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Existence of the Uniform Attractors for the Nonautonomous Suspension Bridge Equations with Strong Damping

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Abstract In this paper we show the existence of the uniform attractors for the family of processes corresponding to the suspension bridge equations in $H_0^2 \times L^2$ by a new concept of Condition (C*) and the enegy estimats methods.

Keywords suspension bridge equations; uniform attractors; Condition (C*); uniform condition (C).

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1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$. In this paper we study the following initial-boundary value problem

$$u_{tt} + \triangle^2 u + \triangle^2 u_t + ku^+ + f(u) = g(x,t), \text{ in } \Omega \times \mathbb{R}^+, \tag{1}$$

$$u = \triangle u = 0, \text{ on } \partial\Omega, \tag{2}$$

$$u|_{t=\tau} = u_\tau(x), \ \partial_t u|_{t=\tau} = p_\tau(x), \quad \tau \in \mathbb{R}, \tag{3}$$

where $u(x,t)$ is the unknown function, which represents the deflection of the road bed in the vertical plane, $k > 0$ denotes the spring constant.

The suspension bridge equations were presented by Lazer and McKenna as the new problems in fields of nonlinear analysis [1]. For (1), there are many classical results. For instance, existence, multiplicity and property of the travelling wave solutions etc., were studied by the most of authors, we refer the reader to [1–5] and the references therein. In [9], we investigated the existence of global attractors in $H_0^2(\Omega) \times L^2(\Omega)$ for the autonomous suspension bridge equations, that is, $g(x,t) = g(x)$. In the sequel, using the condition (C) introduced in [6] and combining with techniques of the energy estimates, we also achieved the existence of strong solutions and

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the global attractors in $D(A)\times H_0^2(\Omega)$ [10]. In this paper, what we are interested in is the uniform compact attractors for the non-autonomous systems (1)–(3).

In the last two decades, the autonomous dynamical systems and their attractors have been extensively studied. There are many classical and significant literatures and works. We refer the reader to [6–12]. Recently, the study of the uniform attractors for the non-autonomous dynamical systems has attracted much attention and made fast progress, see [13–15] and the references therein. As we know, the most general method to consider the existence of the uniform attractors for the non-autonomous dynamical systems was presented by Chepyzhov and Vishik [14]. However, their approaches can only be used to deal with the problems with translation compact symbols while in applications symbols of many problems do not satisfy this condition. Motivated by [6], the authors of [15] gave the necessary and sufficient conditions of existence of the uniform attractors for the non-autonomous infinite-dimensional dynamical systems making use of the concept of noncompactness measure, and successfully proved the existence of the uniform attractors for non-autonomous 2D Navier-Stokes equations with normal external forces in $L^2_{\mathrm{loc}}(\mathbb{R},L^2)$ which is translation bounded but not translation compact. Recently, a new class of functions was introduced in [13] for weakly dissipative dynamical systems, which are also more general than translation compact. Moreover, the authors obtained the uniform attractors of the weakly damped non-autonomous hyperbolic equations with this new class of time dependent external forces. Inspired by [13], we show in the present paper the existence of the uniformly absorbing set for the strong damping non-autonomous suspension bridge equations, and then prove the existence of the uniform attractors for the family of processes corresponding to the equation in $H_0^2\times L^2$ using the methods in [13].

Assume that the nonlinear function $f\in C^2(\mathbb{R},\mathbb{R})$ satisfies the following general conditions: There exists a constant $C_1>0$ such that

(F1) $\liminf_{|s|\to\infty}\frac{F(s)}{s^2}\geq 0$, $F(s)=\int_0^s f(\tau)\mathrm{d}\tau$;

(F2) $\limsup_{|s|\to\infty}\frac{|f'(s)|}{|s|^\gamma}=0$, $\forall 0\leq\gamma<\infty$;

(F3) $\liminf_{|s|\to\infty}\frac{sf(s)-C_1F(s)}{s^2}\geq 0$.

