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Knots and Polynomials

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Abstract In this paper, we deal with some corresponding relations between knots and polynomials by using the basic properties of knot polynomials (such as, some special values of knot polynomials, the Arf invariant and derivative of knot polynomials). We give necessary and sufficient conditions that a Laurent polynomial with integer coefficients, whose breadth is less than five, is the Jones polynomial of a certain knot.

Keywords knot; Jones polynomial; Laurent polynomial.

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1. Introduction

A fundamental problem is the classification of knots in the knot theory. The knot invariants play a very key role. So do knot polynomials. The Alexander polynomial $\triangle(t)$ [1] and the Jones polynomial $V(t)$ [2] are two well known knot polynomials. The Alexander polynomial $\triangle(t)$ is characterized by the following algebraic conditions: (1) $\triangle(1) = 1$ and (2) $\triangle(t)$ is a reciprocal polynomial (sometimes abbreviated as $\triangle(t) \doteq \triangle(t^{-1})$). It is well known that a Laurent polynomial with integer coefficients is the Alexander polynomial of a certain knot if and only if it satisfies the two conditions above [3, 4]. How can one tell whether a Laurent polynomial with integer coefficients is the Jones polynomial of a certain knot? In this paper, we will discuss the problem. In Section 2, we introduce the properties of the Jones polynomial and give the properties of derivative of the Jones polynomial, which are used in the next section. In Section 3, we give necessary and sufficient conditions that a Laurent polynomial with integer coefficients is the Jones polynomial of some knot, when the breadth of the Laurent polynomial is less than five.

2. The Jones polynomial and its properties

Let L be an oriented link with n components. So one has the Jones polynomial, denoted by $V(L;t)$, of the link L as follows:

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(1) $V(L;t)$ is an ambient isotopy invariant;

(2) Skein relation: $t^{-1}V(L_+;t) - tV(L_-;t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0;t)$;

(3) $V(U;t) = 1$, where U is trivial.

In (2), L_+, L_- and L_0 are the same except for in small disk where their projections are a positive crossing, a negative crossing and an orientation preserving smoothing of that crossing, respectively, as shown in the following Figure 1.

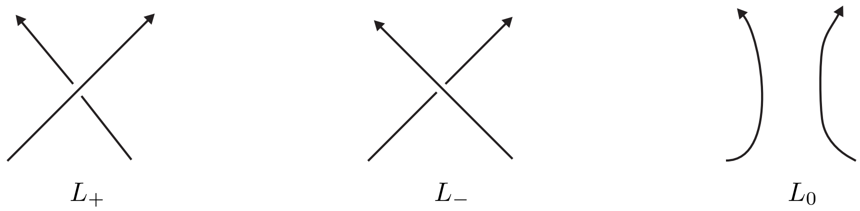

Figure 1 crossings

Lemma 2.1 ([2, 5]) *Let L be an oriented link with n components. Then one can get the following properties of the Jones polynomial:*

(1) $V(L;1) = (-2)^{n-1}$;

(2) $V'(L;1) = 0$, *If* $n = 1$;

(3) $V(L;e^{2\pi i/3}) = (-1)^{n-1}$;

(4) If $Arf(L)$ exits, $V(L;i) = (-\sqrt{2})^{n-1}(-1)^{Arf(L)}$; otherwise $V(L;i) = 0$. Here $Arf(L)$ is the Arf invariant of links. One can find the definition of the invariant in [5,6]. i denotes $\sqrt{-1}$.

To discuss the properties of derivative of the Jones polynomial, we introduce the Kauffman polynomial and its properties . We consider also oriented link though the Kauffman polynomial is defined for unoriented links. Let L be an oriented link with n components. So one can obtain the following recursive relations:

(1) $\langle L_+\rangle = A\langle L_0\rangle + A^{-1}\langle L_\infty\rangle, \langle L_-\rangle = A\langle L_\infty\rangle + A^{-1}\langle L_0\rangle$;

(2) $\langle U;L\rangle = (-A^2 - A^{-2})\langle L\rangle$, where U is trivial;

(3) $\langle U\rangle = 1$.

