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# Annihilation Coefficients, Binomial Expansions and $q$-Analogs

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**Abstract** Let $\{A_n\}_{n=0}^{\infty}$ be an arbitary sequence of natural numbers. We say $A(n,k;A)$ are the Convolution Annihilation Coefficients for $\{A_n\}_{n=0}^{\infty}$ if and only if

$$\sum_{k=0}^{n} A(n,k;A)(x-A_k)^{n-k} = x^n. \tag{0.1}$$

Similary, we define $B(n,k;A)$ to be the Dot Product Annihilation Coefficients for $\{A_n\}_{n=0}^{\infty}$ if and only if

$$\sum_{k=0}^{n} B(n,k;A)(x-A_k)^{k} = x^n. \tag{0.2}$$

The main result of this paper is an explicit formula for $B(n,k;A)$, which depends on both $k$ and $\{A_n\}_{n=0}^{\infty}$. This paper also discusses binomial and $q$-analogs of Equations (0.1) and (0.2).

**Keywords** Annihilation coefficient; Binomial expansion; stirling number of the first kind; stirling number of the second kind; vadermonde convolution.

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## 1. Introduction

Motivated by the identities

$$(x-1)^4 + 4(x-2)^3 + 18(x-3)^2 + 64(x-4)^1 + 125 = x^4. \tag{1.1}$$

$$16(x-1)^1 + 48(x-2)^2 + 16(x-3)^3 + (x-4)^4 = x^4. \tag{1.2}$$

Gould [1] defined the Convolution Annihilation Coefficients by the expansion

$$\sum_{k=0}^{n} A(n,k)(x-k-1)^{n-k} = x^n, \tag{1.3}$$

and the Dot Product Annihilation Coefficients by the expansion

$$\sum_{k=0}^{n} B(n,k)(x-k)^{k} = x^n, \tag{1.4}$$

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for $A(n,k)$ and $B(n,k)$ arbitrary arrays of numbers.

The terminology was chosen in analogy with a dot product $\sum_{i=0}^{n} x_i y_i$ and a convolution product $\sum_{i=0}^{n} x_i y_{n-i}$. The proof that they exist and are unique is an easy consequence of the Fundamental Theorem of Algebra. For a given value of $n$ they may be successively determined by simple computations, annihilating lower powers one at a time.

More generally, Gould considered the following definitions:

**Definition 1.1** *Given an arbitrary sequence of natural numbers $\{A_n\}_{n=0}^{\infty}$, we say $A(n,k;A)$ are the Convolution Annihilation Coefficients for the sequence $\{A_n\}_{n=0}^{\infty}$ if and only if*

$$\sum_{k=0}^{n} A(n,k;A)(x-A_k)^{n-k} = x^n. \tag{1.5}$$

**Definition 1.2** *Given an arbitrary sequence of natural numbers $\{A_n\}_{n=0}^{\infty}$, we say $B(n,k;A)$ are the Dot Product Annihilation Coefficients for the sequence $\{A_n\}_{n=0}^{\infty}$ if and only if*

$$\sum_{k=0}^{n} B(n,k;A)(x-A_k)^{k} = x^n. \tag{1.6}$$

*For brevity we sometimes call them AC's. Most of the time we will assume in our applications that the sequence $\{A_n\}_{n=0}^{\infty}$ is strictly increasing. The proof that they exist and are unique is a consequence of the Fundamental Theorem of Algebra.*

In [1] a detailed study of $A(n,k;A)$ was made. It was shown that

$$A(n,k;A) = \binom{n}{k} C(k), \tag{1.7}$$

where $C(k)$ depends only upon $k$. It was then shown from Definition 1.1 that there exist unique coefficients $C(k)$ such that

$$\sum_{k=0}^{n} \binom{n}{k} C(k)(x-A_k)^{n-k} = x^n. \tag{1.8}$$

Setting $x=0$ in (1.8) we have the basic linear recurrence for $C(k)$ is

$$\sum_{k=0}^{n} \binom{n}{k} C(k)(-A_k)^{n-k} = \begin{cases} 0, & \text{for } n \geq 1; \\ 1, & \text{for } n = 0. \end{cases} \tag{1.9}$$

Its main use is to compute $C(n)$ from $C(0), C(1), C(2), \ldots, C(n-1)$. For this use it is convenient to restate it in the form

$$C(n) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n}{k} C(k) A_k^{n-k}, \quad \text{for } n \geq 1. \tag{1.10}$$

This was used in [1] to compute $C(k)$ in general and a (complicated) general formula was proved giving an explicit construction of $C(k)$.

Here we will study the corresponding results which follow from Definition 1.2, i.e., for general dot product AC's.

## 2. Dot product annihilation coefficients

We begin this section with a theorem that parallels Equation (1.8).

**Theorem 2.1** *There exist unique coefficients $D(n,k)$ such that*

$$\sum_{k=0}^{n} \binom{n}{k} D(n,k)(x-A_k)^k = x^n, \tag{2.1}$$

*where the coefficients $D(n,k)$ are functions of $n$ as well as of $k$.*

Note that, in contrast, the coefficients $C(k)$ in (1.8) were functions of $k$ alone. The proof of Theorem 2.1 follows directly from the Fundamental Theorem of Algebra.

Setting $x = 0$ in Equation (2.1), we have the basic linear recurrence for $D(n,k)$.

