} (1 - |z|^2)^q g^s(z, a) \, dA(z) = 0; \)

(4) \( \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^p} (1 - |z|^2)^q g^{s+s} \, dA(z) \) is a compact \( s \)-Carleson measure.

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