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Generalized Nonlinear Implicit Variational-Like Inequalities

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Abstract A new class of generalized nonlinear implicit variational-like inequality problems (for short, GNIVLIP) in the setting of locally convex topological vector spaces is introduced and studied in this paper. Under suitable conditions, some existence theorems of solutions for (GNIVLIP) are presented by using some fixed point theorems.

Keywords generalized nonlinear implicit variational-like inequality; upper semicontinuity; fixed point theorem; multifunction.

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1. Introduction

The variational-like inequality, also known as the pre-variational inequality, is one of the generalized forms of variational inequalities [1–6]; Variational-like inequality and generalized variational-like inequality problems are powerful tools for studying nonconvex optimization problems, and nonconvex and nondifferentiable optimization problems, respectively [2, 4, 6].

On the other hand, some new problems related to variational-like inequalities have been proposed and studied by many authors. Ahmad and Irfan [7] introduced generalized nonlinear variational-like inequality problems and established some existence results. Liu and Wu [8] considered the penalty method for solving generalized nonlinear variational inequalities and obtained some existence theorems for the variational inequalities in reflexive real Banach spaces. Motivated and inspired by these works, in this paper, we introduce and study a new class of generalized nonlinear implicit variational-like inequality problem in the setting of locally convex topological vector spaces and derive some existence results of its solutions under suitable assumptions.

Let $\langle E, E^*\rangle$ be a dual system of locally convex spaces and K a nonempty convex subset of E. Given a single valued mapping $\varphi : E^* \times E^* \times E^* \to E^*$, a bifunction $\eta : K \times K \to E$, multivalued

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mappings $M, S, T : K \to 2^{E^*}$, and a functional $\Phi : K \times K \to R$. We consider the following generalized nonlinear implicit variational-like inequality problem (GNIVLIP): Find $x \in K$ such that $\forall y \in K$, $\exists u \in M(x)$, $v \in S(x)$, $\omega \in T(x)$ satisfying

$$\langle \varphi(u, v, \omega), \eta(y, x)\rangle \geq \Phi(x, y).$$

If $\varphi(u, v, \omega) = p(u) - (f(v) - g(\omega))$ and $\Phi(x, y) = \phi(x) - \phi(y)$, where $f, g, p : E^* \to E^*$, $\phi : K \to R$, then (GNIVLIP) reduces to (GNVLIP) of finding $x \in K$ such that $\forall y \in K$, $\exists u \in M(x)$, $v \in S(x)$, and $\omega \in T(x)$ satisfying

$$\langle p(u) - (f(v) - g(\omega)), \eta(y, x)\rangle \geq \phi(x) - \phi(y),$$

which has been studied by Ahmad and Irfan [7]. Furthermore, if f, g and M are identity mappings, $\phi = 0$ and $\eta(y, x) = y - q(x)$, where $q : K \to K$ is a nonlinear operator, then (GNVLIP) collapses to the problem of finding $x \in K$ such that $\forall y \in K$, $\exists v \in S(x)$ and $\omega \in T(x)$ satisfying $\langle p(x) - (v - \omega), y - q(x)\rangle \geq 0$.

2. Preliminaries

We first recall some definitions and lemmas which are needed in the main results of this paper.

Definition 2.1 *Let X and Y be topological vector spaces. Let $P : X \to 2^Y$ be a multivalued mapping*

(i) P is called upper semicontinuous (usc, for short) at $x_0 \in \operatorname{dom} P := \{x \in X : P(x) \neq \emptyset\}$ if, for every open set V in Y containing $P(x_0)$, there exists an open neighborhood U of x_0 in X such that $P(x) \subseteq V$, $\forall x \in U$;

(ii) P is called closed if for every net $\{x_\lambda\}$ converging to x^* and $\{y_\lambda\}$ converging to y^*, one has $\forall \lambda, y_\lambda \in P(x_\lambda)$ implies that $y^* \in P(x^*)$;

(iii) The graph of P, denoted by $G(P)$, is given by $G(P) = \{(x, z) \in X \times Y : x \in X, z \in P(x)\}$.

Definition 2.2 *P is called usc if P is usc at each point of $\operatorname{dom} P$.*

Remark 2.2 (i) If K is a compact subset of X, and P is usc and compact-valued, then $P(K)$ is compact.

(ii) If P is usc and compact-valued, then P is closed.

(iii) A multivalued mapping is closed if its graph is closed.

The following fixed-point theorems are useful in the proof of our results.