With the usual notation, we introduce the spaces $H=L^2(\Omega)$, $V=H_0^2(\Omega)$, and endow these spaces with the usual scalar products and norms, $(\cdot,\cdot)$, $|\cdot|$, $((\cdot,\cdot))$, $\|\cdot\|$, respectively, where

$$(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x,\quad ((u,v))=\int_\Omega \triangle u(x)\triangle v(x)\mathrm{d}x.$$

2. Non-Autonomous systems and their attractors

In this section, we iterate some notations and theorems in [13–15], which are important to get our main results.

Let E be a Banach space, and let a two-parameter family of mappings $\{U(t,\tau)\}=\{U(t,\tau)\mid t\geq\tau,\tau\in\mathbb{R}\}$ act on E:

$$U(t,\tau):E\to E, t\geq\tau,\quad \tau\in\mathbb{R}.$$

Definition 1 *Let Σ be a parameter set. $\{U_\sigma(t,\tau)\mid t\geq\tau,\tau\in\mathbb{R}\}$, $\sigma\in\Sigma$ is said to be a family of* *processes in Banach space E, if for each $\sigma \in \Sigma$, $\{U_\sigma(t,\tau)\}$ is a process, that is, the two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ from E to E satisfy:*

$$U_\sigma(t,s) \circ U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geq s \geq \tau,\ \tau \in \mathbb{R}, \tag{4}$$

$$U_\sigma(\tau,\tau) = \mathrm{Id} \textit{ is the identity operator}, \quad \tau \in \mathbb{R}, \tag{5}$$

where Σ is called the symbol space and $\sigma \in \Sigma$ is the symbol.

A set $B_0 \subset E$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$, there exists $t_0 = t_0(\tau, B) \geq \tau$ such that $\cup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq t_0$. A set $Y \subset E$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any fixed $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(E)$

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \mathrm{dist}_E(U_\sigma(t,\tau)B, Y) = 0, \tag{6}$$

where $\mathcal{B}(E)$ is the set of all bounded subset of E.

Assumption I Let $\{T(h)|h \geq 0\}$ be a family of operators acting on Σ and satisfy:

i) $T(h)\Sigma = \Sigma$, $\forall h \in \mathbb{R}^+$;

ii) translation identity:

$$U_\sigma(t+h, \tau+h) = U_{T(h)\sigma}(t,\tau),\ \forall\ \sigma \in \Sigma,\ t \geq \tau,\ \tau \in \mathbb{R},\ h \geq 0. \tag{7}$$

Definition 2 *A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is said to satisfy the uniform (w.r.t. $\sigma \in \Sigma$) condition (C) if for any fixed $\tau \in \mathbb{R}$, $B \in \mathcal{B}(E)$ and $\varepsilon > 0$, there exist a $t_0 = t_0(\tau, B, \varepsilon) \geq \tau$ and a finite dimensional subspace E_m of E such that*

i) $P_m(\cup_{\sigma\in\Sigma} \cup_{t\geq t_0} U_\sigma(t,\tau)B)$ is bounded; and

ii) $\| (I - P_m)(\cup_{\sigma\in\Sigma} \cup_{t\geq t_0} U_\sigma(t,\tau)x) \|_E \leq \varepsilon$, $\forall x \in B$,

where $\dim E_m = m$ and $P_m : E \to E_m$ is a bounded projector.

Theorem 1 *Let Σ be a complete metric space, and under assumption I. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ possesses the compact uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma$ in E satisfying*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0), \quad \forall \tau \in \mathbb{R},$$

if it

i) has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0; and

ii) satisfies uniform (w.r.t. $\sigma \in \Sigma$) condition (C).

Moreover, if E is a uniformly convex Banach space, then the converse is true.

Remark 1 The Theorem 1 is true without any continuous assumption on $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ and $\{T(t)\}_{t\geq 0}$.

Definition 3 *Let X be a Hilbert space. A function $g \in L_b^2(\mathbb{R}; X)$ is said to satisfy Condition (C^*) if for any $\varepsilon > 0$, there exists a finite dimensional subspace X_1 of X such that*

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \| (I - P_m) g(x,s) \|_X^2 \, \mathrm{d}s < \varepsilon,$$

where $P_m : X \to X_1$ is the canonical projector.

