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(f,ω)-Compatible Pair (B,H) for ω-Smash Coproduct Hopf Algebras

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Abstract In this paper we introduce the notion of (f,ω)-compatible pair (B,H), by which we construct a Hopf algebra in the category ${}_H^H\mathrm{YD}$ of Yetter-Drinfeld H-modules by twisting the comultiplication of B. We also study the property of ω-smash coproduct Hopf algebras $B_\omega \bowtie H$.

Keywords ω-smash coproduct Hopf algebras; (f,ω)-compatible pair; Yetter-Drinfeld category.

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1. Introduction

In 1986, braided monoidal categories were introduced by Joyal and Street [1]. Since then this notion has been studied extensively. If H and H^{cop} are Hopf algebras, then the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$ is also a braided monoidal categories [2, 3]. Let H be a Hopf algebra and B a Hopf algebra in the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$, Radford constructed a new Hopf algebra $B\star H$ and stated that constructing a biproduct Hopf algebra is equivalent to constructing a Hopf algebra in the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$ [4]. A Hopf algebra in the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$ has been constructed [5–7].

In this paper R denotes a fixed commutative ring with unit, and we follow the terminology by Sweedler [8]. For a coalgebra C and $c\in C$, we have $\triangle(c)=\sum c_1\otimes c_2$. The antipode of a Hopf algebra H is denoted by S (or S_H). For a left H-comodule (M,ρ) and $m\in M$, we have $\rho(m)=\sum m_{-1}\otimes m_0\in H\otimes M$.

Let H be a Hopf algebra. We call B a bialgebra in the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$. If B is both a left H-module algebra and a left H-module coalgebra, B is both a left H-comodule algebra and a left H-comodule coalgebra, satisfying the following compatibility conditions for $a,b\in B$:

1) $\triangle(ab)=\sum a_1(a_{2-1}\rightarrow b_1)\otimes a_{20}b_2$;

2) $\varepsilon(ab)=\varepsilon(a)\varepsilon(b)$, $\varepsilon(1_B)=1_R$.

If B is a bialgebra in the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$ with antipode S, where S is a morphism in the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$, then B is called a Hopf algebra in the Yetter-Drinfeld category ${}_H^H\mathrm{YD}$.

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Let B and H be R-coalgebras and consider a linear map $\omega : B\otimes H \to H\otimes B$. The ω-smash coproduct coalgebra $B_\omega \ltimes H$ is defined as the R-module $B\ltimes H$ with comultiplication

$$\triangle_{B_\omega\ltimes H} = (I_B\otimes\omega\otimes I_H)\circ(\triangle_B\otimes\triangle_H)$$

and counit $\varepsilon_B\otimes\varepsilon_H$, where certain conditions are to be imposed on ω to ensure the required properties of $\triangle_{B_\omega\ltimes H}$ and $\varepsilon_B\otimes\varepsilon_H$. If B and H are Hopf algebras, we may consider $B\otimes H$ as algebra with componentwise multiplication, and necessary and sufficient conditions are given to make $B_\omega\ltimes H$ with this multiplication a Hopf algebra which we call the ω-smash coproduct Hopf algebra and denote it by $B_\omega \bowtie H$ [9]. As a dual concept of quasitriangular bialgebra, braided bialgebra was introduced by Larson and Towber as a tool for providing solutions to the quantum Yang-baxter equations [10]. The braided structures of ω-smash coproduct Hopf algebras $B_\omega\bowtie H$ were studied by Jiao and Wisbauer [11].

In this paper, we define (f,ω)-compatible Hopf algebra pair (B,H) and present some relative properties. We construct left H-module and left H-comodule structure of B by (f,ω)-compatible Hopf algebra pair (B,H) such that B is in the Yetter-Drinfeld category ${}^H_H\mathrm{YD}$. We also present a new comultiplication of B by twisting the comultiplication of B such that B is a Hopf algebra in the Yetter-Drinfeld category ${}^H_H\mathrm{YD}$.