Lemma 2.1 ([9]) *Assume that X is a Hausdorff topological vector space, $A \subseteq X$ is nonempty and convex and $\varphi : A \to 2^A$ is a multifunction with nonempty convex values, and assume that:*

(i) $\varphi^{-1}(y)$ is open in A for each $y \in A$;

(ii) There exists a nonempty subset X_0 contained in a compact convex set of A such that $A \setminus \bigcup_{y \in X_0} \varphi^{-1}(y)$ is compact or empty.

Then, there exists $\tilde{x} \in A$ such that $\tilde{x} \in \varphi(\tilde{x})$.

Lemma 2.2 ([9]) *Assume that V is a convex set in a Hausdorff topological vector space and that $f : V \to 2^V$ is a multifunction with convex values. Assume that:*

(i) $V = \bigcup_{x \in V} \operatorname{int} f^{-1}(x)$;

(ii) There exists a nonempty compact subset $D \subseteq V$ such that, for all finite subsets $M \subseteq V$, there is a compact convex subset L_M of V, containing M, such that $L_M \setminus D \subseteq \bigcup_{x \in L_M} f^{-1}(x)$.
Then, there is a fixed point of f in V.

Lemma 2.3 ([10]) *Let K be a nonempty convex subset of a Hausdorff topological vector space X, and let $P, Q : K \to 2^K$ be two multivalued mappings. Assume that the following conditions hold:*

(i) For each $x \in K$, $\operatorname{co}(P(x)) \subseteq Q(x)$ and $P(x)$ is nonempty;

(ii) $K = \{\operatorname{int}_K P^{-1}(y) : y \in K\}$;

(iii) If K is not compact, then assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$ there exists $\tilde{y} \in B$ such that $x \in \operatorname{int}_K P^{-1}(\tilde{y})$.
Then there exists $\tilde{x} \in K$ such that $\tilde{x} \in Q(\tilde{x})$.

3. Existence of solutions

In the sequel, we present different sufficient conditions for the existence of solutions to (GNIVLIP).

Theorem 3.1 *Let E be a locally convex Hausdorff topological vector space and K a nonempty convex subset of E. Let $M, S, T : K \to 2^{E^*}$ be upper semicontinuous multivalued mappings with nonempty and compact values. Assume that the following conditions hold:*

(i) $\varphi : E^* \times E^* \times E^* \to E^*$ is continuous in every argument;

(ii) $\eta : K \times K \to E$ is continuous in the second argument with $\eta(x.x) = 0$, $\forall x \in K$;

(iii) $\Phi : K \times K \to R$ is continuous in the first argument with $\Phi(x, x) = 0$, $x \in K$;

(iv) The set $F(x) = \{y \in K : \forall u \in M(x), v \in S(x), \omega \in T(x) s.t. \langle \varphi(u, v, \omega), \eta(y, x)\rangle < \Phi(x, y)\}$ is convex, $\forall x \in K$;

(v) If K is not compact, then assume that there exists a nonempty subset B contained in a nonempty compact convex subset J of K such that $\forall x \in K \backslash J$, $\exists y_x \in B \bigcap F(x)$, $\langle \varphi(u, v, \omega), \eta(y_x, x)\rangle < \Phi(x, y_x)$, $\forall u \in M(x)$, $v \in S(x)$, $\omega \in T(x)$.
Then (GNIVLIP) has a solution.

Proof Assume that the conclusion of the theorem does not hold. Then $\forall x \in K$, the set

$$F(x) = \{y \in K : \forall u \in M(x), v \in S(x), \omega \in T(x) \text{ s.t. } \langle \varphi(u, v, \omega), \eta(y, x)\rangle < \Phi(x, y)\} \neq \emptyset.$$

Define a multivalued mapping $Q : K \to 2^K$ as follows:

$$Q(x) = \{y \in K : \forall u \in M(x), v \in S(x), \omega \in T(x) \text{ s.t. } \langle \varphi(u, v, \omega), \eta(y, x)\rangle < \Phi(x, y)\}, \ \forall x \in K.$$

Obviously, $Q(x) \neq \emptyset$, $\forall x \in K$. By assumption (iv), Q has convex values.

