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Some Results on Graph Products Determined by Their Spectra

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Abstract In this paper, we prove that some Kronecker products of G and K_2 are determined by their spectra where the graph G is also determined by its spectrum. And a problem for further researches is proposed.

Keywords Kronecker product; spectrum of graph; determined by spectra.

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1. Introduction

In this paper, we only consider undirected simple graphs (loops and multiple edges are not allowed). Let G be a graph with n vertices and the adjacency matrix $A(G)$. Let $D(G)$ be the diagonal matrix with the degrees of G. The matrix $L(G)=D(G)-A(G)$ is called the Laplacian matrix of G. The adjacency (resp. Laplacian) spectrum of G is the set of all eigenvalues of $A(G)$ (resp. $L(G)$) together with their multiplicities. The adjacency matrix of the complement of graph G is denoted by $\overline{A(G)}$, that is, $\overline{A(G)}=J-A(G)-I$ where J and I are the all-ones matrix and the identity matrix, respectively.

Two graphs are said to be cospectral with respect to (w.r.t. for short) adjacency (resp. Laplacian) matrix if they share the same adjacency (resp. Laplacian) spectrum. A graph G is said to be determined by its spectrum (DS for short) if any graph H that has the same spectrum as G is isomorphic to G (of course, we should identify the spectrum concerned, such as the adjacency spectrum, the Laplacian spectrum, etc.)

The problem of characterizing the DS graphs goes back for half a century and originates from chemistry. By now few families of DS graphs are known, so finding new families of DS graphs is an interesting problem. For the background and some known results about this problem, we refer the reader to [1] and the references therein.

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The Cartesian product of two graphs G and H, denoted by $G \bigcirc H$, is a new graph that has vertex set $V(G) \times V(H)$ and edge set $\{(a,x)(b,y) : ab \in E(G)$ and $x = y$ or $xy \in E(H)$ and $a = b\}$. The Kronecker product of G and H, denoted by $G \times H$, is a new graph that has vertex set $V(G) \times V(H)$ and edge set $\{(a,x)(b,y) : ab \in E(G)$ and $xy \in E(H)\}$. The strong product of G and H, denoted by $G \otimes H$, is also a new graph that has vertex set $V(G) \times V(H)$ and edge set $E(G \bigcirc H) \bigcup E(G \times H)$.

In [2], some properties of Kronecker products of graphs were given. In this paper, we investigate the DS properties of products of some graphs and K_2, and show that some products of some known DS graphs and K_2 are also DS. Finally, we propose a problem for further researches.

2. Preliminaries

The following are several known results we shall use in the next section.

Theorem 2.1 ([1]) *A regular graph is DS if and only if it is DS w.r.t. the adjacency matrix A, the Laplacian matrix L and the adjacency matrix $\overline{A}$ of the complement.*

Lemma 2.1 ([1]) *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum.*

(i) The number of vertices. (ii) The number of edges.

(iii) Whether G is regular. (iv) Whether G is regular with any fixed girth.

For the adjacency matrix the following follow from the spectrum.

(v) The number of closed walk of any length. (vi) Whether G is bipartite.

For the Laplacian matrix the following follow from the spectrum.

(vii) The number of spanning trees. (viii) The number of components.

Lemma 2.2 *Let G be a regular graph. If H is cospectral with G w.r.t. the adjacency matrix A, then G and H are cospectral w.r.t. the Laplacian matrix L.*

Proof Suppose that G is a k-regular graph of order n. Then, so is the graph H by (i), (ii) and (iii) of Lemma 2.1. Then $D(G) = D(H) = kI_n$. Since $A(G)$ and $A(H)$ are two cospectral symmetric 0,1-matrices, they are similar. It follows that $D(G) - A(G)$ and $D(H) - A(H)$ are also similar ones. This implies that the graphs G and H are cospectral w.r.t. the Laplacian matrix L. □

Theorem 2.2 ([1]) *The complete graph K_n, the regular complete bipartite graph $K_{m,m}$, the cycle C_n and their complements are DS.*

Theorem 2.3 ([1]) *The path P_n with n vertices is DS w.r.t. the adjacency matrix.*