2. (f,ω)-compatible pair (B,H)

Let B and H be R-coalgebras and consider a linear map $\omega : B\otimes H\to H\otimes B$. Then a comultiplication is defined on the R-module $B\otimes H$ by

$$\triangle_{B_\omega\ltimes H} = (I_B\otimes\omega\otimes I_H)\circ(\triangle_B\otimes\triangle_H) \tag{1}$$

and an R-linear map is given by

$$\varepsilon_{B_\omega\ltimes H} := \varepsilon_B\otimes\varepsilon_H : B_\omega\ltimes H\to R. \tag{2}$$

If the triple $(B\otimes H,\ \triangle_{B_\omega\ltimes H},\ \varepsilon_{B_\omega\ltimes H})$ forms a coalgebra, then it is called a smash coproduct of B and H and we denote it by $B_\omega\ltimes H$. This imposes certain conditions on the map ω. To describe these, we write for $b\in B$ and $h\in H$,

$$\omega(b\otimes h) = \sum {}^\omega h\otimes {}^\omega b.$$

Then we get for comultiplication and counit

$$\triangle_{B_\omega\ltimes H}(b\otimes h) = \sum(b_1\otimes^\omega h_1)\otimes({}^\omega b_2\otimes h_2), \tag{3}$$

$$\varepsilon_{B_\omega\ltimes H}(b\otimes h) = \varepsilon_B(b)\varepsilon_H(h). \tag{4}$$

Proposition 2.1 ([9]) *With the notation above, $B_\omega\ltimes H$ is a smash coproduct if and only if the following conditions hold for $b\in B$ and $h\in H$:*

(C.1) $(I_H\otimes\varepsilon_B)\omega(b\otimes h) = \varepsilon_B(b)h$;

(C.2) $(\varepsilon_H\otimes I_B)\omega(b\otimes h) = \varepsilon_H(h)b$;

(C.3) $\sum({}^\omega h)_1\otimes({}^\omega h)_2\otimes^\omega b = \sum {}^\omega h_1\otimes {}^{\overline{\omega}}h_2\otimes {}^{\overline{\omega}}({}^\omega b)$;

(C.4) $\sum {}^\omega h\otimes({}^\omega b)_1\otimes({}^\omega b)_2 = \sum {}^{\overline{\omega}}({}^\omega h)\otimes {}^{\overline{\omega}}b_1\otimes {}^\omega b_2$.

Let B, H be bialgebras and $\omega : B\otimes H \to H\otimes B$ a linear map such that $B_\omega \ltimes H$ is a coalgebra. The canonical multiplication on $B\otimes H$ makes $B_\omega \ltimes H$ an algebra and it becomes a bialgebra provided $\triangle_{B_\omega\ltimes H}$ is a multiplicative map, that is, ω is an algebra map. In this case we call $B_\omega \ltimes H$ an ω-smash coproduct bialgebra and denote it by $B_\omega \bowtie H$. Furthermore, if B and H are Hopf algebras with antipodes S_B and S_H, then $B_\omega \bowtie H$ is a Hopf algebra with an antipode which is, for $b\in B$ and $h\in H$, given by

$$S_{B_\omega\bowtie H}(b\otimes h)=\sum S_B({}^{\omega}b)\otimes S_H({}^{\omega}h).$$

Definition 2.2 *Let B, H be Hopf algebras and $f : H \to B$ a Hopf algebra morphism. Then (B,H) is called an (f,ω)-compatible pair if, for all $h\in H$*

(D.1) $\sum f(h_2)\otimes h_1=\sum {}^{\omega}f(h_1)\otimes {}^{\omega}h_2.$

Remark If (B,H) is an (f,ω)-compatible pair, then we can obtain

(D.2) $f(h)\otimes 1_H=\sum {}^{\omega}f(h_2)\otimes h_1S_H({}^{\omega}h_3).$

(D.3) $\sum f(S_H(h_1))\otimes S_H(h_2)=\sum {}^{\omega}f(S_H(h_2))\otimes {}^{\omega}S_H(h_1).$

Example 2.3 1) Let B and H be Hopf algebras and $f : H\to B$ a Hopf algebra morphism, $\omega = T : B\otimes H\to H\otimes B$ be the switch map. Then $B_\omega \bowtie H = B\otimes H$ is the usual tensor product of Hopf algebras B and H. If H is a cocommutative Hopf algebra, then (B,H) is an (f,ω)-compatible pair.

2) Let B be a commutative Hopf algebra and H a Hopf algebra and $f : H\to B$ a Hopf algebra morphism. For all $b\in B$ and $h\in H$. Let

$$\omega : B\otimes H\to H\otimes B,\quad b\otimes h\to \sum h_2\otimes bfS_H(h_1)f(h_3).$$

Then $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra and (B,H) is an (f,ω)-compatible pair by direct calculation.