In (1), L_+, L_- and L_0 are the same as in the Jones polynomial. L_∞ is obtained by an orientation reserving smoothing of that crossing. $\langle L\rangle$ is said to be the Kauffman polynomial. L_∞ and L_+, L_-, L_0 are the same except for in the small disk.

We call the equivalence generated by move II and move III (the Reidemeister moves) regular isotopy. Thus $\langle L\rangle$ is a regular isotopy invariant [7]. Let $\omega(L)$ be the writhe of L. It is also a regular invariant. We define $f(L;A) = \alpha^{-\omega(L)}\langle L\rangle$, where $\langle\rangle$ forgets the particular orientation ($\alpha = -A^3$).

Lemma 2.2 ([7]) *$f(L;A)$ is an ambient isotopy invariant for oriented knots and links L. And $f(L;t^{-\frac{1}{4}}) = V(L;t)$, $V(L^*;t) = V(L,t^{-1})$. Where L^* is the mirror image of L.*

If L_+ and L_- are knots, then L_0 is a link with two components, denoted by L_{01}, L_{02}, respectively. Set $l = lk(L_{01}, L_{02}) = lk(L_0)$. L_∞ is also a knot.

Lemma 2.3 *Let K_+, K_-, K_0 and K_∞ be knots defined above. Then*

$$V(K_+;t) - tV(K_-;t) = (1-t)t^{3l}V(K_\infty;t),$$

where $\lambda = lk(K_0)$.

Proof By the Kauffman relations above,

$$(1)\ \langle K_+\rangle = A\langle K_0\rangle + A^{-1}\langle K_\infty\rangle,$$

$$(2)\ \langle K_-\rangle = A\langle K_\infty\rangle + A^{-1}\langle K_0\rangle.$$

$(1) * A^{-1} - (2) * A$ gives

$$A^{-1}\langle K\rangle - A\langle K_-\rangle = (A^{-2} - A^2)\langle K_\infty\rangle.$$

Since $f(K;A) = \alpha^{-\omega(K)}\langle K\rangle$, we have

$$A^{-1}\alpha^{\omega(K_+)}f(K_+;A) - A\alpha^{\omega(K_-)}f(K_-;A) = (A^{-2} - A^2)\alpha^{\omega(K_\infty)}f(K_\infty;A).$$

By the definition of the writhe, one gets the following relations: $\omega(K_-) = \omega(K_+) - 2$, $\omega(K_\infty) = \omega(K_+) - 4\lambda - 1$. So

$$A^{-1}f(K_+;A) - A\alpha^{-2}f(K_-;A) = (A^{-2} - A^2)\alpha^{-1}\alpha^{-4\lambda}f(K_\infty;A),$$

$$f(K_+;A) - A^{-4}f(K_-;A) = (1 - A^{-4})\alpha^{-4\lambda}f(K_\infty;A).$$

This completes the proof by setting $A = t^{-\frac{1}{4}}$ and Lemma 2.2.

Remarks In [6] and [8], one can find a proof of the above lemma. Here we give an elementary proof by applying the relations between the Jones polynomial and the Kauffman polynomial.

Lemma 2.4 *Let K be an oriented knot and $\psi_n(K) = V^{(n)}(K;1)$. Then*

(1) $\psi_0(K) = 1$, $\psi_1(K) = 0$;

(2) $\psi_2(K) \in 6Z$;

(3) $\psi_n(K) \in 6nZ$, *for any* $n \geq 3$.

Proof The first is clear by Lemma 2.1.

For (2), assume that $K = K_+$. By Lemma 2.3 and simple computations, one can get

$$\psi_2(K_+) - \psi_2(K_-) = (\psi_1(K_-) - \psi_1(K_\infty) - 6l\psi_0(K_\infty) = -6l.$$

So $\psi_2(K_+) - \psi_2(K_-) \in 6Z$. Furthermore, $\psi_2(K) \in 6Z$ since any knot can be changed into unknot by switching some crossings.