Theorem 2.4 ([1]) *The disjoint union of k complete graphs $K_{m_1} + K_{m_2} + \cdots + K_{m_k}$, the disjoint union of k disjoint paths $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$, the disjoint union of k disjoint cycles $C_{n_1} + C_{n_2} + \cdots + C_{n_k}$ are all DS w.r.t. the adjacency matrix.*

Theorem 2.5 ([2]) *Let G be a connected graph. The Kronecker product $G \times K_2$ is a bipartite graph with partition $\{(x,1)|x \in V(G)\} \bigcup \{(x,2)|x \in V(G)\}$. If G has no odd cycle, then $G \times K_2$ has exactly two connected components isomorphic to G.*

Theorem 2.6 ([3]) *Let G be a bipartite graph with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$ with respective multiplicities m_1, m_2, and m_3. Then $\lambda_1 = -\lambda_3$, $\lambda_2 = 0$, $m_3 = m_1$ and G is the disjoint union of m_1 complete bipartite graphs K_{r_i,s_i} where $r_i s_i = \lambda_1^2$, $i = 1, \ldots, m_1$, and $m_2 - \sum_{i=1}^{m_1}(r_i + s_i - 2)$ isolated vertices.*

3. Main results

In this section we only consider the DS property of some graphs w.r.t. the adjacency matrix. So, the DS property w.r.t. the adjacency matrix is simply denoted by DS.

In [4], it was pointed out that a regular graph G has ± 1 and $\pm r$ as distinct eigenvalues if and only if each connected component is isomorphic to a graph obtained from $K_{r+1,r+1}$ by deleting a complete matching. By the definition of Kronecker product and Cartesian product of graphs, it is obvious that $K_n \bigcirc K_2 = (K_n \times K_2)^c$, where G^c denotes the complement of G. Because of the regularity of $K_n \times K_2$ and Theorem 2.1, $K_n \bigcirc K_2$ is DS. Moreover, we can easily verify that $K_n \otimes K_2 = K_{2n}$. In view of Theorem 2.2, the following theorem immediately holds.

Theorem 3.1 *The products $K_n \bigcirc K_2$, $K_n \times K_2$, $K_n \otimes K_2$ are all DS.*

Theorem 3.2 *Let G be a k-regular bipartite graph of order $2m$. If $k = 1, 2, m-1, m$, then the product $G \times K_2$ is DS.*

Proof For $k = 1$, we have $G = P_2 + P_2 + \cdots + P_2 = mP_2$. It can be easily verified that $G \times K_2 = 2mP_2$. By Theorem 2.4, this theorem follows immediately.

For $k = 2$, G must be a union of t disjoint even cycles, that is, $G = C_{2k_1} + C_{2k_2} + \cdots + C_{2k_t}$ where $k_1 + k_2 + \cdots + k_t = m$. Applying Theorems 2.4 and 2.5, this theorem holds.

For $k = m-1$, G will be denoted by $K_{m,m}^{(1)}$, which is a bipartite graph by removing a complete matching from $K_{m,m}$. Then we have $K_{m,m}^{(1)} \times K_2 = 2K_{m,m}^{(1)}$ with application of Theorem 2.5.

Suppose that the graph H is cospectral with $K_{m,m}^{(1)} \times K_2$. Then H must be an $(m-1)$-regular bipartite graph of order $4m$ with two connected components because of (i), (iii), (vi), (viii) of Lemmas 2.1 and 2.2. So we can assume that $H = H_1 + H_2$ where H_i is an $(m-1)$-regular bipartite graph of order $2m$ for $i = 1, 2$. Otherwise we have $|V(H_1)| \neq |V(H_2)|$, then one of H_i for $i = 1, 2$, without loss of generality, H_1 is $K_{m-1,m-1}$, and its spectrum is $\{m-1, 0^{2m-4}, -(m-1)\}$ (see p.72-74 in [4]). $K_{m,m}^{(1)}$ is isomorphic to $K_m \times K_2$ by Theorem 3.1, so the spectrum of $K_{m,m}^{(1)} \times K_2$ is $\{(m-1)^2, 1^{2m-2}, (-1)^{2m-2}, (-(m-1))^2\}$. This is a contradiction to the fact that $H_1 + H_2$ is cospectral with $K_{m,m}^{(1)} \times K_2$. Since $K_n \times K_2$ is DS, H is isomorphic to $2K_{m,m}^{(1)} = K_{m,m}^{(1)} \times K_2$.