Lemma 2.4 ([11]) *Let $B_\omega \bowtie H$ be an ω-smash coproduct Hopf algebra with a right normal linear map ω, that is, for all $h\in H$, $\omega(1_B\otimes h)=h\otimes 1_B$. Then for all $b\in B$ and $h\in H$,*

(E.1) $\sum {}^{\omega}1_Hh\otimes {}^{\omega}b=\sum {}^{\omega}h\otimes {}^{\omega}b=\sum h\,{}^{\omega}1_H\otimes {}^{\omega}b;$

(E.2) $\sum {}^{\overline{\omega}}1_H{}^{\omega}1_H\otimes {}^{\overline{\omega}}b_1\otimes {}^{\omega}b_2=\sum {}^{\omega}1_H\otimes ({}^{\omega}b)_1\otimes ({}^{\omega}b)_2=\sum {}^{\omega}1_H{}^{\overline{\omega}}1_H\otimes {}^{\overline{\omega}}b_1\otimes {}^{\omega}b_2.$

Proposition 2.5 *Let $B_\omega \bowtie H$ be an ω-smash coproduct Hopf algebra with a right normal linear map ω. For all $h\in H$ and $b\in B$, define:*

$$i_H : H\to B_\omega\bowtie H,\ i_H(h)=1_B\otimes h;$$
$$j_B : B\to B_\omega\bowtie H,\ j_B(b)=b\otimes 1_H.$$

Then 1) i_H is a bialgebra morphism;

2) j_B is an algebra morphism and satisfies

$$\triangle_{B_\omega\bowtie H}j_B(b)=\sum j_B(b_1)i_H({}^{\omega}1_H)\otimes j_B({}^{\omega}b_2).$$

Proof The proof follows by direct calculations.

Let $B_\omega \bowtie H$ be an ω-smash coproduct Hopf algebra with a right normal linear map ω, $\sigma : (B_\omega \bowtie H) \otimes (B_\omega \bowtie H) \to R$ is a bilinear map. For all $a, b \in B$ and $h, g \in H$, define:

$$\tau : H \otimes H \to R,\ \tau(h, g) = \sigma(1_B \otimes h, 1_B \otimes g).$$

If $(B_\omega \bowtie H, \sigma)$ is a braided Hopf algebra, then it can be easily derived from Proposition 2.5 that (H, τ) is a braided Hopf algebra.

Proposition 2.6 *Let B, H be Hopf algebras. Then (B, H) is an (f, ω)-compatible pair if and only if $F : H \to B_\omega \bowtie H$, and $F(h) = \sum f(h_1) \otimes h_2$ is a Hopf algebra morphism.*

Proof Since F is a Hopf algebra morphism, for all $h \in H$, we have

$$\triangle_{B_\omega \bowtie H} F(h) = \sum f(h_1) \otimes {}^\omega h_3 \otimes {}^\omega f(h_2) \otimes h_4,$$
$$(F \otimes F) \triangle_H (h) = \sum f(h_1) \otimes h_2 \otimes f(h_3) \otimes h_4,$$
$$\sum f(h_1) \otimes {}^\omega h_3 \otimes {}^\omega f(h_2) \otimes h_4 = \sum f(h_1) \otimes h_2 \otimes f(h_3) \otimes h_4.$$

Using $\varepsilon_B \otimes \mathrm{id} \otimes \mathrm{id} \otimes \varepsilon_H$ in the equation above, we obtain $\sum f(h_2) \otimes h_1 = \sum {}^\omega f(h_1) \otimes {}^\omega h_2$. Conversely, if $\sum f(h_2) \otimes h_1 = \sum {}^\omega f(h_1) \otimes {}^\omega h_2$, we easily derive F is a Hopf algebra morphism.

3. The construction of Hopf algebra in ${}^H_H\mathrm{YD}$

In this section, let B and H be Hopf algebras with linear map $\omega : B \otimes H \to H \otimes B$ which is right normal such that $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra.

Proposition 3.1 *Let (B, H) be an (f, ω)-compatible pair. For all $b \in B$ and $h \in H$, define:*

$$\alpha : H \otimes B \to B,\ \alpha(h \otimes b) = h \rightarrow b = \sum f(h_1) b f(S_H(h_2));$$
$$\rho : B \to H \otimes B,\ \rho(b) = \sum b_{-1} \otimes b_0 = \sum {}^\omega 1_H \otimes {}^\omega b.$$

Then

1) $(B, \rightarrow)$ is a left H-module algebra;

2) (B, ρ) is a left H-comodule algebra;

3) $(B, \rightarrow, \rho)$ is a left Yetter-Drinfeld module, if it satisfies the condition

(F.1) $\sum S_B({}^\omega b) \otimes {}^\omega h = \sum {}^\omega (S_B(b)) \otimes {}^\omega h$.