For (3), suppose that $K = K_+$ without loss of generality. By Lemma 2.3, $V(K_+;t) - tV(K_-;t) = (1-t)t^{3l}V(K_\infty;t)$.

We will prove the lemma by using induction on n. Taking nth derivatives on both sides gives

$$V^{(n)}(K_+;t) = \sum_{i=0}^{n} C_n^i V^{(n-i)}(K_-;t)t^{(i)} + \sum_{j=0}^{n} C_n^j V^{(n-j)}(K_\infty;t)(t^{3l} - t^{3l+1})^{(j)}$$

$$= tV^{(n)}(K_-;t) + nV^{(n-1)}(K_-;t) + \sum_{j=0}^{n} C_n^j V^{(n-j)}(K_\infty;t)[3l(3l-1)\cdots (3l-j+2)][(3l-j+1)-(3l+1)t]t^{3l-j}.$$

Let $t=1$. Then

$$\psi_n(K_+) - \psi_n(K_-) = n(\psi_{n-1}(K_-) - \psi_{n-1}(K_\infty)) - \sum_{j=2}^{n} jC_n^j \psi_{n-j}(K_\infty) \prod_{m=0}^{j-2}(3l-m)$$
$$= n(\psi_{n-1}(K_-) - \psi_{n-1}(K_\infty)) - \sum_{j=2}^{n-1} jC_n^j \psi_{n-j}(K_\infty) \prod_{m=0}^{j-2}(3l-m) - n\prod_{m=0}^{n-2}(3l-m).$$

(i) $n=3$, by Lemma 2.1, $\psi_3(K_+) - \psi_3(K_-) = 3(\psi_2(K_-) - \psi_2(K_\infty)) - 9l(3l-1)$. Thus $\psi_3(K_+) - \psi_3(K_-) \in 18Z$ by Lemma 2.4 and $l \in Z$. The lemma is true. If $n=4$, the proof of the lemma can be finished by using the same arguments.

(ii) Suppose the lemma is right if $3 \le m \le n-1$, that is $\psi_m(K) \in 6mZ$ (of course, $\psi_m(K) \in Z$). One knows that $jC_n^j = nC_{n-1}^{j-1}$, so $jC_n^j\psi_{n-j}(K_\infty) \in 6nZ$ since $\psi_m(K) \in 6Z$. It is obvious that $n\prod_{m=0}^{n-2} = 3nl(3l-1)\prod_{m=2}^{n-2} \in 6nZ$ if $n \ge 4$. Thus $\psi_n(K_+) - \psi_n(K_-) \in 6nZ$, hence $\psi_n(K) \in 6nZ$. This completes the proof of the lemma. □

Remarks The further research on the properties of derivative of the Jones polynomial contributes to study the polynomial invariant of integral homology 3-spheres [9-11].

3. Main results

In the section, a polynomial of degree n ($n \in Z, n \ge 0$) means that the exponent of each term of the polynomial is a non-negative integer. A polynomial of degree $-n$ ($n \in Z, n \ge 0$) means that the exponent of each term of the polynomial is a non-positive integer. Let $\deg(f(t))$ denote the degree of a polynomial $f(t)$ and let $\mathrm{span}(f(t))$ denote the difference between the maximal and minimal degrees of a polynomial $f(t)$.

Theorem 3.1 *Any polynomial of degree 2 with integral coefficients is not the Jones polynomial of a certain knot.*

Proof Let $f(t) = at^2 + bt + c$, where $a \ne 0$, b, $c \in Z$. If $f(t)$ is the Jones polynomial of some knot, then by Lemmas 2.1 and 2.4, $f(1) = 1$, that is, $a+b+c=1$, $f'(1)=0$. So $2a+b=0$, $f''(1) \in 6Z$, hence $2a \in 6Z$. By (3) of Lemma 2.1, one can obtain that $a=0$. This completes the proof. □

Corollary 3.1 *Any polynomial of degree-2 with integral coefficients is not the Jones polynomial of any knot.*

The result can be proved by Lemma 2.2.

