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Existence Theorem of Solutions for Mixed Variational Inequality in Banach Spaces

Qing Bang ZHANG[1,*], **Liu LIU**[2]
1. *College of Economic Mathematics, Southwestern University of Finance and Economics, Sichuan* 611130*, P. R. China;*
2. *Department of Mathematics, Sichuan University of Arts and Science, Sichuan* 635000*, P. R. China*

Abstract By using the property of generalized f-projection operator and FKKM theorem, the existence theorem of solutions for the mixed variational inequality is proved under weaker assumption in reflexive and smooth Banach space. The results improve and extend the corresponding results shown recently.

Keywords mixed variational inequality; generalized f-projection operator; existence theorem; FKKM theorem.

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1. Introduction

The metric projection operators in Hilbert and Banach space have been used in many different areas of mathematics, such as functional and numerical analysis, optimization theory, fixed point theorem, nonlinear programming, variational inequality and complementarity problem, etc. [1–7, 10].

Recently, Wu and Huang [6, 7] extended the definition of the generalized projection operators introduced by Alber [1] and Li [4], and introduced and studied a new class of generalized f-projection operator in Banach space. In [6], they studied the existence of the solution for the following mixed variational inequality MVI(T, K) in Banach space: Find $x \in K$ such that

$$\langle Tx, y - x\rangle + f(y) - f(x) \geq 0, \quad \forall y \in K, \tag{1.1}$$

which was introduced and studied in Hilbert space by Noor [8], and it was proved that:

Theorem 1 ([6]) *Let K be a nonempty closed convex subset of a reflexive and smooth Banach space X with dual space X^*. Let mappings $T : K \to X^*$ be continuous and $f : K \subset X \to R \cup \{+\infty\}$ be proper, convex and lower semi-continuous. If there exists an element $y_0 \in K$ such*

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* Corresponding author
E-mail address: zhangqingbang@126.com (Q. B. ZHANG)

that

$$\{x \in K : 2\langle Jy_0 - \frac{1}{2}Ty_0), y_0 - x\rangle + \|x\|^2 + f(x) \le \|y_0\|^2 + f(y_0)$$

is a compact subset of K. Then Problem (1.1) has a solution.

In this paper, by using the property of the generalized f-projection operator and FKKM theorem [9], we consider the existence of the solution for the following mixed variational inequality MVI$(T-\xi, K)$: Find $x \in K$ such that

$$\langle Tx - \xi, y - x\rangle + f(y) - f(x) \ge 0, \ \forall y \in K.$$

Our conclusions improve and extend the above theorem 1 and the corresponding results in [5] and [4].

2. Preliminaries

Throughout the paper, let X be a real Banach space with dual space X^* and K be a nonempty closed and convex subset of a real Banach space X, and 2^X stand for the family of all the nonempty subsets of X. Let $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, R be the field of real numbers.

For any fixed $\rho > 0$, let the functional $G : X^* \times K \to R \cup \{+\infty\}$ be defined by

$$G(\varphi, x) = \|\varphi\|^2 - 2\langle\varphi, x\rangle + \|x\|^2 + 2\rho f(x),$$

where $\varphi \in X^*$, $x \in X$ and $f : K \subset X \to R \cup \{+\infty\}$ is proper, convex, lower semi-continuous. It is easy to have the properties of G as follows [7]:

(i) $(\|\varphi\| - \|x\|)^2 + 2\rho f(x) \le G(\varphi, x) \le (\|\varphi\| + \|x\|)^2 + 2\rho f(x)$;

(ii) $G(\varphi, x)$ is convex and continuous with respect to φ when x is fixed;

(iii) $G(\varphi, x)$ is convex and lower semi-continuous with respect to x when φ is fixed.

Definition 1 ([6]) *Let X be a Banach space with dual space X^* and K be a nonempty closed convex subset of X. The operator $\pi_K^f : X^* \to 2^K$ is called the generalized f-projection operator if*

$$\pi_K^f\varphi = \{u \in K : G(\varphi, u) = \inf_{y\in K} G(\varphi, y)\}, \quad \forall\varphi \in X^*.$$

Note that if $f(x) = 0$ for all $x \in K$, the generalized f-projection operator reduces to the generalized projection operator defined by Alber [1] and Li [4].