The proof of the case for $k = m$ is similar to that of the case for $k = m-1$, and is omitted here. □

Remark 3.1 Note that the proof of the case for $m = k$ in Theorem 3.2 can be simplified. A

k-regular graph G is said to be strongly regular with parameters (k, a, c) if each pair of adjacent vertices in G has the same number $a \geq 0$ of common neighbors, and each pair of non-adjacent vertices in G has the same number $c \geq 1$ of common neighbors. It was pointed out in [1] that the disjoint union of t copies of a strongly regular DS graph is also DS. It is obvious that $K_{m,m}$ is a strongly regular graph with parameters $(m, 0, 1)$. We can complete the proof by choosing $t = 2$.

Corollary 3.1 *The product $C_n \times K_2$ is DS.*

Proof By the definition of Kronecker product of graphs, it is obvious that

$$C_n \times K_2 = \begin{cases} C_{2n}, & \text{if } n \text{ is odd;} \\ 2C_n, & \text{if } n \text{ is even.} \end{cases}$$

This corollary holds immediately by Theorems 2.2 and 3.2. □

By a similar method, we can show that the product $P_n \times K_2$ is also DS.

In the next, we will consider the DS property of complete bipartite graphs and always assume that m, n are two positive integers with $m \leq n$ in the complete bipartite graph $K_{m,n}$.

Theorem 3.3 *The complete bipartite graph $K_{m,n}$ $(m < n)$ is DS if and only if for any integer k with $m < k < n$, k is not a factor of mn.*

Proof Let $f(x) = m+n-x-\frac{mn}{x}$, and let $f_0(x) = x + \frac{mn}{x}$, where x and $\frac{mn}{x}$ are always positive integers.

First we consider the necessity of this theorem. Suppose that a graph H is cospectral with $K_{m,n}$. Since the spectrum of $K_{m,n}$ is $\{\sqrt{mn}, 0^{m+n-2}, -\sqrt{mn}\}$, by Theorem 2.6, $H = K_{r_1,s_1} + (m+n-r_1-s_1)K_1$ where $r_1 s_1 = mn$. Because of the DS property of $K_{m,n}$, we may have that $r_1 = m, s_1 = n$. Since the function $f(x) = m+n-x-\frac{mn}{x}$ has the maximum value 0 and reaches its maximum only at $x = m$ or $x = n$, the function $f_0(x) = x + \frac{mn}{x}$ has the minimum value $m+n$ in $[m, n]$ and reaches its minimum only at $x = m$ or $x = n$. Owing to the monotonicity of the function $f_0(x)$ at the integers in the intervals $(m, \sqrt{mn})$ and $(\sqrt{mn}, n)$, any integer x such that $m < x < n$ is not a factor of mn.

Now we turn to the sufficiency of this theorem. Because of the non-divisibility of mn by any integer k with $m < k < n$, the function $f_0(x) = x + \frac{mn}{x}$ reaches the minimum value $m+n$ only at $x = m$ or $x = n$. So we have that $m+n-r_1-s_1 \leq 0$ for all positive integers r_1, s_1 such that $r_1 \leq s_1$ and $r_1 s_1 = mn$ and the equality holds only if $r_1 = m, s_1 = n$. Therefore, by Theorem 2.6, any graph H cospectral with $K_{m,n}$ is isomorphic to $K_{m,n}$, thus this result follows. □

Corollary 3.2 *The star $K_{1,n}$ is DS if and only if n is 1 or prime.*

For any integer k with $1 < k < n$, k is not a factor of n, then n must be prime. Combining it and Theorem 2.2, this corollary follows immediately.

Corollary 3.3 *Let m and n be two positive integers with $n - m \leq 2$. Then the complete bipartite graph $K_{m,n}$ is DS.*

Proof When $n-m=0$, this result is easily obtained by Theorem 2.2. For the case of $n-m=1$, there is no integer k such that $m<k<n$, therefore this result follows immediately from Theorem 3.3. While $n-m=2$, $m+1$ is the unique integer between m and n which is not a factor of $mn=m(m+2)$. In view of Theorem 3.3, the proof is completed. □

Note that for $n-m=3$, this corollary does not hold. It is easy to verify that $K_{1,4}$ and $K_{2,2}+K_1$ are two cospectral but not isomorphic graphs.