Denote by $L^2_{c^*}(\mathbb{R}; X)$ the set of all functions satisfying Condition (C^*).

Lemma 1 *If $h \in L^2_{c^*}(\mathbb{R}; X)$, then for any $\varepsilon > 0$ and $\tau \in \mathbb{R}$, we have*

$$\sup_{t \geq \tau} \int_\tau^t e^{-\alpha(t-s)} \parallel (I - P_m)h(s) \parallel_X^2 \mathrm{d}s \leq \varepsilon,$$

where P_m is the same as that in Definition 3.

3. Uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set and uniform (w.r.t. $\sigma \in \Sigma$) attractor in $V \times H$

For the existence of the solutions for (1)–(3), since the time-dependent term makes no essential complications, we directly give the following results of the existence and uniqueness of the solution without proof. In fact, the proof is based on the Faedo-Galerkin approximation approaches, see [1, 7] for the details. Denote $\mathbb{R}_\tau = [\tau, +\infty]$.

Theorem 2 *If g, u_τ, p_τ are given satisfying $g \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, $u_\tau \in V$, $p_\tau \in H$, then (1)–(3) have a unique solution*

$$u(t) \in C(\mathbb{R}_\tau; V), \quad \partial_t u \in C(\mathbb{R}_\tau; H).$$

Now we will write (1), (3) as an evolutionary system by introducing $y(t) = (u(t), p(t))$ and $y_\tau = (u_\tau, p_\tau)$ for brevity. We denote by $E = V \times H$ the space of vector functions $y(x) = (u(x), p(x))$ with the finite norms $\|y\|_E = \{\|u\|^2 + |p|^2\}^{\frac{1}{2}}$, which is equivalent to $\|y\|_E = \|u\|^2 + |p + \varepsilon u|^2$. Then the system (1), (3) is equivalent to the following system

$$\begin{aligned} &\partial_t u = p, \ \partial_t p = -Ap - Au - ku^+ - f(u) + g(x,t), \ \forall\, t \geq \tau; \\ &u|_{t=\tau} = u_\tau, \quad p|_{t=\tau} = p_\tau, \end{aligned} \tag{9}$$

which can be rewritten in the operator form

$$\partial_t y = A_{\sigma(t)}(y), \quad y|_{t=\tau} = y_\tau, \tag{10}$$

where $\sigma(s) = g(x, s)$ is the symbol of equation (10). Thus if $y_\tau \in E$, then (10) has a unique solution $y(t) \in C_b(\mathbb{R}_\tau; E)$. This implies that the process $\{U_\sigma(t, \tau)\}$ given by the formula $U_\sigma(t, \tau)y_\tau = y(t)$ is well defined in E.

We now give a fixed external force g_0 in $L^2_b(\mathbb{R}; X)$ and define the symbol space $\mathcal{H}(\sigma_0)$ for (10). Let a fixed symbol $\sigma_0(s) = g_0(s) = g_0(\cdot, s)$ satisfy Condition (C*) in $L^2_{loc}(\mathbb{R}; X)$; that is, the family of translation $\{g_0(s+h), h \in \mathbb{R}\}$ forms a function set satisfying Condition (C*). Therefore

$$\mathcal{H}(\sigma_0) = \mathcal{H}(g_0) = [g_0(x, s+h) | h \in \mathbb{R}]_{L^{2,w}_{\mathrm{loc}}(\mathbb{R}; X)},$$

where [ ] denotes the closure of a set in a topological space $L^{2,w}_{\mathrm{loc}}(\mathbb{R}; X)$.

Thus, for any $g(x,t) \in \mathcal{H}(g_0)$, the problem (9) with g instead of g_0 possesses a corresponding processes $\{U_g(t, \tau)\}$ acting on E.