Lemma 1 ([7]) *If X is a reflexive Banach space with dual space X^* and K is a nonempty closed convex subset of X, then the following conclusions hold:*

(i) For any given $\varphi \in X^*$, $\pi_K^f\varphi$ is a nonempty, closed and convex subset of K;

(ii) If X is smooth, then for any given $\varphi \in X^*$, $x \in \pi_K^f\varphi$ if and only if

$$\langle\varphi - Jx, x - y\rangle + \rho f(y) - \rho f(x) \ge 0, \quad \forall\ y \in K;$$

(iii) If $f : K \to R \cup \{+\infty\}$ is positively homogeneous, i.e., $f(tx) = tf(x)$ for all $t > 0$ and $x \in K$ with $tx \in K$, and X is strictly convex, then the operator $\pi_K^f : X^* \to 2^K$ is single-valued.

Lemma 2 ([7]) *Let X be a reflexive strictly convex Banach space with dual space X^* and K*

be a nonempty compact convex subset of X. If $f : K \to R \cup \{+\infty\}$ is proper, convex, lower semi-continuous and positively homogeneous, then $\pi_K^f : X^* \to K$ is continuous.

3. Existence theorem

Proposition 1 *Let K be a nonempty closed convex subset of a reflexive and smooth Banach space X with dual space X^* and $\rho > 0$ be any given constant. If $f : K \to R \cup \{+\infty\}$ is proper, convex and lower semi-continuous, then the following states are equivalent:*

(i) x is a solution of $\mathrm{MVI}(T - \xi, K)$;

(ii) $x \in \pi_K^f(Jx - \rho(Tx - \xi))$;

(iii) $G(Jx - \rho(Tx - \xi), x) \leq G(Jx - \rho(Tx - \xi), y)$, $\forall y \in K$.

Proof For any given positive number $\rho > 0$, by Lemma 1(ii) and Definition 1, it is easy to show that the proposition holds. □

Proposition 2 *Let K be a nonempty closed convex subset of a reflexive and smooth Banach space X with dual space X^*. Let mapping $f : K \subset X \to R \cup \{+\infty\}$ be proper, convex, lower semi-continuous. For any given $\xi \in X^*$ and constant $\rho > 0$, suppose that there exists a nonempty weakly compact subset D of K and $y_0 \in K$ such that $\langle Jy_0 - \rho(T(y_0) - \xi), y_0 - x\rangle > \frac{1}{2}(\|y_0\|^2 - \|x\|^2) + \rho(f(y_0) - f(x))$, for all $x \in K \setminus D$. Then there exists $\bar{x} \in D \subset K$ such that*

$$G(Jy - \rho(T(y) - \xi), \bar{x}) \leq G(Jy - \rho(T(y) - \xi), y), \quad \forall y \in K.$$

Proof Define a multi-valued mapping $F : K \to 2^K$ by

$$F(y) = \{x \in K : G(Jy - \rho(T(y) - \xi), x) \leq G(Jy - \rho(T(y) - \xi), y)\}, \text{ for all } y \in K.$$

Then $F(y)$ is nonempty, since $y \in F(y)$ for each $y \in K$. For each fixed $y \in K$, from the property of G, it follows that the mapping $x \mapsto G(Jy - \rho(Ty - \xi), x)$ is lower semi-continuous and convex. Therefore for all $y \in K$, the set $F(y)$ is closed and convex, and $F(y)$ is weakly closed and convex.

Next, we shall show that the mapping $F : K \to 2^K$ is a KKM mapping in K. Indeed, suppose that $N = \{y_1, y_2, \ldots, y_n\}$ is an arbitrary finite subset in K, then, for any $v \in coN$, there exist $\{\lambda_i\}_{i=1}^n$ satisfying $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^n \lambda_i = 1$ such that $v = \sum_{i=1}^n \lambda_i y_i$. Hence, for any $y_j \in N$, we have

$$\begin{aligned} G(Jy_j - \rho(T(y_j) - \xi), v) &= G(Jy_j - \rho(T(y_j) - \xi), \sum_{i=1}^n \lambda_i y_i) \\ &\leq \sum_{i=1}^n \lambda_i G(Jy_j - \rho(T(y_j) - \xi), y_i) \leq \max_{1 \leq i \leq n} G(Jy_j - \rho(T(y_j) - \xi), y_i). \end{aligned}$$

Hence there is $y_{j_0} \in N$ such that $G(Jy_j - \rho(T(y_j) - \xi), v) \leq G(Jy_j - \rho(T(y_j) - \xi), y_{j_0})$, for all $j = 1, 2, \ldots, n$. In special,

$$G(Jy_{j_0} - \rho(T(y_{j_0}) - \xi), v) \leq G(Jy_{j_0} - \rho(T(y_{j_0}) - \xi), y_{j_0}),$$

i.e., $v \in F(y_{j_0}) \subset \bigcup_{j=1}^n F(y_j)$.