Next, we prove $Q^{-1}(y)$ is open in K. To this end, it suffices to show that $[Q^{-1}(y)]^c$, the complement of $Q^{-1}(y)$ in K, is closed in K. Let net $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq [Q^{-1}(y)]^c$ such that $\{x_\lambda\}$ converges to $x^* \in K$. Then, $\exists u_\lambda \in M(x_\lambda)$, $v_\lambda \in S(x_\lambda)$, $\omega_\lambda \in T(x_\lambda)$ such that

$$\langle \varphi(u_\lambda, v_\lambda, \omega_\lambda), \eta(y, x_\lambda)\rangle \geq \Phi(x_\lambda, y), \quad \forall\lambda.$$

Let $A = \{x_\lambda\} \bigcup \{x^*\}$. Then A is compact and $u_\lambda \in M(A)$, $v_\lambda \in S(A)$ and $\omega_\lambda \in T(A)$. Since $M(A)$, $S(A)$ and $T(A)$ are compact, $\{u_\lambda\}$, $\{v_\lambda\}$ and $\{\omega_\lambda\}$ have a convergent subnet with a limit, say, u^*, v^* and ω^*, respectively. Without loss of generality we may assume that $\{u_\lambda\}$ converges to u^*, $\{v_\lambda\}$ converges to v^*, and $\{\omega_\lambda\}$ converges to ω^*. Then by the upper semicontinuities of M, S and T, we have $u^* \in M(x^*)$, $v^* \in S(x^*)$ and $\omega^* \in T(x^*)$. From assumptions (i)–(iii), we have

$$\langle \varphi(u_\lambda, v_\lambda, \omega_\lambda), \eta(y, x_\lambda)\rangle - \Phi(x_\lambda, y)$$

converges to

$$\langle \varphi(u^*, v^*, \omega^*), \eta(y, x^*)\rangle - \Phi(x^*, y).$$

It follows that

$$\langle \varphi(u^*, v^*, \omega^*), \eta(y, x^*)\rangle \geq \Phi(x^*, y),$$

and therefore $x^* \in [Q^{-1}(y)]^c$. Thus $[Q^{-1}(y)]^c$ is closed in K. Also, we have

$$K = \bigcup_{y\in K} Q^{-1}(y) = \bigcup_{y\in K} \{\mathrm{int}_K Q^{-1}(y)\}.$$

From assumption (v), one has $K \setminus J \subseteq \bigcup_{y\in B} Q^{-1}(y) \subseteq K$ and so $K \setminus \bigcup_{y\in B} Q^{-1}(y) \subseteq J$. Since J is compact, it follows that $J \setminus \bigcup_{y\in B} Q^{-1}(y) = J \bigcap (\bigcup_{y\in B} Q^{-1}(y))^c = J \bigcap (\bigcap_{y\in B} [Q^{-1}(y)]^c)$ is compact. Applying Lemma 2.1, then there exists $\tilde{x} \in K$, such that $\tilde{x} \in Q(\tilde{x})$. That is, $\forall \tilde{u} \in M(\tilde{x})$, $\tilde{v} \in S(\tilde{x})$, $\tilde{\omega} \in T(\tilde{x})$, we have

$$\langle \varphi(\tilde{u}, \tilde{v}, \tilde{\omega}), \eta(\tilde{x}, \tilde{x})\rangle < \Phi(\tilde{x}, \tilde{x}),$$

which is a contradiction since $\eta(\tilde{x}, \tilde{x}) = 0$ and $\Phi(\tilde{x}, \tilde{x}) = 0$, $\forall x \in K$. This completes the proof. □

Remark 3.1 (a) Assumption (v) can be considered as a coercivity condition.

(b) If $\varphi(u, v, \omega) = p(u) - (f(v) - g(\omega))$, $\Phi(x, y) = \phi(x) - \phi(y)$, where f, g, $p : E^* \to E^*$, $\phi : K \to R$, then (GNIVLIP) collapses to (GNVLIP) [7].

Theorem 3.1 is still valid if the coercivity assumption (v) is suitably replaced.

Theorem 3.2 *Assume that assumptions (i)–(iv) of Theorem 3.1 hold and replace (v) of Theorem 3.1 by*

(v′) There exists a nonempty compact subset $P \subseteq K$ such that for all finite subsets $Y \subseteq K$, there is a compact convex subset L_Y of K, containing Y, such that $\forall x \in L_Y \setminus P$, $\exists y_x \in L_Y$, $y_x \in F(x)$ and $\langle \varphi(u, v, \omega), \eta(y_x, x)\rangle < \Phi(x, y_x)$, $\forall u \in M(x)$, $v \in S(x)$, $\omega \in T(x)$.
Then (GNIVLIP) has a solution.

Proof Assume that the conclusion of the theorem does not hold. Let $Q(x)$ be as in the proof of

Theorem 3.1. Similarly, we have $K = \bigcup\{\mathrm{int}_K Q^{-1}(y) : y \in K\}$. By assumption (v′), $\forall x \in L_Y \setminus P$, $\exists y_x \in L_Y \bigcap Q(x)$. Hence $x \in Q^{-1}(y_x) \subseteq \bigcup_{x \in L_Y} Q^{-1}(x)$, i.e., (ii) of Lemma 2.2 is satisfied. Then we may reach the same contradiction as in Theorem 3.1. The proof is completed. □

Theorem 3.3 *Assume that assumptions (i)–(iv) of Theorem 3.1 hold and replace (v) of Theorem 3.1 by*

(v″) If K is not compact, then assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that $\forall x \in K \setminus D$, $\exists \tilde{y} \in B$ such that

$$\langle \varphi(u, v, \omega), \eta(\tilde{y}, x)\rangle < \Phi(x, \tilde{y}), \quad \forall u \in M(x), v \in S(x), \omega \in T(x).$$

Then (GNIVLIP) has a solution.