Corollary 3.4 *If m and n are two distinct prime integers, then the complete bipartite graph $K_{m,n}$ is DS.*

Corollary 3.5 *If m and n are two distinct integers such that $\sqrt{mn}$ is an integer, then the complete bipartite graph $K_{m,n}$ is not DS.*

Corollary 3.6 *For $i\in\{1,2\}$, let m and n be two distinct integers such that $n-m>i$ and $mn=x^2+ix$ for a positive integer x. Then the complete bipartite graph $K_{m,n}$ is not DS.*

Since x and $x+2$ are two factors of mn such that $m<x<x+i<n$ for $i=1,2$ when $n-m>i$, the proof of this corollary is a direct consequence of Theorem 3.3. As examples, $K_{m,n}$ and $K_{x,x+i}+(m+n-2x-i)K_1$ are a pair of cospectral but not isomorphic graphs.

Remark 3.2 Let $k>1$ be an integer. Since there exists an integer xk such that $xk|(mknk)$ when $m<x<n$ and x is a factor of mn, the complete bipartite graph $K_{mk,nk}$ is not DS when $K_{m,n}$ is not DS. However, not all the complete bipartite graphs $K_{mk,nk}$ is DS when $K_{m,n}$ is DS . As an example, $K_{2,3}$ is DS, so is the graph $K_{4,6}$, but $K_{20,30}$ is not DS because of the fact that $20\times 30=600=24\times 25$.

Theorem 3.4 *The complete bipartite graph $K_{m,n}$ is DS if and only if $K_{m,n}\times K_2$ (the Kronecker product of $K_{m,n}$ by K_2) is DS.*

Proof By Theorem 2.5, we have that $K_{m,n}\times K_2=2K_{m,n}$. In view of Theorem 3.3, the graph $K_{m,n}$ is DS if and only if the function $f_0(x)=x+\frac{mn}{x}$, where x and $\frac{mn}{x}$ are positive integers, has the minimum value $m+n$ at $x=m$ or $x=n$. It is equivalent to the fact that the bivariate function $g(x,y)=x+\frac{mn}{x}+y+\frac{mn}{y}$, where x and y are positive integers, has the minimum value $2(m+n)$ at $(x,y)=(m,m),(m,n),(n,m)$ or (n,n). Considering that the spectrum of $2K_{m,n}$ is $\{\sqrt{mn}^2,0^{2m+2n-4},(-\sqrt{mn})^2\}$ (see p.72-74 in [4]), by Theorem 2.6, the graph $K_{m,n}\times K_2$ is DS if and only if the bivariate function $g(x,y)=x+\frac{mn}{x}+y+\frac{mn}{y}$, where x and y are positive integers, has the minimum value $2(m+n)$ at $(x,y)=(m,m),(m,n),(n,m)$ or (n,n). Thus we have completed the proof. □

Corollary 3.7 *The product graph $K_{m,n}\times K_2$ is DS if and only if for any integer k with $m<k<n$, k is not a factor of mn.*

In this paper we have provided some Kronecker products of graphs by K_2 determined by their spectra. But for a more general case, it seems difficult to give an exact answer. It may be helpful to investigate the structure and the automorphism group of Kronecker product of graphs. Based on the results on the products of known graph which is DS and K_2, we propose the following problem.

Problem 3.1 Which DS graphs are the ones such that their Kronecker products by K_2 are also DS?

When G is a connected regular graph, the above problem may be easier to deal with. But so far we have not obtained any progress about it. As the interesting problems, some more modest ones may be worth researching, such as:

(1) Given two DS graphs G_0 and G, which graphs are the objects such that $G \bigcirc G_0$, $G \times G_0$ and $G \otimes G_0$ are all DS w.r.t. the adjacency matrix?

(2) Which DS graphs denoted by G_0 are the ones such that $G_0 \bigcirc G_0$, $G_0 \times G_0$ and $G_0 \otimes G_0$ are DS w.r.t. the adjacency matrix?

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