**Theorem 2.2**

$$\sum_{k=0}^{n} \binom{n}{k} D(n,k)(-A_k)^k = \begin{cases} 0, & \text{for } n \geq 1; \\ 1, & \text{for } n = 0. \end{cases} \tag{2.2}$$

As in [1], we may obtain a more general orthogonality relation for the $D$'s than Relation (2.2) by using the binomial expansion of $(x-A_k)^k$. Indeed we have

$$(x-A_k)^k = \sum_{j=0}^{k} \binom{k}{j} x^j (-A_k^{k-j})$$

so that

$$\begin{aligned} x^n &= \sum_{k=0}^{n} \binom{n}{k} D(n,k)(x-A_k)^k = \sum_{k=0}^{n} \binom{n}{k} D(n,k) \sum_{j=0}^{k} \binom{k}{j} x^j (-A_k^{k-j}) \\ &= \sum_{j=0}^{n} x^j \sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} D(n,k)(-A_k^{k-j}), \end{aligned}$$

so that we get, by uniqueness of coefficients,

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} D(n,k)(-A_k^{k-j}) = \delta_j^n. \tag{2.3}$$

Since $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$, we may rewrite the equation as

**Theorem 2.3**

$$\sum_{k=j}^{n} \binom{n-j}{k-j} D(n,k)(-A_k^{k-j}) = \delta_j^n. \tag{2.4}$$

Note Equation (2.2) is just the special case corresponding to $j = 0$. Equation (2.3) may be compared to equation (23) in [1] for the C(k) coefficients:

$$\sum_{k=0}^{n-j} \binom{n}{k}\binom{n-k}{j} C(k)(-A_k^{n-k-j}) = \delta_j^n.$$

We next use Theorem 2.1, to explicitly solve for $D(n,k)$. In particular, we have

**Theorem 2.4**

$$D(n,k) = \frac{1}{\binom{n}{k}} \sum_{j=1}^{n-k} \binom{n}{k+j} D(n,k+j) \binom{k+j}{j} (-A_{k+j})^j. \tag{2.5}$$

Theorem 2.4 is important since it allows us to prove Theorem 2.5 by induction on $k$. Theorem 2.5 allows us to write $D(n,k)$ in terms of $\{A_k\}$. We should note that Theorem 2.5 corresponds to Theorems 6 and 7 in [1].

**Theorem 2.5** *In general, $D(n,k)$ has the form of a sum of $2^{n-k-1}$ terms*

$$\sum_{\substack{\sigma \in 2^{n-k-1} \\ p_{k+1}+p_{k+2}+\cdots+p_n = n-k}} \frac{(n-k)!(-1)^{n-k+s}}{p_{k+1}!p_{k+2}!...p_n!} A^{p_{k+1}}_{\sigma(k+1)} A^{p_{k+2}}_{\sigma(k+2)} \cdots A^{p_n}_{\sigma(n)}, \tag{2.6}$$

*where $s$ is the number of nonzero $p_j'$s and $\sigma$ is an element of the power set of $[n-k-1]$ and acts on the base term $A_{k+1}A_{k+2}\cdots A_n$ via the following algorithm.*

*(i) Write $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_p$, where each the elements $\sigma_i$ forms a subsequence consisting of $t_i$ consecutive integers. Note that $1 \le t_i \le |\sigma|$ and $\sum_{i=1}^{p} t_i = |\sigma|$.*

*(ii) For $1 \le i \le p$, use $\sigma_i$ to locate the corresponding positions in the base term $A_{k+1}A_{k+2}..A_n$. In other words, if the smallest term in $\sigma_i$ is $m_i$, we work with the $A's$ in positions $m_i$ to $m_i+t_i-1$.*

*(iii) Change all the $A's$ located in Step II to the $A$ value located immediately to the right of this subsequence.*

In order to understand Equation (2.6), we will calculate $C(5,1)$. For $C(5,1)$, the base term is $A_2A_3A_4A_5$, and $\sigma$ is an element of the power set of [3]. For example, suppose $\sigma = \{2,3\}$. In this case, the elements of $\sigma$ form an increasing subsequence of two consecutive integers. We locate, in the base term, the $A's$ in positions 2 and 3. In other words, we will change $A_3A_4$ to the $A$ value immediately to the right of these terms, namely $A_5$. Hence, applying $\sigma = \{2,3\}$ turns $A_2A_3A_4A_5$ into $A_2A_5^3$. The following table calculates all the terms in $C(5,1)$.

| $\sigma$ | Adjusted Base Term | Term in $C(5,1)$ |
|---|---|---|
| $\emptyset$ | $A_2A_3A_4A_5$ | $24A_2A_3A_4A_5$ |
| $\{1\}$ | $A_3^2A_4A_5$ | $-12A_3^2A_4A_5$ |
| $\{2\}$ | $A_2A_4^2A_5$ | $-12A_2A_4^2A_5$ |
| $\{3\}$ | $A_2A_3A_5^2$ | $-12A_2A_3A_5^2$ |
| $\{1,2\}$ | $A_4^3A_5$ | $4A_4^3A_5$ |
| $\{1,3\}$ | $A_3^2A_5^2$ | $6A_3^2A_5^2$ |
| $\{2,3\}$ | $A_2A_5^3$ | $4A_2A_5^3$ |
| $\{1,2,3\}$ | $A_5^4$ | $-A_5^4$ |