Theorem 3.2 *Any polynomial of degree-3 with integral coefficients is not the Jones polynomial of any knot.*

Proof Let $f(t) = a_3t^3 + a_2t^2 + a_1t + a_0$ $(a_3 \neq 0, a_i \in Z, i = 0, 1, 2, 3)$. If $f(t)$ is the Jones polynomial of some knot, by Lemmas 2.1, 2.4 and 2.5, one can get the following properties:

(1) $f(1) = 1$, so, $a_3 + a_2 + a_1 + a_0 = 1$;

(2) $f'(1) = 0$, so, $3a_3 + 2a_2 + a_1 = 0$;

(3) $f''(1) \in 6Z$, so, $6a_3 + 2a_2 \in 6Z$;

(4) $f'''(1) \in 18Z$, so, $6a_3 \in 18Z$.

From (4) and (3), one obtains that $a_3 = 3k(k \in Z)$, $a_2 = 3l(l \in Z)$. By (2) and (1), $a_1 = -9k - 6l$, $a_0 = 6k + 3l + 1$. Hence $f(t) = 3kt^3 + 3lt^2 + (-9k - 6l)t + 6k + 3l + 1$. By Lemma 2.1, $f(e^{2\pi i/3}) = 1$, which implies that $k = 0, l = 0$. Therefore $a_3 = 0$. The proof is completed. □

Corollary 3.2 *Any polynomial of degree -3 with integral coefficients is not the Jones polynomial of any knot.*

One knows that $\triangle(\text{trefoil}; t) = t^2 - t + 1, \triangle(\text{figure} - \text{eight}; t) = t^2 - 3t + 1, \triangle(\text{stevedore}; t) = 2t^2 - 5t + 2$ ([4]). But these polynomials are not the Jones polynomial of knots. We comment again that the Alexander polynomial is only determined up to multiplication by $\pm t^k$. In fact, the polynomials multiplied by $\pm t^k$ are not the Jones polynomials of knots either, which can be proved by using Theorem 3.4 below. Here some differences between the Alexander polynomial and the Jones polynomial of knots are exhibited . Can one conclude that if a polynomial $f(t)$ (perhaps the Alexander polynomial) is not the Jones polynomial, then so is the case for the polynomial multiplied $\pm t^k$? We will answer the question below.

Theorem 3.3 *Let* $f(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ $(a_i \in Z, i = 0, 1, 2, 3, 4, a_4 \neq 0)$. *Then* $f(t)$ *is the Jones polynomial of some knot if and only if* $a_4 = -1$, $a_3 = 1$, $a_2 = 0$, $a_1 = 1$, $a_0 = 0$.

Proof The sufficiency is clear since the Jones polynomial of the trefoil satisfies the conditions of the theorem.

Suppose that $f(t)$ is the Jones polynomial of a certain knot K. By Lemmas 2.1, 2.4 and 2.5, we have the following properties:

(1) $f(1) = 1$, so, $a_4 + a_3 + a_2 + a_1 + a_0 = 1$;

(2) $f'(1) = 0$, so, $4a_4 + 3a_3 + 2a_2 + a_1 = 0$;

(3) $f''(1) \in 6Z$, so, $12a_4 + 6a_3 + 2a_2 \in 6Z$;

(4) $f'''(1) \in 18Z$, so, $24a_4 + 6a_3 \in 18Z$.