From the assumption, it follows that there is $y_0 \in K$ such that $F(y_0) \subset D \subset K$. Indeed, for all $x \in K \setminus D$, we have that

$$\begin{aligned}&\|Jy_0 - \rho(Ty_0 - \xi)\|^2 - 2\langle Jy_0 - \rho(Ty_0 - \xi), x\rangle + \|x\|^2 + 2\rho f(x)\\ &\quad > \|Jy_0 - \rho(Ty_0 - \xi)\|^2 - 2\langle Jy_0 - \rho(Ty_0 - \xi), y_0\rangle + \|y_0\|^2 + 2\rho f(y_0).\end{aligned}$$

By the definition of G, we have $x \notin F(y_0)$.

Since $F(y_0)$ is weakly closed and D is weakly compact, $F(y_0)$ is also weakly compact. Therefore, it follows from FKKM Theorem [9] that $\bigcap_{y\in K} F(y) \neq \emptyset$, i.e., there exists $\bar{x} \in D \subset K$ such that

$$G(Jy - \rho(T(y) - \xi), \bar{x}) \leq G(Jy - \rho(T(y) - \xi), y), \quad \forall y \in K.$$

This completes the proof. □

Proposition 3 *Let X be a reflexive smooth Banach space with the dual space X^* and K be a nonempty convex subset of X. Let $\rho > 0$ be any given constant and $T : K \to X^*$ be continuous from the line segments in K to the weak* topology of X^*. Let*

$$A = \{x \in K : G(Jx - \rho(Tx - \xi), x) \leq G(Jx - \rho(Tx - \xi), y)\}.$$

Then for each fixed $y \in K$, the intersection of A with any line segment is closed in K.

Proof For $x_1, x_2 \in K$, let $[x_1, x_2]$ denote the line segment $[x_1, x_2] = \{tx_1 + (1-t)x_2 : t \in [0, 1]\}$. Let $\{x_n\} \subset A \cap [x_1, x_2]$ such that $x_n \to x_0 \in [x_1, x_2]$. Since $T : K \to X^*$ is continuous from the line segments in K to the weak* topology of X^*, $\{Tx_n\}$ converges to Tx_0 in the weak* topology. Since J is continuous from the strong topology of X to the weak* topology of X^* in $K \subset X$, $\{Jx_n\}$ converges to Jx_0 in the weak* topology. Hence, $\{Jx_n - \rho(Tx_n - \xi)\}$ converges to $Jx_0 - \rho(Tx_0 - \xi)$ in the weak* topology. Thus, it is bounded in the weak* topology and so is bounded in norm by the uniform boundedness principle. Now, for any fixed $y \in K$, from $x_n \in A$ and the definition of G, we obtain

$$\begin{aligned}&\|Jx_n - \rho(Tx_n - \xi)\|^2 - 2\langle Jx_n - \rho(Tx_n - \xi), x_n\rangle + \|x_n\|^2 + 2\rho f(x_n)\\ &\quad \leq \|Jx_n - \rho(Tx_n - \xi)\|^2 - 2\langle Jx_n - \rho(Tx_n - \xi), y\rangle + \|y\|^2 + 2\rho f(y),\end{aligned}$$

which is equivalent to the following inequality:

$$2\langle Jx_n - \rho(Tx_n - \xi), y - x_n\rangle + \|x_n\|^2 + 2\rho f(x_n) \leq \|y\|^2 + 2\rho f(y).$$

We observe that

$$\begin{aligned}&|\ \langle Jx_n - \rho(Tx_n - \xi), y - x_n\rangle - \langle Jx_0 - \rho(Tx_0 - \xi), y - x_0\rangle\ |\\ &\quad \leq |\ \langle Jx_n - \rho(Tx_n - \xi), y - x_n - (y - x_0)\rangle\ | +\\ &\qquad |\ \langle Jx_n - \rho(Tx_n - \xi) - (Jx_0 - \rho(Tx_0 - \xi)), y - x_0\rangle\ |\\ &\quad \leq \|Jx_n - \rho(Tx_n - \xi)\|\|x_n - x_0\| +\\ &\qquad |\ \langle Jx_n - \rho(Tx_n - \xi) - (Jx_0 - \rho(Tx_0 - \xi)), y - x_0\rangle\ |\ .\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \langle Jx_n - \rho(Tx_n - \xi), y - x_n\rangle = \langle Jx_0 - \rho(Tx_0 - \xi), y - x_0\rangle.$$