Proposition 1 ([13, 14]) *If X is a reflexive separable, then*

1) *For all* $g_1 \in \mathcal{H}(\phi)$, $\|g_1\|^2_{L^2_b(\mathbb{R};X)} \leq \|g\|^2_{L^2_b(\mathbb{R};X)}$;
2) *The translation group* $\{T(t)\}$ *is weakly continuous on* $\mathcal{H}(g)$;
3) $T(t)\mathcal{H}(g) = \mathcal{H}(g)$ *for all* $t \in \mathbb{R}$.

Therefore, the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$: $U_g(t,\tau):\ E \to E,\ t \geq \tau,\ \tau \in \mathbb{R}$ are defined. Furthermore, the translation semigroup $\{T(h)|h \in \mathbb{R}^+\}$ satisfies that $\forall h \in \mathbb{R}^+$, $T(h)\mathcal{H}(g_0) = \mathcal{H}(g_0)$, and the following translation identity holds:

$$U_g(t+h, \tau+h) = U_{T(h)g}(t,\tau),\ \forall\ g \in \mathcal{H}(g_0),\ t \geq \tau,\ \tau \in \mathbb{R},\ h \geq 0.$$

For (10), we give a fixed external force $g_0 \in L^2_{C*}(\mathbb{R}; H)$ and $\mathcal{H}(\sigma_0) = \mathcal{H}(g_0) = [g_0(x, s+h)|h \in \mathbb{R}]_{L^{2,w}_{\mathrm{loc}}(\mathbb{R};H)}$.

3.1 A priori estimates

By (F1) and (F3), for any $0 < \varepsilon < \frac{1}{2}$, there exist constants $K_1,\ K_2 > 0$ such that

$$\int_\Omega F(u(x))\mathrm{d}x + \frac{1-\varepsilon}{4}\|u\|^2 \geq -K_1, \quad \forall u \in V, \tag{11}$$

$$\int_\Omega uf(u)\mathrm{d}x - C_1\int_\Omega F(u(x))\mathrm{d}x + \frac{1-\varepsilon}{4}\|u\|^2 \geq -K_2, \quad \forall u \in V. \tag{12}$$

Taking the scalar product in H of equation (1) with $v = \partial_t u + \varepsilon u$, after a computation, we find

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big((1-\varepsilon)\|u\|^2 + |v|^2 + k|u^+|^2 + 2\int_\Omega F(u(x))\mathrm{d}x\Big) + \varepsilon(1-\varepsilon)\|u\|^2 + \\ &\|v\|^2 - \varepsilon|v|^2 + \varepsilon^2(u,v) + \varepsilon k|u^+|^2 + \varepsilon(f(u),u) = (g(t),v).\end{aligned} \tag{13}$$

Using the Poincaré inequality

$$\|v\| \geq \lambda_1|v|, \quad \forall v \in V, \tag{14}$$

where λ_1 is the first eigenvalue of $\triangle^2$ in V, together with (12)–(13) and exploiting the Young inequality and the Hölder inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big((1-\varepsilon)\|u\|^2 + |v|^2 + k|u^+|^2 + 2\int_\Omega F(u(x))\mathrm{d}x\Big) + \frac{3\varepsilon(1-\varepsilon)}{4}\|u\|^2 + \\ &(\frac{\lambda_1}{2} - \varepsilon)|v|^2 + \varepsilon^2(u,v) + \varepsilon k|u^+|^2 + \varepsilon C_1\int_\Omega F(u(x))\mathrm{d}x \\ &\leq \varepsilon K_2 + \frac{|g(t)|^2}{2\lambda_1}.\end{aligned} \tag{15}$$