By (4) and (3), one can obtain that $a_3 = 3k, a_2 = 3l$, so $a_1 = -4a_4 - 9k - 6l$ and $a_0 = 3a_4 + 6k + 3l + 1$ from (2) and (1). Hence

$$f(t) = a_4t^4 + 3kt^3 + 3lt^2 - (4a_4 + 9k + 6l)t + 3a_4 + 6k + 3l + 1.$$

By Lemma 2.1, $f(e^{2\pi i/3}) = 1$ implies that $\frac{\sqrt{3}}{2}i(-3a_4 - 9k - 9l) + \frac{3}{2}(3a_4 + 9k + 3l) = 0$. Therefore,

$$a_4 + 3k + 3l = 0,$$

$$a_4 + 3k + l = 0.$$

We get that $l = 0$, $a_3 = -a_4$. Furthermore, $a_2 = 0$, $a_1 = -a_4$, $a_0 = 1 + a_4$, $f(t) = a_4t^4 - a_4t^3 - a_4t + 1 + a_4$. So $f(i) = 2a_4 + 1$. By Lemma 2.1, $f(i) = (-1)^{\mathrm{Arf}(K)} = -1$ since $f(t)$ is the Jones polynomial of a certain knot and $a_4 \neq 0$. This implies that $a_4 = -1$. This completes the proof. □

Corollary 3.3 *Let* $f(t) = a_{-4}t^{-4} + a_{-3}t^{-3} + a_{-2}t^{-2} + a_{-1}t^{-1} + a_0$ $(a_{-i} \in Z, i = 0, 1, 2, 3, 4)$. *Then* $f(t)$ *is the Jones polynomial of a knot if and only if* $a_{-4} = -1$, $a_{-3} = 1$, $a_{-2} = 0$, $a_{-1} = 1$, $a_0 = 0$.

Corollary 3.4 *Let* K *be a knot in* S^3. *If* $V(K;t) = -t^4 + t^3 + t$, *then* $Arf(K) = 1$.

Corollary 3.5 Arf(trefoil) = 1.

Corollary 3.6 *Let* K *be an alternating knot in* S^3. *Then* $V(K;t) = -t^4 + t^3 + t$ *if and only if* K *is the trefoil.*

Lemma 3.1 ([4]) *Let* $f(x) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$ $(a_0 \neq 0)$ *be a polynomial with integer coefficients. If* $f(x)$ *is the Alexander polynomial of some knot, then* $a_i = a_{n-i}$ $(i = 0, 1, ..., n)$.

By Theorem 3.3, we know that the polynomial $f(t) = -t^4 + t^3 + t$ is the Jones polynomial of some knot (for example, trefoil). But the polynomial is not the Alexander polynomial of a certain knot by Lemma 3.1. It is clear that $-t^3 + t^2 + 1$ is neither the Alexander polynomial nor the Jones polynomial. However $t(-t^3 + t^2 + 1)$ remains the Jones polynomial of a knot.

Conjecture 3.1 *If* K *is a non-alternating knot, then* $V(K;t) \neq -t^4 + t^3 + t$.

Theorem 3.4 *Let* $f(t) = a_{-1}t^{-1} + a_0 + a_1t$ $(a_{-1} \neq 0)$ *be a polynomial with integer coefficients. Then* $f(t)$ *is not the Jones polynomial of any knot.*

Proof Suppose that $f(t)$ is the Jones polynomial of some knot. Then one obtains the following properties by Lemmas 2.1 and 2.4.

(1) $f(1) = a_{-1} + a_0 + a_1 = 1$.

(2) $f'(1) = -a_{-1} + a_1 = 0$, so $a_{-1} = a_1$.

(3) $f''(1) = 2a_{-1} \in 6Z$, so $a_{-1} = 3k, k \in Z$.

Hence $f(t) = 3kt^{-1} + (1 - 6k) + 3kt$. Thus $f(i) = 1 - 6k = -1$ or 1 by Lemma 2.1. This is in contradiction to $k \in Z$ and $a_{-1} \neq 0$.

Theorem 3.5 *Let* $f(t) = a_{-2}t^{-2} + a_{-1}t^{-1} + a_0 + a_1t$ $(a_{-2} \neq 0)$ *be a polynomial with integer coefficients. Then* $f(t)$ *is not the Jones polynomial of any knot.*

Proof Suppose that $f(t)$ is the Jones polynomial of a certain knot. Then one obtains the following properties by Lemmas 2.1, 2.4 and 2.5.