Proof 1) The proof follows by direct calculations.

2) From the equation (C.3),we have $(I \otimes \rho)\rho(b) = \sum {}^\omega 1_H \otimes {}^{\overline{\omega}} 1_H \otimes {}^{\overline{\omega}} ({}^\omega b) = \sum ({}^\omega 1_H)_1 \otimes ({}^\omega 1_H)_2 \otimes {}^\omega b = (\triangle_H \otimes I)\rho(b)$.

Since ω is an algebra map, for all $a, b \in B$, $\omega(ab \otimes 1_H) = \omega(a \otimes 1_H)\omega(b \otimes 1_H)$, that is $\sum {}^\omega 1_H \otimes {}^\omega (ab) = \sum {}^\omega 1_H {}^{\overline{\omega}} 1_H \otimes {}^\omega a {}^{\overline{\omega}} b$. We obtain

$$\rho(ab) = \sum {}^\omega 1_H {}^{\overline{\omega}} 1_H \otimes {}^\omega a {}^{\overline{\omega}} b = \rho(a)\rho(b).$$

Thus (B, ρ) is a left H-comodule algebra.

3) Since (F.1) $\sum S_B({}^\omega b) \otimes {}^\omega h = \sum {}^\omega (S_B(b)) \otimes {}^\omega h$, that is

(D.1) $\sum f(h_2) \otimes h_1 = \sum {}^\omega f(h_1) \otimes {}^\omega h_2 \Leftrightarrow$

(F.2) $\sum f(S_H(h_2))\otimes h_1=\sum {}^{\omega}f(S_H(h_1))\otimes {}^{\omega}h_2$.

We get

$$\begin{aligned}\sum(h_1\to b)_{-1}h_2\otimes(h_1\to b)_0&=\sum {}^{\omega}1_H h_3\otimes {}^{\omega}(f(h_1)bf(S_H(h_2)))\\&=\sum {}^{\omega}1_H{}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}h_3\otimes {}^{\omega}f(h_1)^{\overline{\omega}}b^{\overline{\overline{\omega}}}f(S_H(h_2))\\&=\sum {}^{\omega}1_H{}^{\overline{\omega}}1_H h_2\otimes {}^{\omega}f(h_1)^{\overline{\omega}}bf(S_H(h_3))\\&=\sum {}^{\overline{\omega}}1_H{}^{\omega}1_H h_2\otimes {}^{\omega}f(h_1)^{\overline{\omega}}bf(S_H(h_3))\\&=\sum {}^{\overline{\omega}}1_H h_1\otimes f(h_2)^{\overline{\omega}}bf(S_H(h_3))\\&=\sum h_1b_{-1}\otimes(h_2\to b_0).\end{aligned}$$

Finally, we conclude that $(B,\to,\rho)$ is a left Yetter-Drinfeld module.

Theorem 3.2 *Let (B,H) be an (f,ω)-compatible pair and ω satisfies the condition (F.1). Then there exists a bialgebra $\overline{B}$ in ${}^H_H YD$. In particular, $\overline{B}=B$ as vector space and the left H-module structure map α and left H-comodule structure map ρ of $\overline{B}$ in ${}^H_H YD$ as in Proposition 3.1, the multiplication and the unit and the counit of $\overline{B}$ coincide with bialgebra B, respectively. The comultiplication of $\overline{B}$ is defined as follows:*

$$\overline{\triangle}(b)=\sum b_1fS_H({}^{\omega}1_H)\otimes^{\omega}b_2,$$

where the comultiplication of B is defined by $\triangle(b)=\sum b_1\otimes b_2$.

Furthermore, the bialgebra $\overline{B}$ is a Hopf algebra in ${}^H_H YD$ with an antipode which is given by

$$\overline{S}(b)=\sum f({}^{\omega}1_H)S_B({}^{\omega}b).$$

Proof From Proposition 3.1, we easily derive $\overline{B}$ is an object in the category ${}^H_H$YD and $(\overline{B},\to)$ is H-module algebra and $(\overline{B},\rho)$ is H-comodule algebra. First we establish that $\overline{B}$ is an R-coalgebra. We compute:

$$\begin{aligned}(\overline{\triangle}\otimes I)\overline{\triangle}(b)&=\sum b_1fS_H(({}^{\omega}1_H)_2)fS_H({}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}1_H)\otimes^{\overline{\omega}}b_2{}^{\overline{\overline{\omega}}}fS_H(({}^{\omega}1_H)_1)\otimes^{\omega}b_3\\&\overset{(E.1)}{=}\sum b_1fS_H({}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}(({}^{\omega}1_H)_2))\otimes^{\overline{\omega}}b_2{}^{\overline{\overline{\omega}}}fS_H(({}^{\omega}1_H)_1)\otimes^{\omega}b_3\\&\overset{(F.2)}{=}\sum b_1fS_H({}^{\overline{\omega}}1_H({}^{\omega}1_H)_1)\otimes^{\overline{\omega}}b_2fS_H(({}^{\omega}1_H)_2)\otimes^{\omega}b_3\\&\overset{(C.3)}{=}\sum b_1fS_H({}^{\overline{\omega}}1_H{}^{\omega}1_H)\otimes^{\overline{\omega}}b_2fS_H({}^{\overline{\overline{\omega}}}1_H)\otimes^{\overline{\overline{\omega}}}({}^{\omega}b_3)\\&\overset{(E.1)}{=}\sum b_1fS_H({}^{\overline{\omega}}({}^{\omega}1_H))\otimes^{\overline{\omega}}b_2fS_H({}^{\overline{\overline{\omega}}}1_H)\otimes^{\overline{\overline{\omega}}}({}^{\omega}b_3)\\&\overset{(C.4)}{=}\sum b_1fS_H({}^{\omega}1_H)\otimes({}^{\omega}b_2)_1fS_H({}^{\overline{\omega}}1_H)\otimes^{\overline{\omega}}({}^{\omega}b_2)_2\\&=(I\otimes\overline{\triangle})\overline{\triangle}(b).\end{aligned}$$

From the equation (C.1) we easily derive that counit of B is counit of $\overline{B}$. Hence we get $(\overline{B},\overline{\triangle})$ is an R-coalgebra.

On the other hand, we have

$$(I\otimes\overline{\triangle})\rho(b)\overset{(C.4)}{=}\sum {}^{\overline{\omega}}({}^{\omega}1_H)\otimes^{\overline{\omega}}b_1fS_H({}^{\overline{\overline{\omega}}}1_H)\otimes{}^{\overline{\overline{\omega}}}({}^{\omega}b_2)$$

$$\overset{(E.1)}{=} \sum {}^{\overline{\omega}}1_H{}^{\omega}1_H \otimes^{\overline{\omega}} b_1 f S_H({}^{\overline{\overline{\omega}}}1_H) \otimes {}^{\overline{\overline{\omega}}}({}^{\omega}b_2)$$
$$\overset{(C.3)}{=} \sum {}^{\overline{\omega}}1_H({}^{\omega}1_H)_1 \otimes^{\overline{\omega}} b_1 f S_H(({}^{\omega}1_H)_2) \otimes {}^{\omega}b_2$$
$$\overset{(F.2)}{=} \sum {}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}({}^{\omega}1_H)_2 \otimes^{\overline{\omega}} b_1{}^{\overline{\overline{\omega}}} f S_H(({}^{\omega}1_H)_1) \otimes {}^{\omega}b_2$$
$$\overset{(E.1)}{=} \sum {}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}1_H({}^{\omega}1_H)_2 \otimes^{\overline{\omega}} b_1{}^{\overline{\overline{\omega}}} f S_H(({}^{\omega}1_H)_1) \otimes {}^{\omega}b_2$$
$$\overset{(C.3)}{=} \sum {}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}1_H{}^{\overline{\overline{\overline{\omega}}}}1_H \otimes^{\overline{\omega}} b_1{}^{\overline{\overline{\omega}}} f S_H({}^{\omega}1_H) \otimes {}^{\overline{\overline{\overline{\omega}}}}({}^{\omega}b_2)$$
$$= (m_H \otimes I \otimes I)(I \otimes T \otimes I)(\rho \otimes \rho)\overline{\triangle}(b),$$
$$(I \otimes \varepsilon_B)\rho(b) \overset{(C.1)}{=} \varepsilon_B(b)1_H,$$
$$\overline{\triangle}(h \to b) = \sum f(h_1)b_1 f S_H(h_4) f S_H({}^{\omega}1_H) \otimes^{\omega} (f(h_2)b_2 f S_H(h_3))$$
$$\overset{(E.1)}{=} \sum f(h_1)b_1 f S_H({}^{\omega}1_H{}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}h_4) \otimes^{\omega} f(h_2)^{\overline{\omega}}b_2{}^{\overline{\overline{\omega}}} f S_H(h_3)$$
$$\overset{(F.2)}{=} \sum f(h_1)b_1 f S_H({}^{\omega}1_H{}^{\overline{\omega}}1_H h_3) \otimes^{\omega} f(h_2)^{\overline{\omega}}b_2 f S_H(h_4)$$
$$\overset{(E.1)}{=} \sum f(h_1)b_1 f S_H({}^{\omega}h_3{}^{\overline{\omega}}1_H) \otimes^{\omega} f(h_2)^{\overline{\omega}}b_2 f S_H(h_4)$$
$$\overset{(D.1)}{=} \sum f(h_1)b_1 f S_H(h_2{}^{\omega}1_H) \otimes f(h_3)^{\omega}b_2 f S_H(h_4)$$
$$= \sum (h_1 \to b_1 f S_H({}^{\omega}1_H)) \otimes (h_2 \to^{\omega} b_2),$$
$$\varepsilon(h \to b) = \varepsilon_H(h)\varepsilon_B(b).$$