Since

$$\begin{aligned}&\frac{3\varepsilon(1-\varepsilon)}{4}\|u\|^2 + (\frac{\lambda_1}{2} - \varepsilon)|v|^2 + \varepsilon^2(u,v) \\ &\geq \frac{3\varepsilon(1-\varepsilon)}{4}\|u\|^2 + (\frac{\lambda_1}{2} - \varepsilon)|v|^2 - \frac{\varepsilon^2}{\lambda_1}\|u\|\cdot|v| \\ &\geq \frac{\varepsilon(1-\varepsilon)}{2}\|u\|^2 + (\frac{\lambda_1}{2} - \frac{\varepsilon^3}{\lambda_1^2(1-\varepsilon)} - \varepsilon)|v|^2 \\ &\geq \frac{\varepsilon(1-\varepsilon)}{2}\|u\|^2 + (\frac{\lambda_1}{2} - \frac{\varepsilon^2}{\lambda_1^2} - \varepsilon)|v|^2,\end{aligned} \tag{16}$$

we take ε small enough such that $\frac{\lambda_1}{2} - \frac{\varepsilon^2}{\lambda_1^2} - \varepsilon > \frac{\lambda_1}{4}$. Collecting with (15) and (16) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big((1-\varepsilon)\|u\|^2 + |v|^2 + k|u^+|^2 + 2\int_\Omega F(u(x))\mathrm{d}x\Big) + \varepsilon(1-\varepsilon)\|u\|^2 + \\ &\frac{\lambda_1}{2}|v|^2 + 2\varepsilon k|u^+|^2 + 2\varepsilon C_1\int_\Omega F(u(x))\mathrm{d}x \le 2\varepsilon K_2 + \frac{|g(t)|^2}{\lambda_1}.\end{aligned} \tag{17}$$

Take $\alpha = \min\{\varepsilon,\ \frac{\lambda_1}{2},\ \varepsilon C_1\}$, then it follows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big((1-\varepsilon)\|u\|^2 + |v|^2 + k|u^+|^2 + 2\int_\Omega F(u(x))\mathrm{d}x\Big) + \\ &\qquad \alpha\Big((1-\varepsilon)\|u\|^2 + |v|^2 + k|u^+|^2 + 2\int_\Omega F(u(x))\mathrm{d}x\Big) \\ &\le 2\varepsilon K_2 + \frac{|g(t)|^2}{\lambda_1}.\end{aligned} \tag{18}$$

Denote $Y(t) = (1-\varepsilon)\|u\|^2 + |v|^2 + k|u^+|^2 + 2\int_\Omega F(u(x))\mathrm{d}x + 2K_1$, in the light of (11) we have $Y(t) > 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + \alpha Y(t) \le C + \frac{|g(t)|^2}{\lambda_1}, \quad C = 2(\alpha K_1 + \varepsilon K_2).$$

From Proposition 1, recall that

$$\|g\|^2_{L^2_b} \le \|g_0\|^2_{L^2_b}, \text{ for all } g \in \mathcal{H}(g_0),$$

and using the Gronwall lemma [14, p35, Lemma 1.3], we obtain

$$Y(t) \le Y(\tau)\exp(-\alpha(t-\tau)) + (1+\alpha^{-1})(C + \frac{1}{\lambda_1}\|g_0\|^2_{L^2_b}). \tag{19}$$

According to (11) again, we achieve

$$(1-\varepsilon)\|u\|^2 + |v|^2 \le 2Y(t) \le 8(1+\alpha^{-1})(C + \frac{1}{\lambda_1}\|g_0\|^2_{L^2_b}),$$

that is

$$(1-\varepsilon)\|u\|^2 + |v|^2 \le \mu_0^2, \tag{20}$$

where $\mu_0^2 = 8(1+\alpha^{-1})(C + \frac{1}{\lambda_1}\|g_0\|^2_{L^2_b})$.

Therefore, we obtain a bounded uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set $B_0 = \{y = (u,p) :\ \|y\|^2_E \le \mu_0^2\}$ in E, i.e., for every $B \in \mathcal{B}(E)$ and for all $g \in \mathcal{H}(g_0)$, there exists a $t_0 = t_0(\tau, B) \ge \tau$ such that

$$\bigcup_{g\in\mathcal{H}(g_0)} U_g(t,\tau)B \subset B_0, \quad \forall t \ge t_0.$$

Thus, we have the following results immediately.

