Journal of Mathematical Research & Exposition
Mar., 2010, Vol. 30, No. 2, pp. 309–318
DOI:10.3770/j.issn:1000-341X.2010.02.014
Http://jmre.dlut.edu.cn

On the Expected Discounted Penalty Function for a Risk Process with Stochastic Return on Investments

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Abstract This paper considers the expected discounted penalty function $\Phi(u)$ for the perturbed compound Poisson risk model with stochastic return on investments. After presenting an integro-differential equation that the expected discounted penalty function satisfies, the paper derives the closed form solution by constructing an identical equation. The exact expression for $\Phi'(0)$ is given using the Laplace transform technique when interest rate is constant. Applications of the results are given to the ruin probability and moments of the deficit at ruin.

Keywords expected discounted penalty function; integro-differential equation; Laplace transform; ruin.

Document code A
MR(2000) Subject Classification 91B30; 60J75
Chinese Library Classification O211.6; F224.7

1. Introduction

We assume that the surplus process of an insurance company follows the perturbed compound Poisson risk process of Dufresne and Gerber [1],

$$U_t = ct - \sum_{i=1}^{N(t)} Y_i + \sigma_p W_t, \quad t \geq 0,$$

where $c > 0$ is the rate of premium; $\{N(t), t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$, representing the total number of claims up to time t; $\sigma_p > 0$ is the dispersion parameter and the claim size $\{Y_i,\ i = 1, 2, \ldots\}$, independent of $\{N(t), t \geq 0\}$ and standard Brownian motion $\{W(t), t \geq 0\}$, is a sequence of independent and identically distributed positive random variables with distribution function F.

Suppose that the insurer is allowed to invest in an asset or investment portfolio. The return on investment generating process is $R_t = \delta t + \sigma B_t$, $t \geq 0$, where $\{B_t, t \geq 0\}$ is another standard Brownian motion. For simplicity, we assume throughout the paper that the process U and R are independent. Then the current surplus at time t with an initial surplus u is

$$X(t) = e^{R_t}\left(u + \int_0^t e^{-R_s} \mathrm{d}U_s\right), \quad t \geq 0. \tag{1}$$

Received March 21, 2008; Accepted July 4, 2008
Supported by Key Project of National Social Science Fund (Grant No. 06&ZD039) and "Mathematics+X" Project of DUT.
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Let $\tau = \inf\{t \geq 0,\ X(t) \leq 0\}$ be the bankruptcy time and $w = w(x,y)$ be a non-negative measurable function on $[0,\infty) \times [0,\infty)$. Then the expected discounted penalty function under risk process (1) is

$$\Phi(u) = E\big[e^{-\rho\tau} w(X(\tau-), |X(\tau)|) I(\tau < \infty) \mid X(0) = u\big], \tag{2}$$

where $I(A)$ is the indicator function of event A and $\rho > 0$ is a discount factor.

The concept of the expected discounted penalty function was first introduced by Gerber and Shiu [2–4] and it has proven to be a powerful analytical tool to study ruin quantities. Many important results of ruin related functions can be obtained using this tool [5–9].

Since an insurance company often invests part of its capital in a risky portfolio, it has become an important topic for studying risk models with stochastic return. Although some elegant results have been obtained for $\Phi(u)$, the case with stochastic return still needs further investigation. For a compound Poisson risk process, Cai [10] studied $\Phi(u)$ with stochastic interest process being a Brownian motion with drift, while Yuen and Wang [11] considered the problem when stochastic return is a compound Poisson process with drift, and Yuen et al. [12] extended the analysis to a risk model with stochastic return under barrier strategy.

In this paper, we shall discuss the expected discounted penalty function in the classical risk model perturbed by diffusion with stochastic return being a Brownian motion with drift. It is slightly more general than discussed in Cai [10] since we add a Brownian motion to perturb the aggregate claims process. The idea and methods we use are motivated by Sundnt and Teugels [13] and Cai [10, 14]. In Section 2, we show the expected discounted penalty function $\Phi(u)$ satisfies an integro-differential equation, which generalizes results of Cai [10] and Wang and Wu [9]. In Section 3, we derive general solutions for $\Phi(u)$ in the form of an infinite series. The exact solution for $\Phi'(0)$ when interest rate is constant is found using Laplace transforms in Section 4. Section 5 presents two examples to illustrate applications of the results.