Thus $\overline{B}$ is both a left H-comodule coalgebra and a left H-module coalgebra. Finally,

$$\overline{\triangle}(ab) = \sum a_1 f S_H({}^{\omega}1_H)({}^{\overline{\omega}}1_H \to b_1 f S_H({}^{\overline{\overline{\omega}}}1_H)) \otimes^{\overline{\omega}} ({}^{\omega}a_2)^{\overline{\overline{\omega}}}b_2.$$

Indeed,

$$\overline{\triangle}(ab) = \sum a_1 b_1 f S_H({}^{\omega}1_H{}^{\overline{\omega}}1_H) \otimes^{\omega} a_2{}^{\overline{\omega}}b_2$$
$$\overset{(C.2)}{=} \sum a_1 f(S_H(({}^{\omega}1_H)_1)({}^{\omega}1_H)_2) b_1 f S_H({}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}1_H) \otimes^{\overline{\omega}} ({}^{\omega}a_2)^{\overline{\overline{\omega}}}b_2$$
$$\overset{(C.3)}{=} \sum a_1 f(S_H({}^{\omega}1_H)^{\overline{\omega}}1_H) b_1 f S_H({}^{\overline{\overline{\omega}}}1_H{}^{\overline{\overline{\overline{\omega}}}}1_H) \otimes^{\overline{\overline{\omega}}} ({}^{\overline{\omega}}({}^{\omega}a_2))^{\overline{\overline{\overline{\omega}}}}b_2$$
$$\overset{(C.3)}{=} \sum a_1 f S_H({}^{\omega}1_H) f(({}^{\overline{\omega}}1_H)_1) b_1 f S_H({}^{\overline{\overline{\omega}}}1_H) f S_H(({}^{\overline{\omega}}1_H)_2) \otimes^{\overline{\omega}} ({}^{\omega}a_2)^{\overline{\overline{\omega}}}b_2$$
$$= \sum a_1 f S_H({}^{\omega}1_H)({}^{\overline{\omega}}1_H \to b_1 f S_H({}^{\overline{\overline{\omega}}}1_H)) \otimes^{\overline{\omega}} ({}^{\omega}a_2)^{\overline{\overline{\omega}}}b_2.$$

Thus $\overline{B}$ is a bialgebra in ${}^H_H\mathrm{YD}$. To show that $\overline{S}$ is a morphism of left-left Yetter-Drinfeld modules,

$$\rho\overline{S}(b) \overset{(F.1)}{=} \sum {}^{\overline{\omega}}1_H{}^{\overline{\overline{\omega}}}1_H \otimes^{\overline{\omega}} f({}^{\omega}1_H) S_B({}^{\overline{\overline{\omega}}}({}^{\omega}b))$$
$$\overset{(C.3)}{=} \sum {}^{\overline{\omega}}1_H({}^{\omega}1_H)_2 \otimes^{\overline{\omega}} f(({}^{\omega}1_H)_1) S_B({}^{\omega}b)$$
$$\overset{(E.1)}{=} \sum {}^{\overline{\omega}}({}^{\omega}1_H)_2 \otimes^{\overline{\omega}} f(({}^{\omega}1_H)_1) S_B({}^{\omega}b)$$
$$\overset{(D.1)}{=} \sum ({}^{\omega}1_H)_1 \otimes f(({}^{\omega}1_H)_2) S_B({}^{\omega}b)$$
$$\overset{(C.3)}{=} \sum {}^{\omega}1_H \otimes f({}^{\overline{\omega}}1_H) S_B({}^{\overline{\omega}}({}^{\omega}b))$$