(1) $f(1) = a_{-2} + a_{-1} + a_0 + a_1 = 1$.

(2) $f'(1) = -2a_{-2} - a_{-1} + a_1 = 0$.

(3) $f''(1) = 6a_{-2} + 2a_{-1} \in 6Z$, so $a_{-1} = 3l$ $(l \in Z)$.

(4) $f'''(1) = -24a_{-2} - 6a_{-1} \in 18Z$, so $a_{-2} = 3k$ $(k \in Z)$ and $a_1 = 6k + 3l, a_0 = 1 - 9k - 6l$. Hence $f(t) = 3kt^{-2} + 3lt^{-1} + 1 - 9k - 6l + (6k+3l)t$, $f(i) = 6ki + 1 - 12k - 6l$. One can complete the proof similarly to Theorem 3.4.

Corollary 3.7 *Let $f(t) = a_{-1}t^{-1} + a_0 + a_1t + a_2t^2$ be a polynomial with integer coefficients. Then the polynomial is not the Jones polynomial of any knot.*

Corollary 3.8 *Let $f(t)$ be a polynomial with integer coefficients. If $|\deg f(\pm t)| \le 3$, and $\mathrm{span}(f(t)) \le 3$, then the polynomial is not the Jones polynomial of any knot.*

Theorem 3.6 *Let $f(t) = a_{-2}t^{-2}+a_{-1}t^{-1}+a_0+a_1t+a_2t^2$ $(a_{-2} \neq 0, a_2 \neq 0)$ be a polynomial with integer coefficients. Then $f(t)$ is the Jones polynomial of some knot if and only if $a_{-2} = a_2 = 1$, $a_{-1} = a_1 = -1$, $a_0 = 1$.*

Proof The sufficiency is clear since the Jones polynomial of the figure-eight knot satisfies the conditions of the theorem.

Assume that $f(t)$ is the Jones polynomial of a certain knot K. By Lemmas 2.1, 2.4 and 2.5, we have the following properties.

(1) $f(1) = a_{-2} + a_{-1} + a_0 + a_1 + a_2 = 1$.

(2) $f'(1) = -2a_{-2} - a_{-1} + a_1 + 2a_2 = 0$.

(3) $f''(1) = 6a_{-2} + 2a_{-1} + 2a_2 \in 6Z$.

(4) $f'''(1) = -24a_{-2} - 6a_{-1} \in 18Z$.

So one cannot determine the properties of the coefficients a_{-2} from (4) as in the proof of Theorem 3.5. There are three situations as follows.

Case 1 If $a_{-2} = 3k$ $(k \in Z)$, that is, the coefficient is divided by 3. Then $a_{-1} = 3l$ $(l \in Z)$ from (4), so $a_2 = 3m$ $(m \in Z)$ by the property (3) and $a_1 = 6k+3l-6m, a_0 = 1-9k-6l+3m$ from (2) and (1). Therefore $f(t) = 3kt^{-2} + 3lt^{-1} + 1 - 9k - 6l + 3m + (6k + 3l - 6m)t + 3mt^2$. By Lemma 2.1, $f(e^{2\pi i/3}) = 3ke^{-4\pi i/3} + 3le^{-2\pi i/3} + 1 - 9k - 6l + 3m + (6k + 3l - 6m)e^{2\pi i/3} + 3me^{4\pi i/3} = 1$. So $\frac{\sqrt{3}}{2}(9k - 9m)i - \frac{3}{2}(9k + 6l - 3m) = 0$. This implies that $k = m, k + l = 0$ and $a_0 = 1$. One gets $f(t) = 3kt^{-2} + 3lt^{-1} + 1 + 3lt + 3kt^2$. Then $f(i) = 1 - 6k = (-1)^{\mathrm{Arf}(K)}$ by Lemma 2.1. This is a contradiction as in Theorem 3.4. So the situation is impossible.

