Journal of Mathematical Research & Exposition
Mar., 2010, Vol. 30, No. 2, pp. 217–226
DOI:10.3770/j.issn:1000-341X.2010.02.003
Http://jmre.dlut.edu.cn

A Class of Iterative Formulae for Solving Equations

Sheng Feng LI[1,2,3,*], **Jie Qing TAN**[1,2], **Jin XIE**[1,2,4], **Xing HUO**[1,2]

1. *School of Computer & Information, Hefei University of Technology, Anhui* 230009*, P. R. China;*
2. *Institute of Applied Mathematics, Hefei University of Technology, Anhui* 230009*, P. R. China;*
3. *Department of Mathematics & Physics, Bengbu College, Anhui* 233030*, P. R. China;*
4. *Department of Mathematics & Physics, Hefei University, Anhui* 230601*, P. R. China*

Abstract Using the forms of Newton iterative function, the iterative function of Newton's method to handle the problem of multiple roots and the Halley iterative function, we give a class of iterative formulae for solving equations in one variable in this paper and show that their convergence order is at least quadratic. At last we employ our methods to solve some non-linear equations and compare them with Newton's method and Halley's method. Numerical results show that our iteration schemes are convergent if we choose two suitable parametric functions $\lambda(x)$ and $\mu(x)$. Therefore, our iteration schemes are feasible and effective.

Keywords Non-linear equation; iterative function; order of convergence; Newton's method; Halley's method.

Document code A
MR(2000) Subject Classification 34A34; 26A18; 39B12
Chinese Library Classification O241.7

1. Introduction

When solving a non-linear equation $f(x)=0$, we can easily consider constructing an iterative function to solve the equation. Usually, there are different ways to build the iterative functions and a great diversity of iteration schemes can be obtained at present, such as Newton's scheme, Halley's scheme, etc.

The Newton's method is one of most powerful and well-known numerical methods for solving a root-finding problem $f(x)=0$. Its iterative function is

$$\varphi(x)=x-\frac{f(x)}{f'(x)}. \tag{1}$$

As we all know, Newton's method converges quadratically [1–3].

Received January 1, 2009; Accepted May 20, 2009
Supported by the National Natural Science Foundation of China (Grant Nos. 60773043;60473114), the Key Project Foundation of Scientific Research, Ministry of Education of China (Grant No. 309017), the Doctoral Program Foundation of Ministry of Education of China (Grant No. 20070359014), the Natural Science Key Foundation of Education Department of Anhui Province (Grant No. KJ2010A237), the Research Funds for Young Innovation Group of Education Department of Anhui Province (Grant No. 2005TD03), the Provincial Foundation for Excellent Young Talents of Colleges and Universities of Anhui Province (Grant No. 2010SQRL118) and the Research Funds for Young Teachers in the College of Education Department of Anhui Province (Grant No. 2008jq1158).
* Corresponding author
E-mail address: lsf7679@yahoo.com.cn (S. F. LI)

If an equation $f(x) = 0$ has multiple roots, then the iterative function of Newton's method to handle the problem of multiple roots is defined by

$$\varphi(x) = x - \frac{f(x)f'(x)}{f'^2(x) - f(x)f''(x)}. \tag{2}$$

If $\varphi(x)$ is provided with the required continuity conditions, then functional iteration applied to $\varphi(x)$ will be quadratically convergent regardless of the multiplicity of the roots [1–3].

In the paper [4], Halley iterative function for solving an equation $f(x) = 0$ by means of Padé approximation was presented by

$$\varphi(x) = x - \frac{2f(x)f'(x)}{2f'^2(x) - f(x)f''(x)}. \tag{3}$$

If $f(x)$ has fifth derivative on $[a, b]$, then it is easy to show that Halley's method converges cubically.

In the present paper, a class of new iteration formulae are constructed by applying the forms of the iterative functions (1), (2) and (3). And it shows that their convergence order is at least two. At last numerical examples are given to show that our iteration scheme is feasible and effective.

2. A class of new iterative formulae

Let x^* be a solution of the nonlinear equation $f(x) = 0$. In order to find approximate value of x^*, we construct an iterative function $\varphi(x)$ by observing the forms of the iterative functions (1), (2) and (3). Then $\varphi(x)$ can be written as follows:

$$\varphi(x) = x - \frac{\lambda(x)f(x)}{\nu(x)f'(x) - \mu(x)f(x)f''(x)}, \tag{4}$$

where three arbitrary parametric functions $\lambda(x)$, $\mu(x)$ and $\nu(x)$ are continuous on the interval $[a, b]$.