Table 1 The terms of $C(5,1)$

**Proof of Theorem 2.5** We fix $n$ and induct on $k$. Using Equation (2.1), it is easy to show

$D(n,n)=1$ and $D(n,n-1)=A_n$. By Equation (2.6), we also have

$$D(n,n-1)=\sum_{\sigma\in 2^0}(-1)^{1+s}\frac{1!}{p_n!}A_{\sigma(n)}^{p_n}=A_n.$$

Now assume, for some positive integer $k$, that Equation (2.6) is true for $D(n,n-i)$, where $0\le i\le k$. We need to compute $D(n,n-k-1)$. By Equation (2.5), we have

$$\begin{aligned}
D(n,n-k-1)=&\frac{1}{\binom{n}{k+1}}\sum_{j=1}^{k+1}(-1)^j D(n,n-k-1+j)\binom{n}{n-k-1+j}\cdot\\
&(A_{n-k-1+j})^j\binom{n-k-1+j}{j}\\
=&\sum_{j}^{k+1}\frac{(-1)^j(k+1)!}{j!(k+1-j)!}(A_{n-k-1+j})^j D(n,n-k-1+j)\\
=&\sum_{j}^{k+1}\frac{(-1)^j(k+1)!}{j!(k+1-j)!}(A_{n-k-1+j})^j\cdot\\
&\sum_{\substack{\sigma\in 2^{k-j}\\ p_{n-k+j}+\cdots+p_n=k+1-j}}\frac{(-1)^{k+1-j+s}(k+1-j)!}{p_{n-k+j}!\ldots p_n!}A_{\sigma(n-k+j)}^{p_{n-k+j}}\ldots A_{\sigma(n)}^{p_n}\\
=&\sum_{j=1}^{k+1}\sum_{\substack{\sigma\in 2^{k-j}\\ p_{n-k+j}+\cdots+p_n=k+1-j}}\frac{(-1)^{k+1+s}(k+1)!}{j!p_{n-k+j}!\ldots p_n!}A_{n-k-1+j}^{j}A_{\sigma(n-k+j)}^{p_{n-k+j}}\ldots A_{\sigma(n)}^{p_n}\\
=&\sum_{\substack{\sigma\in 2^{k}\\ p_{n-k}+\cdots+p_n=k+1}}\frac{(-1)^{k+1+s}(k+1)!}{p_{n-k}!p_{n-k+1}!\ldots p_n!}A_{\sigma(n-k)}^{p_{n-k}}A_{\sigma(n-k+1)}^{p_{n-k+1}}\ldots A_{\sigma(n)}^{p_n}.
\end{aligned}$$

Note that third equality follows from the induction hypothesis. The final equality is what Equation (2.6) provides for $D(n,n-k-1)$. Thus, Theorem 2.5 is true via induction on $k$. □

## 3. Annihilation coefficients in binomial expansions

In this section we generalize Equations (1.5) and (1.6) as follows. For Equation (1.5), we replace the factors of $(x-A_k)^{n-k}$ and $x^n$ with appropriate binomial coefficients to obtain

$$\sum_{k=0}^{n}A(n,k;A)x-A_kn-k=\binom{x}{n}. \tag{3.1}$$

In a similar manner, we transform Equation (1.6) to obtain

$$\sum_{k=0}^{n}B(n,k;A)\binom{x-A_k}{k}=\binom{x}{n}. \tag{3.2}$$

### 3.0.1 Expansions of Equation (3.1)

The goal of this section is to find an expression that will allows us to inductively compute the $A(n,k;A)$ of Equation (3.1) and the $B(n,k;A)$ of Equation (3.2). We begin by finding the formula for the $A(n,k;A)$ of Equation (3.1). The first step to finding such a formula is to rewrite Equation (3.1) as

$$\sum_{k=0}^{n} A(n,k;A)\prod_{j=0}^{k-1}(n-j)\prod_{p=0}^{n-k-1}(x-A_k-p)=\prod_{q=0}^{n-1}(x-q). \tag{3.3}$$

It is well known [2] that

$$\prod_{q=0}^{n-1}(x-q)=\sum_{k=0}^{n} s(n,k)x^k, \tag{3.4}$$

where $s(n,k)$ is appropriate Stirling number of the first kind. We next use Lemma 3.1 to expand the inner product on the left side of Equation (3.3).

**Lemma 3.1** *Let $a_0$ be a fixed number. Then,*

$$\sum_{n=0}^{p+1}(-1)^{p+1-n}x^n\sum_{k=n}^{p+1}|s(p+1,k)|\binom{k}{n}a_0^{k-n}=\prod_{q=0}^{p}(x-a_0-q). \tag{3.5}$$

**Proof of Lemma 3.1** We will use induction on $p$. If $p=0$, the right side of Equation (3.5) becomes

$$(x-a_0)=-1|s(1,1)|a_0+x|s(1,1)|=\sum_{n=0}^{1}(-1)^{1-n}x^n\sum_{k=n}^{1}|s(1,k)|\binom{k}{n}a_0^{k-n}.$$