Proof Suppose to the contrary that the conclusion of the theorem does not hold. Define $Q(x)$ as in the proof of Theorem 3.1 and argue as for Theorem 3.1 to see that $K = \bigcup\{\mathrm{int}_K Q^{-1}(y) : y \in K\}$. To apply Lemma 2.3 instead of Lemma 2.1, we verify assumption (iii) of Lemma 2.3. By (v″) $\forall x \in K \setminus D$, $\exists \tilde{y} \in B$ such that

$$\langle \varphi(u, v, \omega), \eta(\tilde{y}, x)\rangle < \Phi(x, \tilde{y}), \quad \forall u \in M(x), v \in S(x), \omega \in T(x).$$

Hence $x \in \mathrm{int}_K Q^{-1}(y)$, i.e., (iii) is satisfied. Then, by using Lemma 2.3 in the same way as employing Lemma 2.1 for Theorem 3.1, we complete the proof. □

Corollary 3.1 *Let E be a locally convex Hausdorff topological vector space and K a nonempty convex subset of E. Let M, S, $T : K \to 2^{E^*}$ be upper semicontionuous multivalued mappings with nonempty and compact values and $f, g, p : E^* \to E^*$ be continuous. Assume that the following conditions hold:*

(i) $\phi : K \to R$ is continous on K.

(ii) $\eta : K \times K \to E$ is continuous in the second argument with $\eta(x, x) = 0$, $\forall x \in K$.

(iii) The set $F(x) = \{y \in K : \forall u \in M(x), V \in S(x)$ and $\omega \in T(x)$ s.t. $\langle p(u) - (f(v) - g(\omega)), \eta(y, x)\rangle < \phi(x) - \phi(y)\}$ is convex, $\forall x \in K$.

(iv) If K is not compact, then assume that there exists a nonempty subset A contained in a nonempty compact convex subset J of K such that $\forall x \in K \setminus J$, $\exists y_x \in A \bigcap F(x)$, $\langle p(u) - (f(v) - g(\omega)), \eta(y_x, x)\rangle < \phi(x) - \phi(y_x)$, $\forall u \in M(x)$, $v \in S(x)$, $\omega \in T(x)$.
Then (GNVLIP) has a solution.

Proof Apply Theorem 3.1 with $\varphi(u, v, \omega) = p(u) - (f(v) - g(\omega))$ and $\Phi(x, y) = \phi(x) - \phi(y)$.

Corollary 3.2 *Assume assumptions (i)–(iii) of Corollary 3.1 hold and replace assumption (iv) there by*

(iv′) There exists a nonempty compact subset $P \subseteq K$ such that for all finite subsets $Y \subseteq K$, there is a compact convex subset L_Y of K, containing Y, such that

$$\forall x \in L_Y \setminus P, \exists y_x \in L_Y, y_x \in F(x) \text{ and } \langle p(u) - (f(v) - g(\omega)), \eta(y_x, x)\rangle < \phi(x) - \phi(y_x),$$
$$\forall u \in M(x), v \in S(x), \omega \in T(x).$$

Then (GNVLIP) has a solution.

Proof Apply Theorem 3.2 with $\varphi(u,v,\omega)=p(u)-(f(v)-g(\omega))$ and $\Phi(x,y)=\phi(x)-\phi(y)$.

Corollary 3.3 *Assume assumptions (i)-(iii) of Corollary 3.1 hold and replace assumption (iv) there by*

(iv$''$) If K is not compact, assume that there exists a nonempty compact convex subset B of K and a nonempty compact subset D of K such that $\forall x\in K\setminus D$, $\exists\tilde{y}\in B$ such that

$$\langle p(u)-(f(v)-g(\omega)),\eta(\tilde{y},x)\rangle<\phi(x)-\phi(\tilde{y}),\quad \forall u\in M(x),v\in S(x),\omega\in T(x).$$

Then (GNVLIP) has a solution.

Proof Apply Theorem 3.3 with $\varphi(u,v,\omega)=p(u)-(f(v)-g(\omega))$ and $\Phi(x,y)=\phi(x)-\phi(y)$.

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