Now, we require $\varphi'(x^*) = 0$ so that the iterative function (4) is convergent. Computing first derivative of (4) gives

$$\varphi'(x) = 1 - \frac{(\lambda' f + \lambda f')(\nu f' - \mu f f'') - \lambda f(\nu' f' + \nu f'' - \mu' f f'' - \mu f' f'' - \mu f f''')}{(\nu f' - \mu f f'')^2}. \tag{5}$$

Taking into account $f(x^*) = 0$ and above expression (5), we can get the following equation

$$\varphi'(x^*) = 1 - \frac{\lambda(x^*)}{\nu(x^*)} = 0, \tag{6}$$

which implies that parametric functions $\lambda(x)$ and $\nu(x)$ satisfy $\dfrac{\lambda(x^*)}{\nu(x^*)} = 1$ at the root x^*. Unfortunately, we do not know the exact root x^* of the equation at the beginning. In order to avoid bringing troubles to select parametric functions $\lambda(x)$ and $\nu(x)$ such that $\dfrac{\lambda(x^*)}{\nu(x^*)} = 1$, we set $\dfrac{\lambda(x)}{\nu(x)} = 1$ at every point of the interval $[a, b]$. This can make the problem simple and the expression (4) can be rewritten as the form

$$\varphi(x) = x - \frac{\lambda(x)f(x)}{\lambda(x)f'(x) - \mu(x)f(x)f''(x)}. \tag{7}$$

This is a class of iterative functions and the iteration formulae can be set up by

$$x_{k+1} = x_k - \frac{\lambda(x_k)f(x_k)}{\lambda(x_k)f'(x_k) - \mu(x_k)f(x_k)f''(x_k)}. \tag{8}$$

It is obvious that we can get different iterative functions by selecting different parametric functions $\lambda(x)$ and $\mu(x)$. The expression (7) becomes Newton iterative function (1) if $\lambda(x) = 1$, $\mu(x) = 0$. (7) is the iterative function (2) if $\lambda(x) = f'(x)$, $\mu(x) = 1$. Halley iterative function (3) is got if we select $\lambda(x) = 2f'(x)$ and $\mu(x) = 1$. If $\lambda(x) = 2f'^2(x) - f(x)f''(x)$ and $\mu(x) = f'(x)$, then (7) turns out to be

$$\varphi(x) = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x) - 2f(x)f'(x)f''(x)}. \tag{9}$$

It was shown in [5] that the order of convergence of the iterative function is at least three. If $\lambda(x) = 2f'^4(x) + f(x)f'^2(x)f''(x) + f^2(x)f''^2(x)$ and $\mu(x) = f(x)f'(x)f''(x) + f'^3(x)$, then (7) is the iterative function in [6] and [7]

$$\varphi(x) = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x)} - \frac{f^3(x)f''^2(x)}{2f'^5(x)}. \tag{10}$$

It was shown in [6] that the convergence order is at least three. If $\lambda(x) = 2f'^2(x) + f(x)f''(x)$ and $\mu(x) = f'(x)$, then (7) is the Householder iterative function in the paper [7]

$$\varphi(x) = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x)}. \tag{11}$$

If $\lambda(x) = [6f'^3(x)+3f(x)f'(x)f''(x)+f^2(x)f'''(x)]f''(x)$ and $\mu(x) = f(x)f'(x)f''(x)+3f'^2(x)f''(x)$, then (7) is another iterative function given in the paper [7]

$$\varphi(x) = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{2f'^3(x)} - \frac{f^3(x)f'''(x)}{6f'^4(x)}. \tag{12}$$

It was pointed out in [7] that the iterative function has nearly supercubic convergence.

3. Discussion of convergence

Definition 1 *Suppose that the sequence $\{x_k\}$ is generated by some iterative scheme $x_k = \varphi(x_{k-1})$. If $\{x_k\}, k = 1, 2, \dots$, converges to x^*, i.e., $\lim_{k\to\infty} x_k = x^*$, then x^* is called a root of the equation $f(x) = 0$.*

Lemma 1 *Suppose an iterative function $\varphi(x)$ satisfies the following conditions:*

(i) $\varphi(x) \in [a, b]$ for all $x \in [a, b]$;

(ii) These exists a Lipschitz positive constant $L < 1$ such that $|\varphi(x) - \varphi(y)| \le L|x - y|$ for $\forall x, y \in [a, b]$.