Therefore,

$$\begin{aligned}
&2\langle Jx_0-\rho(Tx_0-\xi),y-x_0\rangle+\|x_0\|^2+2\rho f(x_0)\\
&\quad\le \lim_{n\to\infty}2\langle Jx_n-\rho(Tx_n-\xi),y-x_n\rangle+\lim_{n\to\infty}\|x_n\|^2+2\rho\liminf_{n\to\infty}f(x_n)\\
&\quad=\liminf_{n\to\infty}\{2\langle Jx_n-\rho(Tx_n-\xi),y-x_n\rangle+\|x_n\|^2+2\rho f(x_n)\}\\
&\quad\le\|y\|^2+2\rho f(y),
\end{aligned}$$

that is,

$$G(Jx_0-\rho(Tx_0-\xi),x_0)\le G(Jx_0-\rho(Tx_0-\xi),y).$$

Thus $x_0\in A$. Hence the intersection of A with any line segment is closed in K. This completes the proof. □

Theorem 2 *Let X be a reflexive and smooth Banach space with dual space X^*, K be a nonempty closed convex subset of X. Let $\rho>0$ be any given constant and $f:K\subset X\to R\cup\{+\infty\}$ be a proper, convex and lower semi-continuous mapping. Suppose that the following conditions are satisfied:*

(i) $T:K\to X^*$ is continuous from the line segments in K to the weak* topology of X^*;

(ii) There exists a nonempty weakly compact subset D of K and $y_0\in K$ such that

$$\langle Jy_0-\rho(Ty_0-\xi),y_0-x\rangle>\frac{1}{2}(\|y_0\|^2-\|x\|^2)+\rho(f(y_0)-f(x)),$$

for all $x\in K\setminus D$. Then Problem $\mathrm{MVI}(T-\xi,K)$ has a solution $\bar{x}\in D\subset K$.

Proof From Proposition 2, there exists $\bar{x}\in D\subset K$ such that

$$G(Jy-\rho(T(y)-\xi),\bar{x})\le G(Jy-\rho(T(y)-\xi),y),\quad\forall y\in K.\tag{1}$$

We claim that

$$G(J\bar{x}-\rho(T(\bar{x})-\xi),\bar{x})\le G(J\bar{x}-\rho(T(\bar{x})-\xi),y),\quad\forall y\in K.\tag{2}$$

Indeed, suppose that there exists $\bar{y}\in K$ such that

$$G(J\bar{x}-\rho(T(\bar{x})-\xi),\bar{x})>G(J\bar{x}-\rho(T(\bar{x})-\xi),\bar{y}).\tag{3}$$

Let $y_t=t\bar{y}+(1-t)\bar{x}\in K$, $\forall t\in[0,1]$. In view of (1), we have

$$G(Jy_t-\rho(T(y_t)-\xi),\bar{x})\le G(Jy_t-\rho(T(y_t)-\xi),y_t),\tag{4}$$

for all $t\in[0,1]$. From the definition of G and (4), it follows that

$$\begin{aligned}
&2t\langle Jy_t-\rho(Ty_t-\xi),\bar{y}-\bar{x}\rangle+\|\bar{x}\|^2+2t\rho f(\bar{x})\\
&\quad=2\langle Jy_t-\rho(Ty_t-\xi),y_t-\bar{x}\rangle+\|\bar{x}\|^2+2t\rho f(\bar{x})\\
&\quad\le\|y_t\|^2+2t\rho f(\bar{y})\le 2t\|\bar{y}\|^2+2(1-t)\|\bar{x}\|^2+2t\rho f(\bar{y}).
\end{aligned}$$

Hence,

$$2t\langle Jy_t-\rho(Ty_t-\xi),\bar{y}-\bar{x}\rangle+(2t-1)\|\bar{x})\|^2+2t\rho f(\bar{x})\le 2t\|\bar{y}\|^2+2t\rho f(\bar{y}),\ \forall t\in[0,1].\tag{5}$$