2. Integro-differential equation

The purpose of this section is to show that the expected discounted penalty function for risk model (1) satisfies an integro-differential equation. The result is obtained by utilizing the regenerative property of Poisson process at claim instants and conditioning on the amount of the first claim. In order to prove our theorem, we need the following results stated here as a lemma. For $t \geq 0$, define

$$h(t) = u(e^{R_t} - 1) + c\int_0^t e^{R_s}\mathrm{d}s$$

and

$$\widetilde{h}(t) = u(e^{R_t} - 1) + c\int_0^t e^{R_s}\mathrm{d}s + \sigma_p\int_0^t e^{R_t - R_s}\mathrm{d}W_s.$$

Lemma 1 *For any $t \geq 0$,*

$$E[\widetilde{h}(t)] = E[h(t)] = [e^{(\delta+\frac{1}{2}\sigma^2)t} - 1](u + \frac{c}{\delta + \frac{\sigma^2}{2}}), \tag{3}$$

$$E[\widetilde{h}^2(t)] = E[h^2(t)] + \sigma_p^2 \cdot \frac{e^{(\delta+\sigma^2)t} - 1}{2(\delta+\sigma^2)}, \tag{4}$$

$$E[\int_0^t e^{R_t - R_s} \mathrm{d}W_s]^3 = 0, \tag{5}$$

$$\lim_{t\to 0} \frac{E[h(t)(\int_0^t e^{R_t - R_s} \mathrm{d}W_s)^2]}{t} = 0. \tag{6}$$

Proof It is easy to check that equation (3) holds.

By Ito isometry, we have

$$E[\int_0^t e^{R_t - R_s} \mathrm{d}W_s]^2 = E[\int_0^t e^{2(R_t - R_s)} \mathrm{d}s] = \frac{e^{2(\delta+\sigma^2)t} - 1}{2(\delta+\sigma^2)}.$$

Therefore,

$$\begin{aligned} E[\widetilde{h}^2(t)] &= E[h^2(t) + \sigma_p^2 (\int_0^t e^{R_t - R_s} \mathrm{d}W_s)^2 + 2h(t)\sigma_p \int_0^t e^{R_t - R_s} \mathrm{d}W_s] \\ &= E[h^2(t)] + \sigma_p^2 \cdot \frac{e^{(\delta+\sigma^2)t} - 1}{2(\delta+\sigma^2)}, \end{aligned}$$

which implies that equation (4) holds.

Let $M_t^{(1)} = (\int_0^t e^{R_t - R_s} \mathrm{d}W_s)^3$, $Y_t^{(1)} = e^{3R_t}$ and $Z_t^{(1)} = \int_0^t e^{-R_s} \mathrm{d}W_s$. By Ito's formula, we obtain

$$\mathrm{d}M_t^{(1)} = 3M_t^{(1)}[(\delta + \frac{3}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}B_t] + 3Y_t^{(1)}(Z_t^{(1)})^2 e^{-R_t} \mathrm{d}W_t + 3Y_t^{(1)} Z_t^{(1)} e^{-2R_t} \mathrm{d}t.$$

Integrating the above equation and noting that

$$\int_0^t M_u^{(1)} \sigma \mathrm{d}B_u,\ \int_0^t Y_u^{(1)}(Z_u^{(1)})^2 e^{-R_u} \mathrm{d}W_u,\ Y_u^{(1)} Z_u^{(1)} e^{-2R_u} = e^{R_u} \int_0^u e^{-R_s} \mathrm{d}W_s$$

are martingales with zero-expectation, we get

$$E[M_t^{(1)}] - M_0^{(1)} = 3\int_0^t E[M_u^{(1)}(\delta + \frac{3}{2}\sigma^2)]\mathrm{d}u.$$

Denoting $E[M_t^{(1)}] = g(t)$, the above equation can be re-expressed as

$$g(t) - M_0^{(1)} = 3(\delta + \frac{3}{2}\sigma^2)\int_0^t g(u)\mathrm{d}u$$

with solution $g(t) = C_1 e^{(3\delta + \frac{9}{2}\sigma^2)t}$. Since $g(0) = C_1 = M_0^{(1)} = 0$, we get $g(t) \equiv 0$, which implies that equation (5) holds.