Now assume Equation (3.5) is true for all non-negative integers less than or equal to $p$. Then, for $p+1$, Equation (3.5) implies

$$\begin{aligned}\prod_{q=0}^{p+1}(x-a_0-q) =&(x-a_0-(p+1))\prod_{q=0}^{p}(x-a_0-q)\\
=&\sum_{n=0}^{p+1}(-1)^{p+2-n}x^n\sum_{k=n}^{p+1}(p+1)|s(p+1,k)|\binom{k}{n}a_0^{k-n}\ (0)+\\
&\sum_{n=0}^{p+1}(-1)^{p+2-n}x^n\sum_{k=n}^{p+1}(p+1)|s(p+1,k)|\binom{k}{n}a_0^{k+1-n}+\\
&\sum_{n=0}^{p+1}(-1)^{p+1-n}x^{n+1}\sum_{k=n}^{p+1}(p+1)|s(p+1,k)|\binom{k}{n}a_0^{k-n}\\
=&\sum_{n=0}^{p+2}(-1)^{p+2-n}x^n\sum_{k=n}^{p+1}(p+1)|s(p+1,k)|\binom{k}{n}a_0^{k-n}+\\
&\sum_{n=0}^{p+2}(-1)^{p+2-n}x^n\sum_{k=n}^{p+1}(p+1)|s(p+1,k)|\left[\binom{k}{n}+\binom{k}{n-1}\right]a_0^{k+1-n}\\
=&\sum_{n=0}^{p+2}(-1)^{p+2-n}x^n\sum_{k=n}^{p+2}(p+1)|s(p+1,k)|\binom{k}{n}a_0^{k-n}+\\
&\sum_{n=0}^{p+2}(-1)^{p+2-n}x^n\sum_{k=n}^{p+2}(p+1)|s(p+1,k)|\binom{k}{n}a_0^{k-n}\ (1)\end{aligned}$$

$$= \sum_{n=0}^{p+2} (-1)^{p+2-n} x^n \sum_{k=n}^{p+2} |s(p+2,k)| \binom{k}{n} a_0^{k-n} \ (2).$$

The equality at (0) follows from the inductive hypothesis. At (1), we used Pascal's Identity while at (2), we used the well-known recurrence for the Stirling numbers of the first kind. Note that the final expression at (2) is the left side of Equation (3.5) for $p+2$. □

Using Equation (3.4) and Lemma 3.1, we can rewrite Equation (3.3) as

$$\sum_{k=0}^{n} A(n,k;A) \prod_{j=0}^{k-1} (n-j) \sum_{j=0}^{n-k} (-1)^{n-k-j} x^j \sum_{i=j}^{n-k} |s(n-k,i)| \binom{i}{j} A_k^{i-j} = \sum_{j=0}^{n} s(n,j) x^j. \tag{3.6}$$

By interchanging the order of summation, we obtain

$$\sum_{j=0} x^j \sum_{k=0}^{n-j} (-1)^{n-k-j} A(n,k;A) \prod_{j=0}^{k-1} (n-j) \sum_{i=j}^{n-k} |s(n-k,i)| \binom{i}{j} A_k^{i-j} = \sum_{j=0}^{n} s(n,j) x^j. \tag{3.7}$$

Finally, by comparing the coefficient of $x^j$, we obtain the following formula.

$$\sum_{k=0}^{n-j} (-1)^{n-k-j} A(n,k;A) \prod_{j=0}^{k-1} (n-j) \sum_{i=j}^{n-k} |s(n-k,i)| \binom{i}{j} A_k^{i-j} = s(n,j). \tag{3.8}$$

We use Equation (3.8) to inductively define $A(n,m;A)$. In particular, $A(n,m;A)$ is

$$\frac{s(n,n-m) - \sum_{k=0}^{m-1} (-1)^{m-k} A(n,k) \prod_{p=0}^{k-1} (n-p) \sum_{i=n-m}^{n-k} |s(n-k,i)| \binom{i}{n-m} A_k^{i-n+m}}{\prod_{q=0}^{m-1} (n-q)}. \tag{3.9}$$

Thus, if we know $A(n,i;A)$, for $0 \le i \le m$, Equation (3.9) allows us to compute $A(n,m;A)$.

In order to see how Equation (3.9) works, we will compute $A(n,1;A)$ and $A(n,2;A)$. Using Equation (3.3), we easily show $A(n,0;A) = 1$. Then, via Equation (3.9), we have

$$\begin{aligned} A(n,1;A) &= \frac{1}{n} \Big[ s(n,n-1) + \sum_{i=n-1}^{n} |s(n,i)| \binom{i}{n-i} A_0^{i-n+1} \Big] \\ &= \frac{1}{n} \left[ s(n,n-1) + |s(n,n-1)| + n|s(n,n)| A_0 \right] = A_0. \end{aligned}$$

We also have

$$\begin{aligned} A(n,2;A) &= \frac{1}{n(n-1)} \Big[ s(n,n-2) - \sum_{k=0}^{1} (-1)^{2-k} \prod_{p=0}^{k-1} (n-p) \sum_{i=n-2}^{n-k} |s(n-k,i)| \binom{i}{n-2} A_k^{i-n+2} \Big] \\ &= \frac{1}{n(n-1)} \Big[ s(n,n-2) - \sum_{i=n-2}^{n} |s(n,i)| \binom{i}{n-2} A_0^{i-n+2} + nA_0 \sum_{i=n-2}^{n-1} |s(n-1,i)| \binom{i}{n-2} A_1^{i-n+2} \Big] \\ &= \frac{1}{n(n-1)} \Big[ [-(n-1)|s(n,n-1)| + n|s(n-1,n-2)|] A_0 - \binom{n}{n-2} A_0^2 + n(n-1) A_0 A_1 \Big]. \end{aligned}$$