$$
\begin{aligned}
&= (I\otimes \overline{S})\rho(b).\\
\overline{S}(h\to b) &= \sum f({}^{\omega}1_H)f({}^{\overline{\omega}}1_H)S_B({}^{\overline{\omega}}fS_H(h_2))S_B({}^{\omega}(f(h_1)b))\\
&\overset{(D.2)}{=} \sum f({}^{\omega}1_H)f({}^{\overline{\omega}}(h_2S_H({}^{\overline{\overline{\omega}}}h_4)))S_B^2({}^{\overline{\omega}}({}^{\overline{\overline{\omega}}}f(h_3)))S_B({}^{\omega}(f(h_1)b))\\
&= \sum f({}^{\omega}1_H)f(h_2)f({}^{\overline{\omega}}(S_H({}^{\overline{\overline{\omega}}}h_4)))S_B^2({}^{\overline{\omega}}({}^{\overline{\overline{\omega}}}f(h_3)))S_B({}^{\omega}(f(h_1)b))\\
&\overset{(D.1)}{=} \sum f({}^{\omega}1_H)f(h_2)f({}^{\overline{\omega}}(S_H(h_3)))S_B^2({}^{\overline{\omega}}f(h_4))S_B({}^{\omega}(f(h_1)b))\\
&\overset{(D.3)}{=} \sum f({}^{\omega}1_H)f(h_2)f(S_H(h_4))S_B(fS_H(h_3))S_B({}^{\omega}(f(h_1)b))\\
&\overset{(E.1)}{=} \sum f({}^{\omega}h_2)f({}^{\overline{\omega}}1_H)S_B({}^{\overline{\omega}}b)S_B({}^{\omega}f(h_1))\\
&\overset{(D.1)}{=} \sum f(h_1)f({}^{\omega}1_H)S_B({}^{\omega}b)fS_H(h_2)\\
&= h\to \overline{S}(b).
\end{aligned}
$$

Thus $\overline{S}$ is both a left H-comodule morphism and left H-module morphism.

To finish the proof we only need to show

$$(I*\overline{S})(b)=\varepsilon_B(b)=(\overline{S}*I)(b).$$

In fact we have

$$
\begin{aligned}
(I*\overline{S})(b) &\overset{(C.3)}{=} \sum b_1fS_H(({}^{\omega}1_H)_1)f(({}^{\omega}1_H)_2)S_B({}^{\omega}b_2) \overset{(C.2)}{=} \varepsilon_B(b)1_B,\\
(\overline{S}*I)(b) &= \sum f({}^{\omega}1_H)f({}^{\overline{\omega}}1_H)S_B({}^{\overline{\omega}}fS_H({}^{\overline{\overline{\omega}}}1_H))S_B({}^{\omega}b_1){}^{\overline{\overline{\omega}}}b_2\\
&\overset{(D.2)}{=} \sum f({}^{\omega}1_H)f({}^{\overline{\omega}}(({}^{\overline{\overline{\omega}}}1_H)_1S_H({}^{\overline{\overline{\overline{\omega}}}}({}^{\overline{\overline{\omega}}}1_H)_3)))S_B^2({}^{\overline{\omega}}({}^{\overline{\overline{\overline{\omega}}}}f(({}^{\overline{\overline{\omega}}}1_H)_2)))S_B({}^{\omega}b_1){}^{\overline{\overline{\omega}}}b_2\\
&\overset{(F.1)}{=} \sum f({}^{\omega}1_H)f(({}^{\overline{\overline{\omega}}}1_H)_1)f({}^{\overline{\omega}}S_H({}^{\overline{\overline{\overline{\omega}}}}({}^{\overline{\overline{\omega}}}1_H)_3))S_B({}^{\overline{\omega}}({}^{\overline{\overline{\overline{\omega}}}}f(S_H({}^{\overline{\overline{\omega}}}1_H)_2)))S_B({}^{\omega}b_1){}^{\overline{\overline{\omega}}}b_2\\
&\overset{(F.2)}{=} \sum f({}^{\omega}1_H)f(({}^{\overline{\overline{\omega}}}1_H)_1)f({}^{\overline{\omega}}S_H({}^{\overline{\overline{\omega}}}1_H)_2)S_B({}^{\overline{\omega}}fS_H(({}^{\overline{\overline{\omega}}}1_H)_3))S_B({}^{\omega}b_1){}^{\overline{\overline{\omega}}}b_2\\
&\overset{(D.3)}{=} \sum f({}^{\omega}1_H)f(({}^{\overline{\omega}}1_H)_1)fS_H(({}^{\overline{\omega}}1_H)_3)S_B(fS_H(({}^{\overline{\omega}}1_H)_2))S_B({}^{\omega}b_1){}^{\overline{\omega}}b_2\\
&= \sum f({}^{\omega}1_H{}^{\overline{\omega}}1_H)S_B({}^{\omega}b_1){}^{\overline{\omega}}b_2\\
&\overset{(E.2)}{=} \sum f({}^{\omega}1_H)S_B(({}^{\omega}b)_1)({}^{\omega}b)_2\\
&\overset{(C.1)}{=} \varepsilon_B(b)1_B.
\end{aligned}
$$