Let $Y_t^{(2)} = h(t)e^{2R_t}$, $Z_t^{(2)} = (\int_0^t e^{-R_s} \mathrm{d}W_s)^2$ and $M_t^{(2)} = Y_t^{(2)} Z_t^{(2)}$. By Ito's formula, we have

$$\begin{aligned} \mathrm{d}M_t^{(2)} =& Y_t^{(2)}(2\int_0^t e^{-R_t - R_s} \mathrm{d}W_s)\mathrm{d}W_t + Y_t^{(2)} e^{-2R_t} \mathrm{d}t + Z_t^{(2)} e^{2R_t} \sigma[2h(t) + ue^{R_t}]\mathrm{d}B_t + \\ & Z_t^{(2)}[2h(t)\delta + 2h(t)\sigma^2 + e^{R_t}(\delta u + \frac{5}{2}u\sigma^2 + c)]\mathrm{d}t. \end{aligned}$$

Integrating the above equation and taking expectations on both sides, we have

$$E[M_t^{(2)}] = \int_0^t E\big[Y_s^{(2)} e^{-2R_s}\big]\mathrm{d}s + E\big\{\int_0^t Z_s^{(2)}[2h(s)\delta + 2h(s)\sigma^2 + e^{R_s}(\delta u + \frac{5}{2}u\sigma^2 + c)]\mathrm{d}s\big\}.$$

Hence, from L'Hospital rule, we get

$$\begin{aligned}\lim_{t\to 0}\frac{E[M_t^{(2)}]}{t} &= \lim_{t\to 0}\big\{E\big[Y_t^{(2)}e^{-2R_t}\big]+E\big[Z_t^{(2)}h(t)(2\delta+2\sigma^2)\big]+E\big[Z_t^{(2)}e^{R_t}(\delta u+\frac{5}{2}u\sigma^2+c)\big]\big\}\\ &\triangleq \lim_{t\to 0}\big[I_1(t)+I_2(t)+I_3(t)\big]. \end{aligned}\tag{7}$$

From equation (3), we get

$$\lim_{t\to 0} I_1(t)=\lim_{t\to 0}\big[e^{(\delta+\frac{1}{2}\sigma^2)t}-1\big](u+\frac{c}{\delta+\frac{\sigma^2}{2}})=0. \tag{8}$$

By Ito's formula again, we have

$$\mathrm{d}[Z_t^{(2)}h(t)]=[Z_t^{(2)}e^{R_t}(\delta u+\frac{1}{2}u\sigma^2+c)+h(t)e^{-2R_t}]\mathrm{d}t+Z_t^{(2)}e^{R_t}u\sigma\mathrm{d}B_t+2h(t)(\int_0^t e^{-R_t-R_s}\mathrm{d}W_s)\mathrm{d}W_t.$$

Integrating the above equation and taking expectations on both sides one more time, we can write

$$E\big[Z_t^{(2)}h(t)\big]=\int_0^t E\big[Z_s^{(2)}e^{R_s}(\delta u+\frac{1}{2}u\sigma^2+c)+h(s)e^{-2R_s}\big]\mathrm{d}s.$$

Notice that the integrand on the right-hand side of the above equation is continuous and bounded in $(0,t]$, which implies that the integral is also continuous. Therefore,

$$\lim_{t\to 0} I_2(t)=\lim_{t\to 0} E\big[Z_t^{(2)}h(t)(2\delta+2\sigma^2)\big]=0. \tag{9}$$

Similarly, we can obtain

$$\lim_{t\to 0} I_3(t)=\lim_{t\to 0} E\big[Z_t^{(2)}e^{R_t}(\delta u+\frac{5}{2}u\sigma^2+c)\big]=0. \tag{10}$$

Hence, equation (6) follows from (7)–(10).

We can now give equations satisfied by $\Phi(u)$.