### 3.0.2 Specific example of equation (3.1)

In order to help the reader comprehend Equation (3.9), we provide a simple example of

Equation (3.1). In particular, we let $A_k = k+1$. Then, Equation (3.1) becomes

$$\sum_{k=0}^{n} A(n,k)\binom{x-k-1}{n-k} = \binom{x}{n},$$

where, $A(n,k;A) \equiv A(n,k)$. For this situation, Equation (3.9) implies $A(n,m)$ is

$$\frac{s(n,n-m) - \sum_{k=0}^{m-1}(-1)^{m-k}A(n,k)\prod_{p=0}^{k-1}(n-p)\sum_{i=n-m}^{n-k}|s(n-k,i)|\binom{i}{n-m}(k+1)^{i-n+m}}{\prod_{q=0}^{m-1}(n-q)}. \tag{3.11}$$

Using Equation (3.11), we find that

$$A(n,0) = 1 = A(n,1)$$
$$A(n,2) = \frac{1}{n(n-1)}\Big[-(n-1)|s(n,n-1)| + n|s(n-1,n-2)| + \frac{3n(n-1)}{2}\Big].$$

In order to further understand the implications of Equation (3.11), we let $n = 2$. Recalling that $s(1,0) = 0$ while $s(2,1) = -1$, we verify, via Equation (3.11), that $A(2,0) = A(2,1) = A(2,2) = 1$. The reader may check

$$\sum_{k=0}^{2} A(2,k)\binom{x-k-1}{2-k} = \binom{x}{2}. \tag{3.12}$$

### 3.1 Expansions of Equation (3.2)

We now derive a formula for the $B(n,k;A)$ of Equation (3.2). In order to derive this formula, we first write Equation (3.2) in the following equivalent form, namely,

$$n!\sum_{k=0}^{n}\frac{B(n,k;A)}{k!}\prod_{p=0}^{k-1}(x-A_k-p) = \sum_{j=0}^{n}s(n,j)x^j. \tag{3.13}$$

Applying Lemma 3.1 to Equation (3.13) yields

$$n!\sum_{k=0}^{n}\frac{B(n,k;A)}{k!}\sum_{j=0}^{k}(-1)^{k-n}x^j\sum_{i=j}^{k}|s(k,i)|\binom{i}{j}A_k^{i-j} = \sum_{j=0}^{n}s(n,j)x^j. \tag{3.14}$$

Then, by interchanging the order of summation in the left side of Equation (3.11), we are able to compare the coefficients of $x^j$. This comparison implies

$$n!\sum_{k=j}^{n}(-1)^{k-n}\frac{B(n,k;A)}{k!}\sum_{i=j}^{k}|s(k,i)|\binom{i}{j}A_k^{i-j} = s(n,j). \tag{3.15}$$

Note that Equation (3.15) is the counterpart to Equation (3.8). More importantly, Equation (3.15) provides a formula that allows us to inductively define $B(n,p;A)$ in terms of $B(n,m;A)$, where $p+1 \le m \le n$. In particular, we can show that $B(n,p;A)$ is

$$\frac{(-1)^{n-p}p!}{n!}\Big[s(n,p) - n!\sum_{k=p+1}^{n}(-1)^{k-n}\frac{B(n,k;A)}{k!}\sum_{i=p}^{k}|s(k,i)|\binom{i}{p}A_k^{i-p}\Big]. \tag{3.16}$$

By inspection of Equation (3.2), we easily find that $B(n,n;A) = 1$. Then, by utilizing Equation (3.16), we can readily compute $B(n,m;A)$ for all $0 \le m \le n-1$. As an example, we provide the first two such computations. In particular, Equation (3.16) implies that $B(n,n-1;A) = A_n$ while

$$B(n,n-2;A) = \frac{nA_n(|s(n-1,n-2)| + (n-1)A_{n-1}) - (n-1)|s(n,n-1)|A_n - \frac{n(n-1)}{2}A_n^2}{n(n-1)}.$$

### 3.1.1 Specific example of Equation (3.2)

Once again, we believe it would be helpful to analyze a specific example of Equation (3.2). In our example, we let $A_k = k\beta$, where $\beta \neq 0$. In this case, Equation (3.2) becomes

$$\sum_{k=0}^{n} B(n,k)\binom{x-k\beta}{k} = \binom{x}{n}, \tag{3.17}$$

where, $B(n,k;A) \equiv B(n,k)$. Note that the binomial coefficient in the left side of Equation (3.17) is a Hagen-Rothe type coefficient [3].