Thus we complete the proof. □

To end our paper, we construct an explicit example of a Hopf algebra in ${}^H_H\mathrm{YD}$.

Example 3.3 Let B be a Hopf algebra and H a commutative Hopf algebra, $R=\sum R^{(1)}\otimes R^{(2)}\in B\otimes H$ an invertible element such that

1) $\sum \varepsilon_B(R^{(1)})R^{(2)}=1_H$, $\sum R^{(1)}\varepsilon_H(R^{(2)})=1_B$;
2) $\sum \triangle_B(R^{(1)})\otimes R^{(2)}=\sum R^{(1)}\otimes r^{(1)}\otimes R^{(2)}r^{(2)}$;
3) $\sum R^{(1)}\otimes \triangle_H(R^{(2)})=\sum R^{(1)}r^{(1)}\otimes r^{(2)}\otimes R^{(2)}$.

Consider the map

$$\omega: B\otimes H\to H\otimes B,\ b\otimes h\to \sum R^{(2)}hU^{(2)}\otimes R^{(1)}bU^{(1)},$$

where $R^{-1} = U = \sum U^{(1)} \otimes U^{(2)}$. Then linear map ω is right normal and $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra.

Let bilinear form $\langle\ |\ \rangle : H \otimes H \to R$ in $\mathrm{Hom}_R(H \otimes H, R)$, satisfy that for all h, k, $l \in H$,

4) $\sum \langle h_1|k_1\rangle h_2 k_2 = \sum k_1 h_1 \langle h_2|k_2\rangle$;

5) $\langle h|kl\rangle = \sum \langle h_1|k\rangle\langle h_2|l\rangle$;

6) $\langle hk|l\rangle = \sum \langle h|l_2\rangle\langle k|l_1\rangle$;

Consider the map $f : H \to B$, $h \to \sum \langle R^{(2)}|h\rangle R^{(1)}$. Then f is a Hopf algebra morphism. To this end we compute

$$\begin{aligned}
{}^{\omega}f(h_1) \otimes {}^{\omega}h_2 &= \sum R^{(1)} f(h_1) U^{(1)} \otimes R^{(2)} h_2 U^{(2)} \\
&= \sum R^{(1)} \langle r^{(2)}|h_1\rangle r^{(1)} U^{(1)} \otimes R^{(2)} h_2 U^{(2)} \\
&= \sum R^{(1)} U^{(1)} \otimes \langle (R^{(2)})_1|h_1\rangle (R^{(2)})_2 h_2 U^{(2)} \\
&= \sum R^{(1)} U^{(1)} \otimes h_1 (R^{(2)})_1 \langle (R^{(2)})_2|h_2\rangle U^{(2)} \\
&= \sum R^{(1)} r^{(1)} U^{(1)} \langle R^{(2)}|h_2\rangle \otimes h_1 r^{(2)} U^{(2)} \\
&= \sum \langle R^{(2)}|h_2\rangle R^{(1)} \otimes h_1 = f(h_2) \otimes h_1.
\end{aligned}$$

Thus (B, H) is an (f, ω)-compatible pair. By Theorem 3.2, we obtain a Hopf algebra $\overline{B}$ in the Yetter-Drinfeld category ${}^{H}_{H}\mathrm{YD}$. However, in fact the multiplication and the unit and the counit of $\overline{B}$ coincide with Hopf algebra B, respectively. The comultiplication of $\overline{B}$ is defined as follows:

$$\overline{\triangle}(b) = \sum \langle r^{(2)}|S_H(R^{(2)}U^{(2)})\rangle b_1 r^{(1)} \otimes R^{(1)} b_2 U^{(1)}.$$

The expression for the antipode of $\overline{B}$ is $\overline{S}(b) = \langle r^{(2)}|R^{(2)}U^{(2)}\rangle r^{(1)} S_B(R^{(1)} b U^{(1)})$.

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