Then the sequence $\{x_k\}$ generated by the iterative scheme $x_k = \varphi(x_{k-1})$ is convergent for an arbitrarily initial number $x_0 \in [a, b]$ ([1–3]).

Lemma 2 *If the second condition in Lemma 1 is replaced with following condition:*

(iii) $\varphi'(x)$ is continuous on (a,b) with $|\varphi'(x)| \le L < 1$ for all $x \in [a,b]$,

then the conclusion of Lemma 1 holds [1–3].

Theorem 1 *Suppose that x^* is one solution of the equation $f(x) = 0$ and let $\lambda(x^*)f'(x^*) \neq 0$. If $f(x)$ is sufficiently smooth in the neighborhood of x^*, then the sequence $\{x_k\}$ generated by the iterative scheme (8) is convergent.*

Proof Computing first derivative of (7) gives

$$\varphi'(x) = 1 - \frac{(\lambda' f + \lambda f')(\lambda f' - \mu f f'') - \lambda f(\lambda' f' + \lambda f'' - \mu' f f'' - \mu f' f'' - \mu f f''')}{(\lambda f' - \mu f f'')^2}.$$

It is obvious that $\varphi'(x^*) = 0$. Based on the property of continuous function, there exists a constant $\varepsilon > 0$ such that $|\varphi'(x)| \le L < 1$ for all $x \in [x^* - \varepsilon, x^* + \varepsilon]$.
Now, by means of Mean Value Theorem, $\forall x \in [x^* - \varepsilon, x^* + \varepsilon]$, we have

$$\varphi(x) - \varphi(x^*) = \varphi'(\xi)(x - x^*), \;\; \xi \in [x, x^*].$$

Therefore, we obtain $|\varphi(x) - \varphi(x^*)| = |\varphi'(\xi)|\,|x - x^*| < |x - x^*| \le \varepsilon$, that is, $|\varphi(x) - x^*| < \varepsilon$.

It is clear that $\varphi(x) \in [x^* - \varepsilon, x^* + \varepsilon]$ for any $x \in [x^* - \varepsilon, x^* + \varepsilon]$. By Lemmas 1 and 2, we draw the conclusion that the sequence $\{x_k\}$ generated by the iterative scheme (8) is convergent when the initial value x_0 is sufficiently near to x^*. The proof is completed. □

Definition 2 *Suppose that the sequence $\{x_k\}$ is generated by some iterative scheme $x_k = \varphi(x_{k-1})$. Let $e_k = x_k - x^*$. If there exist two positive constants C and m such that*

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^m} = \lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^m} = C,$$

then the iterative method is said to converge to C of order m.

Theorem 2 *Suppose that x^* is one solution of equation $f(x) = 0$ and assume that $\lambda(x^*)f'(x^*) \neq 0$. If $f^{(3)}(x)$ is bounded in some neighborhood of x^*, then the convergence order of the iterative scheme (8) is at least 2.*

Proof Let $e_k = x_k - x^*$. By means of Taylor's expansion, one can get

$$\begin{aligned} f(x^*) &= f(x_k) + f'(x_k)(x^* - x_k) + \frac{1}{2} f''(x_k)(x^* - x_k)^2 + O((x^* - x_k)^3) \\ &= f(x_k) - f'(x_k)e_k + \frac{1}{2} f''(x_k)e_k^2 + O(e_k^3). \end{aligned}$$

Since $f(x^*) = 0$, then the above expression may be rewritten as follows

$$f(x_k) = f'(x_k)e_k - \frac{1}{2} f''(x_k)e_k^2 + O(e_k^3). \tag{13}$$

According to (8), we have

$$[\lambda(x_k)f'(x_k) - \mu(x_k)f(x_k)f''(x_k)](x_{k+1} - x_k) = -\lambda(x_k)f(x_k), \tag{14}$$

i.e.,

$$[\lambda(x_k)f'(x_k) - \mu(x_k)f(x_k)f''(x_k)][(x_{k+1} - x^*) - (x_k - x^*)] = -\lambda(x_k)f(x_k). \tag{15}$$

Rearranging the above equation (15), we get

$$\begin{aligned}&[\lambda(x_k)f'(x_k) - \mu(x_k)f(x_k)f''(x_k)]e_{k+1}\\ &\quad = [\lambda(x_k)f'(x_k) - \mu(x_k)f(x_k)f''(x_k)]e_k - \lambda(x_k)f(x_k).\end{aligned} \tag{16}$$