In this situation, Equation (3.16) becomes

$$B(n,m) = \frac{(-1)^{n-p}p!}{n!}\Big[s(n,p) - n!\sum_{k=p+1}^{n}(-1)^{k-n}\frac{B(n,k)}{k!}\sum_{i=p}^{k}|s(k,i)|\binom{i}{p}(k\beta)^{i-p}\Big]. \tag{3.18}$$

Using Equation (3.18), we find that

$$B(n,n) = 1,$$
$$B(n,n-1) = \beta n,$$
$$B(n,n-2) = \frac{n\beta(|s(n-1,n-2)| + (n-1)(n-1)\beta) - (n-1)|s(n,n-1)|\beta - \frac{n(n-1)}{2}n\beta^2}{(n-1)}.$$

To further understand the previous three equations, we let $n = 2$ and obtain $B(2,2) = 1$, $B(2,1) = 2\beta$, and $B(2,0) = -\beta$. The reader should verify that

$$\binom{x-2\beta}{2} + 2\beta\binom{x-\beta}{1} - \beta\binom{x}{0} = \binom{x}{2}, \tag{3.19}$$

which is Equation (3.17) with $n = 2$.

## 4. $q$-number annihilation coefficients

In this section, we generalize the work Gould started in Section 6 of [1]. Recall that the $q$-numbers are defined by

$$[x] = [x]_q = \frac{q^x - 1}{q - 1} \quad \text{with} \quad [0] = 0. \tag{4.1}$$

Note that when $q = 1$, we let $[x] = x$. The goal of this section is to analyze formulas for the Convolution AC associated with

$$\sum_{k=0}^{n} Q(n,k;A)[x - A_k]^{n-k} = [x]^n, \tag{4.2}$$

and for the Dot Product AC associated with

$$\sum_{k=0}^{n} P(n,k;A)[x - A_k]^k = [x]^n. \tag{4.3}$$

Note that Equations (4.2) and (4.3) are the $q$-analogs of Equations (1.5) and (1.6). Thus, it will not be surprising to find that the formulas for $Q(n,k;A)$ and $P(n,k;A)$ closely resemble the formulas for $A(n,k;A)$ found in Section 2 of [1] and $B(n,k;A)$ found in Section 2 of this paper.

We begin our analysis by rewriting Equation (4.2). Recalling that

$$[x+y] = q^y[x] + [y], \quad [-x] = -q^{-x}[x], \tag{4.4}$$

we find that Equation (4.2) is equivalent to

$$\sum_{k=0}^{n} q^{-A_k(n-k)}([x]-[A_k])^{n-k} Q(n,k;A) = [x]^n. \tag{4.5}$$

The Binomial Theorem allows us to transform Equation (4.5) into an equivalent expression involving a double sum.

$$\sum_{j=0}^{n} [x]^j \sum_{k=0}^{n-j} (-1)^{n-k-j} q^{A_k(n-k)} Q(n,k;A) \binom{n-k}{j} [A_k]^{n-k-j} = [x]^n. \tag{4.6}$$

By comparing the coefficients of $[x]^j$ on both sides of Equation (4.6), we are able to show that

$$Q(n,0;A) = q^{nA_0} \tag{4.7}$$

and for $0 < j \le n$,

$$\frac{Q(n,j;A)}{q^{A_j(n-j)}} = \sum_{k=0}^{j-1} (-1)^{n-j-k-1} q^{-A_k(n-k)} \binom{n-k}{n-j} [A_k]^{j-k} Q(n,k;A). \tag{4.8}$$

Equations (4.6) and (4.7) are the $q$-analogs of Equation (15) in [1].

We now provide Theorem 4.1, the $q$-analog of Theorem 7 in [1]. Since the proof of Theorem 4.1 is basically identical to that of Theorem 7 given in [1], (i.e., we perform induction on $k$ and use Equation (4.8) in place of Equation (15)), we will omit the details and simply state the theorem.

**Theorem 4.1** *Suppose* $\sum_{k=0}^{n} Q(n,k;A)[x-A_k]^{n-k} = [x]^n$. *Then,*

$$\frac{Q(n,k;A)}{q^{A_k(n-k)}\binom{n}{k}} = \sum_{\{p_0,p_1,\ldots,p_{k-1}\}\in P_k} (-1)^{\epsilon} \frac{k!}{p_0!p_1!..p_{k-1}!} [A_0]^{p_0}[A_1]^{p_1}..[A_{k-1}]^{p_{k-1}}, \tag{4.9}$$

*where the index set* $P_k$ *is defined as follows:* $p_0 + p_1 + \cdots + p_{k-1} = k$, *with* $0 \le p_i \le k-i$ *for* $i \ge 1$ *and* $1 \le p_0 \le k$. *Also* $p_i = r \ge 0$ *implies* $p_{i+1} = p_{i+2} = \cdots = p_{i+r-1} = 0$. *The exponent* $\epsilon = (-1)^k +$ *the number of non-zero* $p_j's$ *in* $P_k$.

We now turn our attention to Equation (4.3). Using the $q$-number identities provided by Equation (4.4), we write Equation (4.3) as follows, namely

$$\sum_{k=0}^{n} q^{-A_k k}([x]-[A_k])^k P(n,k;A) = [x]^n. \tag{4.10}$$

Once again, the Binomial Theorem allows us to transform Equation (4.10) into Equation (4.11).

$$\sum_{j=0}^{n} [x]^j \sum_{k=j}^{n} (-1)^{k-j} a^{-kA)k} \binom{k}{j} [A_k]^{k-j} P(n,k;A) = [x]^n. \tag{4.11}$$

By comparing the coefficients of $[x]^j$, we find that

$$P(n,n;A) = q^{nA_n}, \tag{4.12}$$

and for $0 \le j < n$,

$$\frac{P(n,j;A)}{q^{A_j j}} = \sum_{k=j+1}^{n} (-1)^{k-j-1} q^{-A_k k} \binom{k}{j} [A_k]^{k-j} P(n,k;A). \tag{4.13}$$

Note that Equations (4.12) and (4.13) are the $q$-analogs of Theorem 2.4.