Theorem 1 *Consider risk process (1) and the expected discounted penalty function $\Phi(u)$ defined in (2). If F is a continuous distribution and $w(x,y)$ is bounded for $x\geq 0,\ y\geq 0$, then $\Phi(u)$ satisfies the integro-differential equation*

$$\begin{aligned}&\frac{1}{2}({\sigma_p}^2+\sigma^2u^2)\Phi''(u)+(\delta u+\frac{1}{2}\sigma^2u+c)\Phi'(u)-(\lambda+\rho)\Phi(u)+\\ &\lambda\int_0^u\Phi(u-y)\mathrm{d}F(y)+\lambda\int_u^\infty w(u,y-u)\mathrm{d}F(y)=0\end{aligned}\tag{11}$$

and the following boundary conditions:

$$\Phi(\infty)=0, \tag{12}$$

$$\Phi(0)=w(0,0). \tag{13}$$

Proof Using similar arguments to Gerber and Shiu [3] and Willmot and Dickson [15], we condition on the amount, y, of the first claim in a very short interval $(0,\Delta t]$. We note that if there is no claim in $(0,\Delta t]$, then $X(\Delta t)=\widetilde{h}(\Delta t)+u$; if there is one claim in $(0,\Delta t]$, then ruin occurs when $y\geq\widetilde{h}(\Delta t)+u$. Thus,

$$\Phi(u)=(1-\lambda\Delta t)e^{-\rho\Delta t}E\big[\Phi(\widetilde{h}(\Delta t)+u)\big]+\lambda\Delta t e^{-\rho\Delta t}E\big[\int_0^{u+\widetilde{h}(\Delta t)}\Phi(\widetilde{h}(\Delta t)+u-y)\mathrm{d}F(y)\big]+$$

$$\lambda\Delta t e^{-\rho\Delta t} E\big[\int_{u+\widetilde{h}(\Delta t)}^{\infty} w(\widetilde{h}(\Delta t)+u, y-\widetilde{h}(\Delta t)-u)\mathrm{d}F(y)\big] + o(\Delta t). \tag{14}$$

By Taylor's expansion, we have

$$E[\Phi(\widetilde{h}(t)+u)] = \Phi(u) + \Phi'(u)E[\widetilde{h}(t)] + \frac{1}{2}\Phi''(u)E[\widetilde{h}(t)]^2 + \frac{1}{6}E[\Phi'''(\widetilde{u})\widetilde{h}^3(t)],$$

where $u \le \widetilde{u} \le u + \widetilde{h}(t)$.

In view of equations (3), (4) and (5), we can write

$$\begin{aligned} E[\Phi(\widetilde{h}(t)+u)] =& \Phi(u) + \Phi'(u)E[h(t)] + \frac{1}{2}\Phi''(u)E\big[h^2(t) + \sigma_p^2 \frac{e^{2(\delta+\sigma^2)t}-1}{2(\delta+\sigma^2)}\big] + \\ & \frac{1}{6}\Phi'''(\widetilde{u})E\big[h^3(t) + 3h(t)\sigma_p^2(\int_0^t e^{R_t-R_s}\mathrm{d}W_s)^2\big]. \end{aligned}$$

Substituting the above equation into (14) gives

$$\begin{aligned} \Phi(u) =& \Phi(u) + \Phi'(u)E[h(\Delta t)] + \frac{1}{2}\Phi''(u)E\big[h^2(\Delta t) + \sigma_p^2 \cdot \frac{e^{2(\delta+\sigma^2)\Delta t}-1}{2(\delta+\sigma^2)}\big] + \\ & \frac{1}{6}\Phi'''(\widetilde{u})E\big[h^3(\Delta t) + 3h(\Delta t)\sigma_p^2(\int_0^{\Delta t} e^{R_{\Delta t}-R_s}\mathrm{d}W_s)^2\big] - (\lambda+\rho)\Delta t\cdot\Phi(u) + \\ & \lambda\Delta t e^{-\rho\Delta t}E\big[\int_{u+\widetilde{h}(\Delta t)}^{\infty} w(u+\widetilde{h}(\Delta t), y-u-\widetilde{h}(\Delta t))\mathrm{d}F(y)\big] + \\ & \lambda\Delta t e^{-\rho\Delta t}E\big[\int_0^{u+\widetilde{h}(\Delta t)} \Phi(u+\widetilde{h}(\Delta t)-y)\mathrm{d}F(y)\big] + o(\triangle t). \end{aligned} \tag{15}$$

Results in the appendix of [10] have shown that

$$\lim_{t\to 0}\frac{E[h(t)]}{t} = u(\delta + \frac{\sigma^2}{2}) + c, \quad \lim_{t\to 0}\frac{E[h^2(t)]}{t} = \sigma^2 u^2, \quad \lim_{t\to 0}\frac{E[h^3(t)]}{t} = 0.$$

Thus, by dividing Δt on both sides of equation (15), letting $\Delta t \to 0$, and using the above results, we obtain equation (11).