Case 2 If $a_{-2} = 3k + 2$ $(k \in Z)$, that is, $a_{-2} \equiv 2 \bmod 3$. Then $a_{-1} = 3l + 1$, $a_2 = 3m + 2$, $a_1 = 6k + 3l - 6m + 1$, $a_0 = -9k - 6l + 3m - 5$ $(l, m \in Z)$ from (4) to (1). So $f(t) = (3k + 2)t^{-2} + (3l + 1)t^{-1} - 9k - 6l + 3m - 5 + (6k + 3l - 6m + 1)t + (3m + 2)t^2$. $f(e^{2\pi i/3}) = 1$ implies that $\frac{\sqrt{3}}{2}(9k - 9m)i - \frac{3}{2}(9k + 6l - 3m + 6) = 0$. Hence $k = m$, $k + l + 1 = 0$ and $a_0 = 1$, $a_1 = a_{-1} = 3l+1$, $a_2 = a_{-2} = 3k+2$. So $f(t) = (3k+2)t^{-2}+(3l+1)t^{-1}+1+(3l+1)t+(3k+2)t^2$. Then $f(i) = -6k - 3 = (-1)^{\mathrm{Arf}(K)}$ by Lemma 2.1. This is in contradiction with $k \in Z$. The situation is impossible too.

Case 3 If $a_{-2} = 3k + 1$ $(k \in Z)$, that is, $a_{-2} \equiv 1 \bmod 3$. Then $a_{-1} = 3l + 2$, $a_2 = 3m + 1$,

$a_1 = 6k+3l-6m+2$, $a_0 = -9k-6l+3m-5$. So $f(t) = (3k+1)t^{-2}+(3l+2)t^{-1}-9k-6l+3m-5+(6k+3l-6m+2)t+(3m+1)t^2$. $f(e^{2\pi i/3}) = 1$ implies that $\frac{\sqrt{3}}{2}(9k-9m)i-\frac{3}{2}(9k+6l-3m+6) = 0$. Hence $k = m$, $k + l + 1 = 0$ and $a_0 = 1$, $a_1 = a_{-1} = 3l + 2$, $a_2 = a_{-2} = 3k + 1$. So $f(t) = (3k+1)t^{-2} + (3l+2)t^{-1} + 1 + (3l+2)t + (3k+1)t^2$. Then $f(i) = -6k-1 = (-1)^{\mathrm{Arf}(K)}$ by Lemma 2.1. So $f(i) = 1$ or -1 since $f(t)$ is the Jones polynomial of a certain knot. One can determine that $f(i) = -6k-1 = (-1)^{\mathrm{Arf}(K)} = -1$ since $k \in Z$. So $k = 0$, $l = -1$, $m = 0$ and $a_{-2} = a_2 = 1$, $a_{-1} = a_1 = -1$, $a_0 = 1$. This completes the proof. □

Corollary 3.9 *Let K be a knot in S^3. If $V(K;t) = t^{-2} - t^{-1} + 1 - t + t^2$, then* $\mathrm{Arf}(K) = 1$.

Corollary 3.10 Arf(the figure knot) = 1.

Corollary 3.11 *Let K be an alternating knot in S^3. Then $V(K;t) = t^{-2} - t^{-1} + 1 - t + t^2$ if and only if K is the figure knot.*

One knows that $V(4_1;t) = \triangle(5_1;t) = t^{-2} - t^{-1} + 1 - t + t^2$. It is right that $\triangle(5_1;t) = 1+t+t^2-t^3+t^4$ (not the Jones polynomial by Theorem 3.3) by the properties of the Alexander polynomial. But $t^{-2}\triangle(5_1;t)$ is the Jones polynomial of some knot (for example, the figure-eight).

Conjecture 3.2 *If K is a non-alternating knot, then $V(K;t) \neq t^{-2} - t^{-1} + 1 - t + t^2$.*

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