We end this section by describing the $q$-analog of Theorem 2.5. This is Theorem 4.2. Since the proof of Theorem 4.2 directly parallels the induction proof of Theorem 2.5 (with Equation (2.5) replaced with Equation (4.13)), we omit the details and simply state theorem.

**Theorem 4.2** *Suppose* $\sum_{k=0}^{n} P(n,k;A)[x-A_k]^k = [x]^n$. *Then,*

$$\frac{P(n,k;A)}{q^{kA_k}\binom{n}{k}} = \sum_{\substack{\sigma\in 2^{n-k-1} \\ p_{k+1}+p_{k+2}+\cdots+p_n=n-k}} \frac{(n-k)!(-1)^{n-k+s}}{p_{k+1}!p_{k+2}!...p_n!} [A_{\sigma(k+1)}]^{p_{k+1}} [A_{\sigma(k+2)}]^{p_{k+2}} ... [A_{\sigma(n)}]^{p_n}, \tag{4.14}$$

*where $s$ is the number of nonzero $p_j' s$ and $\sigma$ is an element of the power set of $2^{n-k-1}$ and acts on the base term $[A_{k+1}][A_{k+2}]..[A_n]$ via the following algorithm.*

*(i) Write $\sigma = \sigma_1\sigma_2...\sigma_p$, where each the elements $\sigma_i$ forms a subsequence consisting of $t_i$ consecutive integers. Note that $1 \le t_i \le |\sigma|$ and $\sum_{i=1}^{p} t_i = |\sigma|$.*

*(ii) For $1 \le i \le p$, use $\sigma_i$ to locate the corresponding positions in the base term $[A_{k+1}][A_{k+2}]..[A_n]$. In other words, if the smallest term in $\sigma_i$ is $m_i$, we work with the $[A's]$ in positions $m_i$ to $m_i+t_i-1$.*

*(iii) Change all the $[A's]$ located in Step II to the $[A]$ value located immediately to the right of this subsequence.*

## 5. $q$-Binomial annihilation coefficients

In this section, we expand the results of Section 7 of [1]. These results parallel the results provided in Section 3.

Recall that $q$-binomial is defined by

$$\begin{bmatrix} x \\ n \end{bmatrix} = \prod_{i=1}^{n} \frac{q^{x-i+1}-1}{q^1-1}, \quad \begin{bmatrix} x \\ 0 \end{bmatrix} = 1. \tag{5.1}$$

Equivalently, we could define the $q$-binomial by

$$\begin{bmatrix} x \\ n \end{bmatrix} = \frac{[x][x-1]\cdots[x-n+1]}{[n]!}, \tag{5.2}$$

where $[n]! = [n][n-1]...[2][1]$. Note that $[0]! = 1$. With these definitions in place, it is only natural to generalize Equations (3.1) and (3.2) as follows

$$\sum_{k=0}^{n} A(n,k;A) \begin{bmatrix} x-A_k \\ n-k \end{bmatrix} = \begin{bmatrix} x \\ n \end{bmatrix}, \tag{5.3}$$

$$\sum_{k=0}^{n} B(n,k;A) \begin{bmatrix} x-A_k \\ k \end{bmatrix} = \begin{bmatrix} x \\ n \end{bmatrix}. \tag{5.4}$$

Our goal is to determine formulas, similar to Equations (3.8) and (3.12), that allow us to recursively define $A(n,k;A)$ and $B(n,k;A)$, respectively. The techniques to find these formulas require that the reader be familiar with the $q$-Stirling number of the first kind, denoted $S_1(n,k,q)$, and a few of their basic properties.

Recall from [2], that $S_1(n,k,q)$ is the sum of the $\binom{n}{k}$ possible products, each with different factors, formed from the first $n$ $q$-natural numbers. From this definition, it can be shown [2] that

$$S_1(n,k,q) = S_1(n-1,k,q) + [n]S_1(n-1,k-1,q), \tag{5.5}$$

$$\prod_{k=0}^{n-1}([x]-[k]) = \sum_{k=0}^{n}(-1)^{n-k}S_1(n-1,n-k,q)[x]^k. \tag{5.6}$$

We are now ready to determine a formula for recursively defining the $A(n,k;A)$. First, using Equation (5.2), we transform Equation (5.3) as follows:

$$\sum_{k=0}^{n} A(n,k:A)\prod_{j=0}^{k-1}[n-j]\prod_{p=0}^{n-k-1}[x-A_k-p] = [x][x-1]...[x-n+1]. \tag{5.7}$$

Using Equations (4.4) and (5.6), we can show that the right side of Equation (5.7) is equal to

$$\begin{aligned}[x][x-1]\cdots[x-n+1] &= q^{\frac{-n(n-1)}{2}}\prod_{k=0}^{n-1}([x]-[k]) \\ &= q^{\frac{-n(n-1)}{2}}\sum_{k=0}^{n}(-1)^{n-k}S_1(n-1,n-k,q)[x]^k. \end{aligned} \tag{5.8}$$