The boundary condition (12) holds from the boundedness of w, and the second boundary condition (13) follows from the fact that zero initial surplus corresponds to the bankruptcy state.

We remark that Theorem 4.1 in [10] is a special case of above results with $\sigma_p = 0$ and $\sigma > 0$. And our results coincide with the results of combination of Theorems 2.3 and 2.6 of Wang and Wu [9] when $\sigma_p > 0$ and $\sigma = 0$.

3. General solutions for $\Phi(u)$

In this section, we give general solutions for $\Phi(u)$. We first give an identical equation of equation (11). Based on this, the closed form expression for $\Phi(u)$ is obtained.

Theorem 2 *The integro-differential equation*

$$\begin{aligned} & \frac{1}{2}({\sigma_p}^2 + \sigma^2u^2)\Phi''(u) + (\delta u + \frac{1}{2}\sigma^2 u + c)\Phi'(u) - (\lambda+\rho)\Phi(u) + \\ & \lambda\int_0^u \Phi(u-y)\mathrm{d}F(y) + \lambda\int_u^{\infty} w(u, y-u)\mathrm{d}F(y) = 0 \end{aligned} \tag{16}$$

is identical to the Volterra equation

$$\Phi(u)=\int_0^u \Phi(y)K(u,y)\mathrm{d}y+g(u), \tag{17}$$

where

$$g(u)=\frac{1}{\sigma_p^2+\sigma^2u^2}\Big[\sigma_p^2\Phi(0)+\sigma_p^2\Phi'(0)u+2\Phi(0)cu-2\lambda\int_0^u\int_0^y\int_x^\infty w(x,z-x)\mathrm{d}F(z)\mathrm{d}x\mathrm{d}y\Big],$$

$$K(u,y)=\frac{2}{\sigma_p^2+\sigma^2u^2}\Big[-y(-2\sigma^2+2\delta+\lambda+\rho)-u(\frac{1}{2}\sigma^2-\delta-\lambda-\rho)-c-\lambda\int_0^{u-y}F(t)\mathrm{d}t\Big].$$

Furthermore, the solution of equation (16) is unique if F is a continuous distribution.

Proof Integrating equation (16) over u twice, integrating by parts and interchanging the order of integration, we have

$$\frac{1}{2}(\sigma_p^2+\sigma^2u^2)\Phi(u)-\frac{1}{2}\sigma_p^2[\Phi(0)+\Phi'(0)u]-\Phi(0)cu+\int_0^u\Phi(y)[y(-2\sigma^2+2\delta+\lambda+\rho)+$$
$$u(\frac{1}{2}\sigma^2-\delta-\lambda-\rho)+c]\mathrm{d}y+\lambda\int_0^u\Phi(y)\int_0^{u-y}F(t)\mathrm{d}t\mathrm{d}y+$$
$$\lambda\int_0^u\int_0^y\int_x^\infty w(x,z-x)\mathrm{d}F(z)\mathrm{d}x\mathrm{d}y=0,$$

which implies that equation (17) holds.

Since equation (16) is identical to (17), we just need to show the uniqueness of the solution to equation (17). And, according to the theory of Volterra equation, this statement holds since $g(u)$ is absolutely integrable and $K(u,y)$ is bounded integrable, which comes from the continuity of F and the boundedness of w. □

From Theorem 4 and the theory of Volterra equation [16], we can get the closed-form solution of equation (16)

$$\Phi(u)=g(u)+\int_0^u\sum_{n=1}^\infty K_n(u,t)g(t)\mathrm{d}t,$$

where

$$K_n(u,t)=\int_t^u K(u,y)K_{n-1}(y,t)\mathrm{d}y,\quad u\geq t,\ n=2,3,\ldots$$
$$K_1(u,t)=K(u,t).$$

Furthermore, $\Phi(u)$ can be approximated recursively by Picard's sequence defined by

$$\Phi_0(u)=g(u),$$
$$\Phi_n(u)=g(u)+\int_0^u\Phi_{n-1}(s)g(s)\mathrm{d}s,\qquad n=1,2,3,\ldots\ .$$

4. The exact solution for $\Phi'(0)$ under constant interest

Section 3 gives the expression and recursive formula for $\Phi(u)$. But, in order to apply these results, we need to know the solution for $\Phi'(0)$. This section will devote to find $\Phi'(0)$ when

$R_t = \delta t,\ \delta > 0$, by using the Laplace transform technique of Sundt and Teugels [13] and Cai and Dickson [14] and conditions given by Theorem 2.