Theorem 3 *Under assumptions (F1)–(F3), if $g_0(x,s) \in L^2_{C*}(\mathbb{R}; H)$, then the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(h_0)$ corresponding to the problem (10) has a bounded uniformly(w.r.t. $g \in \mathcal{H}(h_0)$) absorbing set B_0 in E.*

3.2 Uniform attractor in $V \times H$

In order to obtain the uniform attractors of the family of processes $\{U_g(t,\tau)\},\ g \in \mathcal{H}(g_0)$, we also need the following compactness for the nonlinear term f.

Lemma 2 ([9]) *Let f be a C^2 function from $\mathbb{R}$ to $\mathbb{R}$ satisfying (F2). Then $f : V \to H$ is continuously compact.*

Now we prove the existence of compact uniform (w.r.t. $h \in \mathcal{H}(g_0)$) attractor for system (1)–(3) with external forces $g_0 \in L^2_{C*}(\mathbb{R}; H)$ in E.

Theorem 4 *Let $k > 0$, and assume that (F1)–(F3) hold. If $g_0(x,t) \in L^2_{c^*}(\mathbb{R}, H)$, then the family of processes $\{U_g(t,\tau)\},\ g \in \mathcal{H}(g_0)$ corresponding to the problem (1) possesses a compact uniform(w.r.t. $g \in \mathcal{H}(g_0)$) attractor $\mathcal{A}_{\mathcal{H}(g_0)}$ in E satisfying*

$$\mathcal{A}_{\mathcal{H}(g_0)} = \omega_{0,\mathcal{H}(g_0)}(B_0) = \omega_{\tau,\mathcal{H}(g_0)}(B_0), \tag{21}$$

where B_0 is the uniformly(w.r.t. $h \in \mathcal{H}(g_0)$) absorbing set in E.

Proof By Theorem 1, we need only to verify that the family of processes $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$ satisfies the uniform (w.r.t. $g \in \mathcal{H}(g_0)$) condition (C).

Since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ with

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \lambda_j \to +\infty, \text{ as } j \to \infty,$$

and a family of elements $\{\omega_j\}_{j=1}^{\infty}$ of V which are orthonormal in H with

$$A\omega_j = \lambda_j \omega_j, \quad \forall j \in \mathbb{N}.$$

Let $H_m = \operatorname{span}\{\omega_1, \omega_2, \ldots, \omega_m\}$, and $P_m : H \to H_m$ be an orthogonal projector. For any $u \in V$, we write

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2.$$

Since $f : V \to H$ is a compact operator by Lemma 2, for any $\varepsilon > 0$, there exists some m such that

$$|(I - P_m)f(u)|_H \le \frac{\varepsilon}{8}, \quad \forall u \in B_0(0,\mu_0), \tag{22}$$

where μ_0 is given by (19).

Choosing $0 < \sigma < 1$, and taking the scalar product with $v_2 = \partial_t u_2 + \sigma u_2$ of equation (1) in H, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((1-\sigma)\|u_2\|^2 + |v_2|^2) + \sigma(1-\sigma)\|u_2\|^2 + \|v_2\|^2 - \sigma|v_2|^2 + \\ &\sigma^2(u_2, v_2) + k((u^+)_2, v_2) + (f(u), v_2) = (g(t), v_2).\end{aligned}$$

By (14) we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((1-\sigma)\|u_2\|^2 + |v_2|^2) + \sigma(1-\sigma)\|u_2\|^2 + (\lambda_1 - \sigma)|v_2|^2 + \\ &\sigma^2(u_2, v_2) + k((u^+)_2, v_2) + (f(u), v_2) = (g(t), v_2).\end{aligned} \tag{23}$$

Due to the uniform boundedness of u in V, and the Sobolev embedding inequality, making use of $|u^+| \le |u|$, it follows that $|(u^+)_2| < \varepsilon$, for any $\varepsilon > 0$. Therefore

$$|(k(u^+)_2, v_2)| \le \frac{\lambda_1}{4}|v_2|^2 + \frac{k^2\varepsilon^2}{\lambda_1}. \tag{24}$$