In addition, by Proposition 3 and (3), the set

$$U=\{x\in K:G(Jx-\rho(T(x)-\xi),x)>G(Jx-\rho(T(x)-\xi),\bar{y})\}\bigcap[\bar{y},\bar{x}]$$

is open in $[\bar{y},\bar{x}]$ and contains $\bar{x}$. Since $y_t\to\bar{x}$ as $t\to 0^+$, there exists $t_0\in(0,1]$ such that $y_t\in U$ for all $t\in(0,t_0)$. Thus

$$G(Jy_t-\rho(T(y_t)-\xi),y_t)>G(Jy_t-\rho(T(y_t)-\xi),\bar{y}),\ \forall t\in(0,t_0).$$

From the definition of G, it follows that, for each $t\in(0,t_0)$,

$$\begin{aligned}&\|Jy_t-\rho(Ty_t-\xi)\|^2-2\langle Jy_t-\rho(Ty_t-\xi),y_t\rangle+\|y_t\|^2+2\rho f(y_t)\\&\quad>\|Jy_t-\rho(Ty_t-\xi)\|^2-2\langle Jy_t-\rho(Ty_t-\xi),\bar{y}\rangle+\|\bar{y}\|^2+2\rho f(\bar{y}).\end{aligned}$$

Since $y_t-\bar{y}=(1-t)(\bar{x}-\bar{y})$, it follows that, for each $t\in(0,t_0)$,

$$\begin{aligned}&2(1-t)\langle Jy_t-\rho(Ty_t-\xi),\bar{x}-\bar{y}\rangle+\|\bar{y}\|^2+2(1-t)\rho(f(\bar{y})-f(\bar{x}))\\&\quad=2\langle Jy_t-\rho(Ty_t-\xi),y_t-\bar{y}\rangle+\|\bar{y}\|^2+2(1-t)\rho(f(\bar{y})-f(\bar{x}))\\&\quad<\|y_t\|^2\le 2t\|\bar{y}\|^2+2(1-t)\|\bar{x}\|^2.\end{aligned}$$

Therefore,

$$2t\langle Jy_t-\rho(Ty_t-\xi),\bar{y}-\bar{x}\rangle+(2t-1)\|\bar{x}\|^2+2t\rho f(\bar{x})>2t\|\bar{y}\|^2+2t\rho f(\bar{y}),\quad\forall t\in(0,t_0),$$

which contradicts (5). Hence,

$$G(J\bar{x}-\rho(T(\bar{x})-\xi),\bar{x})\le G(J\bar{x}-\rho(T(\bar{x})-\xi),y),\quad\forall y\in K.$$

Therefore, according to Proposition 1, problem MVI$(T-\xi,K)$ has a solution $\bar{x}\in D\subset K$. This completes the proof. □

Remark Theorem 2 weakens the continuity of T and the compactness assumption in Theorem 3.1 of Wu and Huang [6] and extends the corresponding results of Zeng and Yao [5] and Li [4].

References

[1] ALBER Y. *Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications*[C]. Proceedings of the Israel Seminar on Functional Differential Equations, Ariel, Israel, 1994, **1**: 1-21.

[2] ALBER Y I. *Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications* [M]. Dekker, New York, 1996.

[3] ALBER Y, GUERRE-DELABRIERE S. *On the projection methods for fixed point problems* [J]. Analysis (Munich), 2001, **21**(1): 17–39.

[4] LI Jinlu. *The generalized projection operator on reflexive Banach spaces and its applications* [J]. J. Math. Anal. Appl., 2005, **306**(1): 55–71.

[5] ZENG Liuchuan, YAO J C. *Existence theorems for variational inequalities in Banach spaces* [J]. J. Optim. Theory Appl., 2007, **132**(2): 321–337.

[6] WU Keqing, HUANG Nanjing. *The generalised f-projection operator with an application* [J]. Bull. Austral. Math. Soc., 2006, **73**(2): 307–317.

[7] WU Keqing, HUANG Nanjing. *Properties of the generalized f-projection operator and its applications in Banach spaces* [J]. Comput. Math. Appl., 2007, **54**(3): 399–406.

[8] NOOR M A. *A new iterative method for monotone mixed variational inequalities* [J]. Math. Comput. Modelling, 1997, **26**(7): 29–34.

[9] FAN K. *A generalization of Tychonoff's fixed point theorem* [J]. Math. Ann., 1960/1961, **142**: 305–310.

[10] ZHANG Qingbang, DENG Ruliang, LIU Liu. *Existence theorem and algorithm for a general implicit variational inequality in Banach space* [J]. Applied Math. Comput., 2009, **215**: 636–643.