Substituting (13) into the above expression (16), one has

$$\begin{aligned}&[\lambda(x_k)f'(x_k) - \mu(x_k)f(x_k)f''(x_k)]e_{k+1}\\ &\quad = [\frac{1}{2}\lambda(x_k)f''(x_k) - \mu(x_k)f'(x_k)f''(x_k)]e_k^2 + O(e_n^3),\end{aligned} \tag{17}$$

namely,

$$e_{k+1} = \frac{\frac{1}{2}\lambda(x_k)f''(x_k) - \mu(x_k)f'(x_k)f''(x_k)}{\lambda(x_k)f'(x_k) - \mu(x_k)f(x_k)f''(x_k)}e_k^2 + O(e_k^3). \tag{18}$$

Letting k tend to positive infinity in the equation (18), we obtain $\lim\limits_{k\to\infty} \frac{e_{k+1}}{e_k^2} = M$, where $M = \frac{\frac{1}{2}\lambda(x^*)f''(x^*) - \mu(x^*)f'(x^*)f''(x^*)}{\lambda(x^*)f'(x^*)}$. Therefore, we show that the convergence order of the iterative scheme (8) is at least 2 according to Definition 2. □

Theorem 3 *Suppose that x^* is one solution of equation $f(x) = 0$ and assume that $\lambda(x^*)f'(x^*) \neq 0$. If $f^{(4)}(x)$ is bounded in some neighborhood of x^*, and $\lambda(x)$ and $\mu(x)$ are selected such that $\lambda(x) = 2\mu(x)f'(x) + f(x)K(x)$, where $K(x)$ is an arbitrary continuous function, then the convergence order of the iterative formula (8) is at least 3.*

Proof In the course of the proof in Theorem 2, we can see that the constant $M = 0$ if $\lambda(x) = 2\mu(x)f'(x) + f(x)K(x)$ in the expression (17). So we have $\lim\limits_{k\to\infty} \frac{e_{k+1}}{e_k^3} = N$, where the obtained constant N is similar to M in Theorem 2. By means of Taylor's expansion in the same way, we can get

$$N = -\frac{\frac{1}{6}\lambda(x^*)f'''(x^*) - \frac{1}{2}\mu(x^*)f''^2(x^*)}{\lambda(x^*)f'(x^*)},$$

which means that the convergence order of the iterative scheme (8) is at least 3 according to Definition 2. □

It is obvious that we can get some iterative formulae with third-order convergence according to Theorem 3. For example, we can get the Halley iterative function (3) if $\mu(x) = 1$ and $K(x) = 0$, the iterative formula (9) if $\mu(x) = f'(x)$ and $K(x) = -f''(x)$ and the iterative formula (10) if $\mu(x) = f(x)f'(x)f''(x) + f'^3(x)$ and $K(x) = f(x)f''^2(x) - f'^2(x)f''(x)$, etc.

Moreover, if we select the suitable $\lambda(x)$, $\mu(x)$ such that the constant $N = 0$, then some iterative formulae are obtained and we can prove their convergence order is at least four. For example, if $\lambda(x) = 6f'^2(x)f''(x) - 3f(x)f''^2(x) + 2f(x)f'(x)f'''(x)$ and $\mu(x) = 3f'(x)f''(x)$, then we can obtain the following iterative function with fourth-order convergence in [8]

$$\varphi(x) = x - \frac{f(x)(6f'^2(x)f''(x) - 3f(x)f''^2(x) + 2f(x)f'(x)f'''(x))}{2f'(x)(3f'^2(x)f''(x) - 3f(x)f''^2(x) + f(x)f'(x)f'''(x))}.$$

4. Numerical examples

Now, we employ our iterative scheme (7) to find zeros of some non-linear equations and compare them with Newton's method (1) and Halley iterative scheme (3). All computations are carried out with double arithmetic precision. All problems are solved with a given initial value x_0. We choose $|f(x_k)| < 10^{-14}$ as stopping criterion so that the iterative process is terminated when the criterion is satisfied. The number of iterations k and the root x^* are displayed in the Tables 1-3. We use the following functions:

$$\begin{aligned} f_1(x) &= x^3 + 4x^2 - 25, & f_2(x) &= x^2 - e^x - 3x + 2, \\ f_3(x) &= \cos(x) - x, & f_4(x) &= x^2 + \sin(x/5) - 1/4, \\ f_5(x) &= x^5 - 5x + 1, & f_6(x) &= 2xe^{-x} - x^2 + 2x - 1/2. \end{aligned}$$