We define an auxiliary function of $\Phi(x)$ as $Z(x) = \Phi(0) - \Phi(x)$. Then $Z(0) = 0$. If the claim size distribution F is sufficiently regular and w is bounded, we have $\Phi(x) \to 0$ as $x \to \infty$ according to the monotone convergence theorem, which implies that $\lim_{x\to\infty} Z(x) = \Phi(0)$. In this case, we can define $\widetilde{Z}(s) = \int_0^\infty e^{-sx}\mathrm{d}Z(x)$, with $\widetilde{Z}(0) = \Phi(0) = w(0,0) < \infty$.

By substituting $\Phi(x) = \Phi(0) - Z(x)$ into equation (11), we have that

$$-\frac{1}{2}\sigma_p^2 Z''(x) - (c+\delta x)Z'(x) + (\lambda+\rho)Z(x) - \lambda\int_0^x Z(x-y)\mathrm{d}F(y) + \Pi(x) = 0, \tag{18}$$

where

$$\Pi(x) = -(\rho+\lambda)\Phi(0) + \lambda\Phi(0)F(x) + \lambda\int_x^\infty w(x, y-x)\mathrm{d}F(y).$$

Taking the Laplace transform of equation (18) yields

$$-\delta\widetilde{Z}'(s) + \widetilde{Z}(s)(L(s) - \frac{\rho}{s}) = \frac{1}{2}\sigma_p^2\widetilde{Z}'(0) + \widetilde{\Pi}(s),$$

where

$$L(s) = \frac{1}{2}\sigma_p^2 s + c - \lambda\int_0^\infty e^{-sx}(1-F(x))\mathrm{d}x,$$

$$\widetilde{\Pi}(s) = \int_0^\infty e^{-sx}\Pi(x)\mathrm{d}x. \tag{19}$$

When $\delta > 0$, we note that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[\widetilde{Z}(s)s^{\frac{\rho}{\delta}}e^{\frac{-1}{\delta}\int_0^s L(t)\mathrm{d}t}\Big] = -\frac{1}{\delta}e^{\frac{-1}{\delta}\int_0^s L(t)\mathrm{d}t}s^{\frac{\rho}{\delta}}\Big[\frac{1}{2}\sigma_p^2 Z'(0) + \widetilde{\Pi}(s)\Big]$$

and by monotone convergence theorem we have that

$$\widetilde{Z}(s)s^{\frac{\rho}{\delta}} = \int_0^\infty e^{-sx}s^{\frac{\rho}{\delta}}\mathrm{d}Z(x) \to 0, \quad s \to \infty,$$

which implies

$$\widetilde{Z}(s)s^{\frac{\rho}{\delta}}e^{\frac{-1}{\delta}\int_0^s L(t)\mathrm{d}t} = \frac{1}{\delta}\int_s^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\Big[\frac{1}{2}\sigma_p^2 Z'(0) + \widetilde{\Pi}(t)\Big]\mathrm{d}t.$$

Put $s = 0$ in the above equation to obtain

$$\frac{1}{2}\sigma_p^2 Z'(0)\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\mathrm{d}t = -\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\widetilde{\Pi}(t)\mathrm{d}t,$$

or, equivalently,

$$Z'(0) = \frac{-2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\widetilde{\Pi}(t)\mathrm{d}t}{\sigma_p^2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\mathrm{d}t}.$$

From the fact that $\Phi'(0) = -Z'(0)$, we obtain

$$\Phi'(0) = \frac{2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\widetilde{\Pi}(t)\mathrm{d}t}{\sigma_p^2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\mathrm{d}t}. \tag{20}$$

Example Under conditions of Theorem 2, if F is an exponential distribution with $F(y) = 1 - e^{-\beta y}$, $\beta > 0$, equation (11) is re-expressed as

$$\frac{1}{2}({\sigma_p}^2 + \sigma^2 u^2)\Phi''(u) + (\delta u + \frac{1}{2}\sigma^2 u + c)\Phi'(u) - (\lambda + \rho)\Phi(u) + \lambda A(u) + \lambda\beta e^{-\beta u}\int_0^u \Phi(y)e^{\beta y}\mathrm{d}y = 0, \tag{21}$$

where $A(u) = \int_u^\infty w(u, y-u)\mathrm{d}F(y)$.