Combining with (23) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((1-\sigma)\|u_2\|^2 + |v_2|^2) + \sigma(1-\sigma)\|u_2\|^2 + (\frac{3\lambda_1}{4} - \sigma)|v_2|^2 + \\ &\sigma^2(u_2, v_2) + (f(u), v_2) \le \frac{k^2\varepsilon^2}{\lambda_1} + (g(t), v_2).\end{aligned} \tag{25}$$

Let σ be small enough such that $\frac{3\lambda_1}{4} - \frac{\sigma^2}{\lambda_1^2} - \sigma \ge \frac{\lambda_1}{4}$. Then

$$\begin{aligned}&\sigma(1-\sigma)\|u_2\|^2 + (\frac{3\lambda_1}{4} - \sigma)|v_2|^2 + \sigma^2(u_2, v_2) \\ &\ge \sigma(1-\sigma)\|u_2\|^2 + (\frac{3\lambda_1}{4} - \sigma)|v_2|^2 - \frac{\sigma^2}{\lambda_1}\|u_2\| \cdot |v_2| \\ &\ge \frac{3\sigma(1-\sigma)}{4}\|u_2\|^2 + (\frac{3\lambda_1}{4} - \frac{\sigma^2}{\lambda_1^2} - \sigma)|v|^2 \\ &\ge \frac{3\sigma(1-\sigma)}{4}\|u_2\|^2 + \frac{\lambda_1}{4}|v_2|^2,\end{aligned} \tag{26}$$

which together with (22)–(26) leads to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((1-\sigma)\|u_2\|^2 + |v_2|^2) + \frac{3\sigma(1-\sigma)}{4}\|u_2\|^2 + \frac{\lambda_1}{4}|v_2|^2 \\ &\le \frac{\varepsilon}{8}|v_2| + \frac{k^2\varepsilon^2}{\lambda_1} + (g(t), v_2) \\ &\le \frac{\varepsilon}{8}|v_2| + \frac{k^2\varepsilon^2}{\lambda_1} + |(I - P_m)g(t)| \cdot |v_2| \\ &\le \frac{\lambda_1}{8}|v_2|^2 + \frac{(4+k^2)\varepsilon^2}{\lambda_1} + \frac{4}{\lambda_1}|(I - P_m)g(t)|^2,\end{aligned}$$

that is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}((1-\sigma)\|u_2\|^2 + |v_2|^2) + \frac{3\sigma(1-\sigma)}{2}\|u_2\|^2 + \frac{\lambda_1}{4}|v_2|^2 \\ &\le \frac{2(4+k^2)\varepsilon^2}{\lambda_1} + \frac{8}{\lambda_1}|(I - P_m)g(t)|^2.\end{aligned} \tag{27}$$

Take $\alpha_0 = \min\{\frac{3\sigma}{2}, \frac{\lambda_1}{4}\}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}((1-\sigma)\|u_2\|^2 + |v_2|^2) + \alpha_0(\|u_2\|^2 + |v_2|^2) \le C\varepsilon^2 + \frac{8}{\lambda_1}|(I - P_m)g(t)|^2, \tag{28}$$

where $C = \frac{2(4+k^2)}{\lambda_1}$.

By the Gronwall Lemma, we obtain

$$\begin{aligned}(1-\sigma)\|u_2(t)\|^2 + |v_2(t)|^2 \le &\left((1-\sigma)\|u_2(\tau)\|^2 + |v_2(\tau)|^2\right) e^{-\alpha_0(t-\tau)} + \\ &\frac{C\varepsilon^2}{\alpha_0} + \frac{8}{\lambda_1}\int_\tau^t e^{-\alpha_0(t-s)}|(I - P_m)g(s)|^2 \mathrm{d}s.\end{aligned} \tag{29}$$

Since $g \in L^2_{c^*}(\mathbb{R}, H)$, by Lemma 1, for any $\varepsilon_1 = \varepsilon_1(\varepsilon) > 0$, there exists an m large enough such that

$$\frac{8}{\lambda_1}\int_\tau^t e^{-\alpha_0(t-s)}|(I-P_m)g(s)|^2\mathrm{d}s \le \frac{\varepsilon_1}{3}, \quad \forall g \in \mathcal{H}(h_0),\ \forall t \ge \tau. \tag{30}$$