Shown in Table 1 is the comparison between the method (7) with second-order convergence and Newton's method (1). And in Table 2, our method (7) with third-order convergence is compared with Halley's method (3). It is easy to find that we can freely choose parametric functions such that our methods are fairly flexible. In Table 3 our method (7) is compared with both Newton's method (1) and Halley's method (3). It can been seen from Table 3 that we can select suitable parametric functions $\lambda(x)$ and $\mu(x)$ such that our iterative scheme (8) can be executed while both Newton's method and Halley's method fail. It is well known that Newton's method and Halley's method converge locally for solving equations, that is to say, the initial value x_0 should be chosen sufficiently near the root x^* such that the sequence generated by one of the two iterative schemes is convergent. One tries to analyze global convergence of Newton's method and Halley's method, that is, one tries to discuss how to select x_0 within given bounds such that the two methods are convergent. For the functions $f_5(x)$ and $f_6(x)$, a lot of $\lambda(x)$ and $\mu(x)$ can be chosen such that our method (8) can be executed. One major reason is that the range of initial value x_0 is extended. Therefore, the sequence generated by our method (8) can converge for initial value x_0 while the one generated by Newton's method or Halley's method diverges when we select the same initial value. As for how much the range of initial value x_0 can be extended, it involves the choices of the two parametric functions in iterative schemes and the function $f(x)$ itself. This is still a challenging work and needs further study. It is a meaningful work to find the choice criteria of two parametric functions and make the iterative schemes converge within as wide a range as possible. But that may be a little difficult at the present stage. As a result, the method (7) presented in this paper is superior to either Newton's method, or Halley's method to some extent.

$f(x)$	x_0	$\lambda(x)$	$\mu(x)$	k	x_k	$\lvert f(x_k)\rvert$
f_1	2	-2	-1	5	2.0352684811819590	0
		-1	1	9	2.0352684811819590	0
		1	2	7	2.0352684811819590	0
		x^3	$-x$	5	2.0352684811819590	0
		$-\sin x$	$\cos x$	5	2.0352684811819590	0
★		1	0	4	2.0352684811819595	3.00×10^{-29}
f_2	0.5	-2	-1	4	0.25753028543986073	2.22×10^{-16}
		-1	1	4	0.25753028543986084	0
		1	2	5	0.25753028543986073	2.22×10^{-16}
		x^3	$-x$	7	0.25753028543986073	2.22×10^{-16}
		$-\sin x$	$\cos x$	5	0.25753028543986073	2.22×10^{-16}
★		1	0	4	0.2575302854398608	0
f_3	1	-2	-1	5	0.739085133215160642	3.68×10^{-29}
		-1	1	5	0.739085133215160642	1.11×10^{-29}
		1	2	5	0.739085133215160705	1.07×10^{-16}
		x^3	$-x$	5	0.739085133215160642	2.42×10^{-24}
		$-\sin x$	$\cos x$	4	0.739085133215160626	2.70×10^{-17}
★		1	0	4	0.739085133215160642	1.07×10^{-20}
f_4	0.5	-2	-1	3	0.40999201798913715	0
		-1	1	5	0.4099920179891371	2.78×10^{-17}
		1	2	5	0.40999201798913715	0
		x^3	$-x$	6	0.4099920179891371388	1.72×10^{-15}
		$-\sin x$	$\cos x$	5	0.4099920179891386	1.50×10^{-15}
★		1	0	4	0.40999201798913715	0

★ Newton's method (1)

Table 1 Comparison of Newton's method (1) ($\lambda(x) = 1$, $\mu(x) = 0$) and the methods (7)

$f(x)$	x_0	$\mu(x)$	$K(x)$	k	x_k	$\lvert f(x_k)\rvert$
f_1	2	-2	-1	2	2.035268481181959	0
		-1	1	2	2.035268481181959	0
		1	2	3	2.035268481181959	0
		x^3	$-x$	2	2.035268481181959	0
		$-\sin x$	$\cos x$	2	2.035268481181959	0
♦		1	0	2	2.035268481181959	0
f_2	0.5	-2	-1	3	0.2575302854398608	0
		-1	1	3	0.2575302854398608	0
		1	2	3	0.2575302854398608	0
		x^3	$-x$	3	0.2575302854398608	0
		$-\sin x$	$\cos x$	3	0.2575302854398608	0
♦		1	0	3	0.2575302854398608	0
f_3	1	-2	-1	3	0.739085133215160642	3.31×10^{-27}
		-1	1	3	0.739085133215160642	2.23×10^{-40}
		1	2	3	0.739085133215160642	3.03×10^{-24}
		x^3	$-x$	3	0.739085133215160642	1.39×10^{-35}
		$-\sin x$	$\cos x$	3	0.7390851332151607	0
♦		1	0	3	0.739085133215160642	5.64×10^{-29}
f_4	0.5	-2	-1	3	0.40999201798913715	0
		-1	1	3	0.40999201798913715	0
		1	2	3	0.40999201798913715	0
		x^3	$-x$	3	0.40999201798913715	0
		$-\sin x$	$\cos x$	3	0.40999201798913715	0
♦		1	0	3	0.4099920179891371	2.78×10^{-17}