Taking the derivative with respect to u on both sides of equation (21), we get

$$\frac{1}{2}({\sigma_p}^2 + \sigma^2 u^2)\Phi'''(u) + \Phi''(u)\sigma^2 u + \Phi''(u)(\delta u + \frac{1}{2}\sigma^2 u + c) + \Phi'(u)(\delta + \frac{1}{2}\sigma^2) - (\lambda + \rho)\Phi'(u) + \lambda A'(u) - \lambda\beta^2 e^{-\beta u}\int_0^u \Phi(y)e^{\beta y}\mathrm{d}y + \lambda\beta\Phi(u) = 0. \tag{22}$$

Multiplying equation (21) by β and adding to equation (22), we can get the differential equation satisfied by $\Phi(u)$

$$\frac{1}{2}({\sigma_p}^2 + \sigma^2 u^2)\Phi'''(u) + (\frac{1}{2}\sigma_p^2\beta + \frac{1}{2}\sigma^2 u^2\beta + \delta u + \frac{3}{2}\sigma^2 u + c)\Phi''(u) + (\beta\delta u + \frac{1}{2}\beta\sigma^2 u + c\beta + \delta + \frac{1}{2}\sigma^2 - \lambda - \rho)\Phi'(u) - (\lambda + \rho)\beta\Phi(u) + \lambda\beta A(u) + \lambda A'(u) + \lambda\beta\Phi(u) = 0.$$

From equation (19) we get $\widetilde{\Pi}(s) = -\frac{\rho}{s}$. Thus according to equation (20), we have

$$\Phi'(0) = -\frac{2\rho\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}-1}\mathrm{d}t}{\sigma_p^2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\mathrm{d}t}.$$

Furthermore, from equation (21) we have

$$\Phi''(0) = \frac{2(\rho - c\Phi'(0))}{\sigma_p^2}.$$

Paulsen and Gjessing [17] and Cai [10] also considered the case for exponential claim distribution. Results in this example coincides with equation (2.11) when $\delta = r - \frac{1}{2}\sigma^2$ in [17]. Also, Example 5.1 in [10] is a special case of our example by taking $\sigma_p = 0$.

5. Applications

In this section, we give two examples to illustrate applications of the derived results.

When interest is constant, the equation satisfied by $\Phi(u)$ is

$$\Phi(u) = \int_0^u \Phi(y)K(u,y)\mathrm{d}y + g(u), \tag{23}$$

where

$$g(u) = w(0,0) + u\Phi'(0) + \frac{1}{\sigma_p^2}\Big[2\ w(0,0)\ c\ u - 2\lambda\int_0^u\int_0^y\int_x^\infty w(x, z-x)\mathrm{d}F(z)\mathrm{d}x\mathrm{d}y\Big],$$

$$K(u,y) = \frac{2}{\sigma_p^2}\Big[y(-2\delta - \lambda) + u(\delta + \lambda) - c - \lambda\int_0^{u-y} F(t)\mathrm{d}t + \rho(u-y)\Big],$$

$$\Phi'(0) = \frac{2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m} t^{\frac{\rho}{\delta}} \widetilde{\Pi}(t)\mathrm{d}t}{\sigma_p^2 \int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m} t^{\frac{\rho}{\delta}}\mathrm{d}t}.$$

5.1 Ruin probability

Denote the ruin probability with initial surplus u as $\Pr(\tau < \infty) = \psi(u)$. Let $w(x, y) = 1$ and $\rho = 0$. We get $\Phi(u) = E[I(\tau < \infty)] = \psi(u)$. From Section 4, we have $\widetilde{\Pi}(s) = 0$, $\Phi'(0) = 0$. Then, equation (23) yields that

$$\psi(u) = \int_0^u \psi(y) K^{(1)}(u, y)\mathrm{d}y + g^{(1)}(u),$$

where

$$g^{(1)}(u) = 1 + \frac{1}{\sigma_p^2}\Big[2 \cdot c \cdot u - 2\lambda \int_0^u \int_0^y 1 - F(x)\mathrm{d}x\mathrm{d}y\Big],$$

$$K^{(1)}(u, y) = \frac{2}{\sigma_p^2}\Big[y(-2\delta - \lambda) + u(\delta + \lambda) - c - \lambda \int_0^{u-y} F(t)\mathrm{d}t\Big].$$