Let $t_1 = \tau + \frac{1}{\alpha_0}\ln\frac{3\mu_0^2}{\varepsilon_1}$. Then we conclude that

$$((1-\sigma)\|u_2(\tau)\|^2 + |v_2(\tau)|^2)e^{-\alpha_0(t-\tau)} \le \mu_0^2 e^{-\alpha_0(t-\tau)} \le \frac{\varepsilon_1}{3}, \quad \forall t \ge t_1. \tag{31}$$

Obviously, we can choose $\varepsilon = \varepsilon(\varepsilon_1)$ such that

$$\frac{C\varepsilon^2}{\alpha_0} \le \frac{\varepsilon_1}{3}. \tag{32}$$

Therefore, combining with (29)–(32) leads to

$$(1-\sigma)\|u_2(t)\|^2 + |v_2(t)|^2 \le \varepsilon_1, \quad \forall t \ge t_1,\ g \in \mathcal{H}(g_0),$$

which indicates that the family of processes $\{U_g(t,\tau)\}, g \in \mathcal{H}(g_0)$ satisfies uniform (w.r.t. $g \in \mathcal{H}(g_0)$) condition (C) in E.

The proof is completed. □

References

[1] LAZER A C, MCKENNA P J. *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis* [J]. SIAM Rev., 1990, **32**(4): 537–578.

[2] HUMPHREYS L D. *Numerical mountain pass solutions of a suspension bridge equation* [J]. Nonlinear Anal., 1997, **28**(11): 1811–1826.

[3] LAZER A C, MCKENNA P J. *Large scale oscillatory behaviour in loaded asymmetric systems* [J]. Ann. Inst. H. Poincaré Anal. Non Linéaire, 1987, **4**(3): 243–274.

[4] MCKENNA P J, WALTER W. *Nonlinear oscillations in a suspension bridge* [J]. Arch. Rational Mech. Anal., 1987, **98**(2): 167–177.

[5] CHOI Q H, JUNG T. *A nonlinear suspension bridge equation with nonconstant load* [J]. Nonlinear Anal., 1999, **35**(6): 649–668.

[6] MA Qingfeng, WANG Shouhong, ZHONG Chengkui. *Necessary and sufficient conditions for the existence of global attractors for semigroups and applications* [J]. Indiana Univ. Math. J., 2002, **51**(6): 1541–1559.

[7] TEMAM R. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics (Second edition)* [M]. Springer-Verlag, New York, 1997.

[8] BALL J M. *Initial-boundary value problems for an extensible beam* [J]. J. Math. Anal. Appl., 1973, **42**: 61–90.

[9] MA Qiaozhen, ZHONG Chengkui. *Existence of global attractors for the suspension bridge equation* [J]. Sichuan Daxue Xuebao, 2006, **43**(2): 271–276.

[10] ZHONG Chengkui, MA Qiaozhen, SUN Chunyou. *Existence of strong solutions and global attractors for the suspension bridge equations* [J]. Nonlinear Anal., 2007, **67**(2): 442–454.

[11] HALE J K. *Asymptotic Behavior of Dissipative Systems* [M]. American Mathematical Society, Providence, RI, 1988.

[12] ROBISON J C. *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors* [M]. Cambridge University Press, 2001.

[13] MA Shan, ZHONG Chengkui. *The attractors for weakly damped non-autonomous hyperbolic equations with a new class of external forces* [J]. Discrete Contin. Dyn. Syst., 2007, **18**(1): 53–70.

[14] CHEPYZHOV V V, VISHIK M I. *Attractors for Equations of Mathematical Physics* [M]. American Mathematical Society, Providence, RI, 2002.

[15] LU Songsong, WU Hongqing, ZHONG Chengkui. *Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces* [J]. Discrete Contin. Dyn. Syst., 2005, **13**(3): 701–719.