♦ Halley's method (3)

Table 2 Comparison of Halley's method (3) ($\mu(x)=1,\ K(x)=0$) and the methods (7)

$f(x)$	x_0	$\lambda(x)$	$\mu(x)$	k	x_k	$\|f(x_k)\|$
f_5	-1	-2	-1	18	-1.5416516841045247	0
		-1	1	11	0.20006410262997543	2.22×10^{-16}
		1	2	54	-1.5416516841045245	7.11×10^{-15}
		x^3	$-x$	18	0.2000641026299754	0
		$-\sin x$	$\cos x$	11	-1.5416516841045247	0
★		1	0	Failure		
⧫		$2f'(x)$	1	Failure		
		$x^8 + 2x^2$	$x \sin x$	8	-1.5416516841045247	0
f_6	1	-2	-1	6	0.1388129811590014	0
		-1	1	8	2.0180630008717833	1.11×10^{-16}
		1	2	13	0.13881298115900145	1.11×10^{-16}
		x^3	$-x$	6	2.0180630008717833	1.11×10^{-16}
		$-\sin x$	$\cos x$	7	2.0180630008717833	1.11×10^{-16}
★		1	0	Failure		
⧫		$2f'(x)$	1	Failure		
		$x^8 + 2x^2$	$x \sin x$	28	0.13881298115900137	1.11×10^{-16}

★ Newton's method (1)

⧫ Halley's method (3)

Table 3 Comparison of Newton's method (1), Halley's method (3) and the methods (7)

Remark We have obtained a class of iterative methods (7) based on the forms of Newton iterative functions involving single roots and multiple roots and the Halley iterative function. In Theorem 2, we prove that the method (7) has second-order convergence. It is also shown that the methods (7) converge cubically if we choose suitable parametric functions $\lambda(x)$, $\mu(x)$ in Theorem 3. Moreover, we can get higher-order iterative formulae by selecting suitable parametric functions $\lambda(x)$, $\mu(x)$. Numerical results show that the methods (8) are more flexible than Newton's method and Halley's method.

References

[1] BURDEN R L, FAIRES J D. *Numerical Analysis (Sixth ed.)* [M]. Brooks/Cole Publishing Company, Calif., 1997.

[2] STOER J, BULIRSCH R. *Introduction to Numerical Analysis (Third ed.)* [M]. Springer-Verlag, New York, 2002.

[3] QUARTERONI A, SACCO R, SALERI F. *Numerical Mathematics* [M]. Springer-Verlag, New York, 2000.

[4] CUYT A. *A class of adaptive multivariate nonlinear iterative methods* [J]. Rocky Mountain J. Math., 1991, **21**(1): 171–185.

[5] BASTO M, SEMIAO V, CALHEIROS F L. *A new iterative method to compute nonlinear equations* [J]. Appl. Math. Comput., 2006, **173**(1): 468–483.

[6] CHUN C. *A new iterative method for solving nonlinear equations* [J]. Appl. Math. Comput., 2006, **178**(2): 415–422.

[7] ABBASBANDY S. *Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method* [J]. Appl. Math. Comput., 2003, **145**(2-3): 887–893.

[8] LI Shengfeng, TAN Jieqing, XIE Chengjun. et al. *An iterative formula with fourth-order convergence for solving equations based on Thiele's continued fraction* [J]. J. Univ. Sci. Technol. China, 2008, **38**(2): 138–140. (in Chinese)

[9] KOU Jisheng. *Some variants of Cauchy's method with accelerated fourth-order convergence* [J]. J. Comput. Appl. Math., 2008, **213**(1): 71–78.

[10] CHUN C, HAM Y. *Some fourth-order modifications of Newton's method* [J]. Appl. Math. Comput., 2008, **197**(2): 654–658.