We now need to simplify the inner product on the left side of Equation (5.7). Using Equation (4.4), we easily show

$$\prod_{p=0}^{n-k-1}[x-A_k-p] = q^{\frac{-(n-k)(2A_k+n-k-1)}{2}}\prod_{i=0}^{n-k-1}([x]-[A_k+i]). \tag{5.9}$$

Using induction on $i$ and the recursion given by Equation (5.6), we prove that

$$\prod_{i=0}^{p}([x]-[A_k+i]) = \sum_{m=0}^{p+1}(-1)^{p+1-m}C(p+1,p+1-k)[x]^m, \tag{5.10}$$

where $C(p+1,m) \equiv S_1(p+1,m,q)$ with $[j] \to [A_k+j-1]$, for $m>0$. Otherwise, $C(p+1,0) \equiv [1]$. For example, $C(2,2,q) = [A_k][A_k+1]$, which is obtained from $S_1(2,2,q) = [1][2]$ by the substitutions of $[1] \to [A_k]$ and $[2] \to [A_k+1]$.

All the previous calculations allow us to write Equation (5.7) as

$$\sum_{k=0}^{n} q^{\frac{-(n-k)(2A_k+n-k-1)}{2}} A(n,k:A) \prod_{j=0}^{k-1}[n-j] \sum_{p=0}^{n-k}(-1)^{n-k-p} C(n-k+1,n-k-p)[x]^p$$
$$= q^{\frac{-n(n-1)}{2}} \sum_{p=0}^{n}(-1)^{n-p} S_1(n-1,n-p,q)[x]^p. \tag{5.11}$$

By rearranging the order of summation in the left side of Equation (5.11) and then comparing the coefficients of $[x]^p$, we obtain the formula

$$\sum_{k=0}^{n-p}(-1)^{n-k-p} q^{\frac{-(n-k)(2A_k+n-k-1)}{2}} A(n,k;A) \prod_{j=0}^{k-1}[n-j] C(n-k+1,n-k-p)$$
$$= q^{\frac{-n(n-1)}{2}}(-1)^{n-p} S_1(n-1,n-p,q). \tag{5.12}$$

Equation (5.12) is the parallel of Equation (3.8). Thus, we can use Equation (5.12) to recursively solve for $A(n,p;A)$. In particular, by letting $p \to n-p$, Equation (5.12) becomes

$$\sum_{k=0}^{p}(-1)^{p-k} q^{\frac{-(n-k)(2A_k+n-k-1)}{2}} A(n,k;A) \prod_{j=0}^{k-1}[n-j] C(n-k+1,p-k)$$
$$= q^{\frac{-n(n-1)}{2}}(-1)^{p} S_1(n-1,p,q). \tag{5.13}$$

Thus, Equation (5.13) implies that

$$A(n,p;A) = \frac{q^{\frac{(n-p)(2A_p+n-p-1)-n(n-1)}{2}}(-1)^p S_1(n-1,p,q)}{[n][n-1]\cdots[n-p+1]} - \frac{q^{\frac{(n-p)(2A_p+n-p-1)}{2}}}{[n][n-1]\cdots[n-p+1]}\cdot$$
$$\sum_{k=0}^{p-1}(-1)^{p-k} q^{\frac{-(n-k)(2A_k+n-k-1)}{2}} A(n,k;A) \prod_{j=0}^{k-1}[n-j] C(n-k+1,p-k). \tag{5.14}$$

Note that Equation (5.14) is the $q$ generalization of Equation (3.9).

For the remainder of this section, we work on determining a recursive formula for $B(n,k;A)$ of Equation (5.4). Just as in the case of Equation (5.3), we use the properties of $q$-binomials to transform Equation (5.4) into the following equivalent equation.

$$\sum_{k=0}^{n} \frac{q^{\frac{-k(2A_k+k-1)}{2}}[n]!B(n,k:A)}{[k]!} \sum_{j=0}^{k}(-1)^{k-j}[x]^j C(k,k-j)$$
$$= q^{\frac{-n(n-1)}{2}} \sum_{j=0}^{n}(-1)^{n-j}[x]^j S_1(n-1,n-j,q). \tag{5.15}$$

After rearranging the order of summation on the right side of Equation (5.5), we are then able to compare the coefficients of $[x]^j$. This comparison shows that

$$\sum_{k=j}^{n} B(n,k:A)\frac{q^{\frac{-k(2A_k+k-1)}{2}}[n]!}{[k]!}(-1)^{k-j}C(k,k-j) = q^{\frac{-n(n-1)}{2}}(-1)^{n-j}S_1(n-1,n-j,q). \tag{5.16}$$

Note that Equation (5.16) is the parallel of Equation (3.12). It can be used to recursively solve for $B(n,p;A)$. In particular, Equation (5.16) implies

$$B(n,p:A) = \frac{[p]!q^{\frac{-n(n-1)+p(2A_p+p-1)}{2}}(-1)^{n-p}}{[n]!}S_1(n-1,n-p,1) - [p]!\sum_{k=p+1}^{n} B(n,k:A)\frac{q^{\frac{-k(2A_k+k-1)}{2}}(-1)^{n-p}}{[k]!}C(k,k-p), \tag{5.17}$$

which is the parallel to Equation (3.13).

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