5.2 Moments of the time of ruin

In the following, we regard ρ as a variable and denote $\Phi(u)$, $K(u,\ y)$, $g(u)$ in Section 3 as $\Phi(u, \rho)$, $K(u, y, \rho)$ and $g(u, \rho)$, respectively. We assume that these functions are infinitely differentiable about ρ.

Denote the m-th order moment of the time of ruin as

$$E\big[\tau^m\ I(\tau < \infty)\big] = (-1)^m R_m(u), \quad m = 1, 2, \ldots .$$

Let $w(x, y) = 1$. We get

$$\Phi(u, \rho) = E\big[e^{-\rho\tau} I(\tau < \infty)\big]$$

and

$$R_m(u) = \frac{\partial^m \Phi(u, \rho)}{\partial \rho^m}\Big|_{\rho=0}.$$

Taking derivatives with respect to ρ on $K(u, y, \rho)$ for m times, $m = 1, 2, 3, \ldots$, we have

$$\frac{\partial K(u, y, \rho)}{\partial \rho} = \frac{2}{{\sigma_p}^2}(u - y),$$

$$\frac{\partial^m K(u, y, \rho)}{\partial \rho^m} = 0, \quad m = 2, 3, 4, \ldots .$$

Then, equation (23) yields that

$$\frac{\partial \Phi(u, \rho)}{\partial \rho} = \int_0^u \frac{\partial \Phi(y, \rho)}{\partial \rho} K(u, y, \rho)\mathrm{d}y + \int_0^u \Phi(u, \rho)\frac{\partial K(u, y, \rho)}{\partial \rho}\mathrm{d}y + \frac{\partial g(u, \rho)}{\partial \rho}, \tag{24}$$

$$\begin{aligned}\frac{\partial^m \Phi(u, \rho)}{\partial \rho^m} = &\int_0^u \frac{\partial^m \Phi(y, \rho)}{\partial \rho^m} K(u, y, \rho)\mathrm{d}y + m\int_0^u \frac{\partial^{m-1}\Phi(y, \rho)}{\partial \rho^{m-1}}\frac{\partial K(u, y, \rho)}{\partial \rho}\mathrm{d}y +\\ &\frac{\partial^m g(u, \rho)}{\partial \rho^m}, \quad m = 2, 3, \ldots .\end{aligned} \tag{25}$$

From equation (19), we have $\widetilde{\Pi}(t)=\frac{-\rho}{t}$. Letting $\rho\to 0+$ in equation (24) and (25), we obtain the recursive equation satisfied by $R_m(u)$

$$R_1(u)=\int_0^u R_1(y)K(u,y)\mathrm{d}y+\frac{2}{{\sigma_p}^2}\int_0^u \Phi(y,0)(u-y)\mathrm{d}y-\frac{2u\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}\frac{1}{t}\mathrm{d}t}{{\sigma_p}^2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}\mathrm{d}t},$$

$$R_m(u)=\int_0^u R_m(y)K^{(2)}(u,y)\mathrm{d}y+\frac{2m}{{\sigma_p}^2}\int_0^u R_{m-1}(y)(u-y)\mathrm{d}y+g^{(2)}(u),\quad m\geq 2,$$

where

$$g^{(2)}(u)=u\cdot\frac{\partial^m}{\partial\rho^m}\Big(\frac{-2\rho\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}-1}\mathrm{d}t}{\sigma_p^2\int_0^\infty e^{\frac{-1}{\delta}\int_0^t L(m)\mathrm{d}m}t^{\frac{\rho}{\delta}}\mathrm{d}t}\Big)\Big|_{\rho=0},$$

$$K^{(2)}(u,y)=\frac{2}{\sigma_p^2}\Big[y(-2\delta-\lambda)+u(\delta+\lambda)-c-\lambda\int_0^{u-y}F(t)\mathrm{d}t\Big].$$

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