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The Generalized Solutions of One Order Periodic Boundary Value Problems in Banach Space

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Abstract In this paper, the authors study the periodic boundary value problems of a class of nonlinear integro-differential equations of mixed type in Banach space with Carathéodory's conditions. We arrive at the conclusion of the existence of generalized solutions between generalized upper and lower solutions, and develop the monotone iterative technique to find generalized extremal solutions as limits of monotone solution sequences in Banach space.

Keywords monotone iterative; partial ordering; generalized solutions; normal cone.

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1. Introduction

We consider the following periodic boundary value problems (PBVP) for nonlinear integro-differential equations of mixed type in Banach space with Carathéodory's conditions

$$u'(t) = H(t, u(t), (Ku)(t), (Tu)(t)), \tag{1.1}$$

$$u(0) = u(2\pi), \tag{1.2}$$

where

$$(Ku)(t) = \int_0^t k(t,s)u(s)\mathrm{d}s, \quad (Tu)(t) = \int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \tag{1.3}$$

$I = [0, 2\pi]$, E is real Banach space, $H : I \times E \times E \times E \to E$ satisfies the Carathéodory's conditions in Banach space, $k : I \times I \to R_+$ satisfies the Carathéodory's conditions and $h : I \times I \to R_+$ is continuous. When H and k are continuous, the existence of extreme solutions for PBVP (1.1)–(1.2) has been studied in [1, 2]. In [3, 4] the author discussed PBVP for nonlinear integro-differential equations of Volterra type

$$u'(t) = H(t, u(t), (Ku)(t)),$$

$$u(0) = u(2\pi),$$

where

$$(Ku)(t) = \int_0^t k(t,s)u(s)\mathrm{d}s,$$

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$H : I \times R^2 \to R, k : I \times I \to R_+$ satisfies the Carathéodory's conditions. In this paper, we deal with PBVP (1.1)–(1.2) for the two aspects relative to generalized lower and upper solutions: $\alpha \leq \beta$, $\alpha \geq \beta$. When we study the integro-differential equations in Banach spaces we must overcome some difficulties, such as differentiability of the function. Though we get the existence of generalized solutions between generalized lower and upper solutions (Theorem 3.1, Theorem 3.2) in this paper, a question may arise if there is any function satisfying the conditions of the theorems. The authors give a positive answer to this question by providing two examples which satisfy all conditions of Theorems 3.1 and 3.2, respectively.

Suppose that E is a real Banach space, and $P \subset E$ is a cone. Then we get a partial ordering produced by P in E: if $y - x \in P$, then $x \leq y$.

The dual cone of P is $P^* = \{\varphi \in E^* \mid \varphi(x) \geq 0, \forall x \in P\}$.

Definition 1.1 ([5]) *An abstract function $x(t) : I \to E$ is a strong bounded variation function if it satisfies following conditions:*

$$\forall(\alpha_k, \beta_k) \subset I(k = 1, 2, \ldots, n),\ (\alpha_i, \beta_i) \cap (\alpha_j, \beta_j) = \emptyset,\ \sup \sum_k \|x(\beta_k) - x(\alpha_k)\| < \infty.$$

While we say x(t) is a strong absolutely continuous function: whenever $\sum_{k=1}^{n} \mid \alpha_k - \beta_k \mid \to 0$ then $\sup \sum_k \|x(\beta_k) - x(\alpha_k)\| \to 0$.

A strong absolutely continuous function must be a strong continuous function. It is also a strong bounded variation function.

Proposition 1.2 ([5]) *Suppose E is reflexive space. If $x(t)$ $(t \in I)$ is a strong bounded variation function, then $x'(t)$ exists for a.e. $t \in I$.*

A cone is normal iff $\forall [x, y] = \{z \in E \mid x \leq z \leq y\} \subset E$ is bounded.

Let $S[I, E]$ denote all strong absolutely continuous functions $u(t) : I \to E$, where E is reflexive space, and $C[I, E]$ denote all strong continuous functions $u(t) : I \to E$.

$\forall u = u(t)$, $v = v(t) \in C[I, E]$, we say $u \leq v$ if $u(t) \leq v(t)$ $(\forall t \in I)$.

Theorem 1.3 ([6]) *Suppose E is reflexive space and P is a normal cone in E. Then a total ordering set M in E is sequentially compact iff M is bounded under normed topology.*

Theorem 1.4 ([7]) *If E is reflexive space, then P is a normal cone iff P is a regular cone.*

Theorem 1.5 ([7]) *Suppose $P \subset E$ is a regular cone and $A : [u_0, v_0] \to E$ is a continuous map, where $[u_0, v_0] \subset E$ and satisfies*

(i) For w_1, $w_2 \in [u_0, v_0]$, $w_1 \leq w_2$ implies $Aw_1 \leq Aw_2$;

(ii) $u_0 \leq Au_0$, $Av_0 \leq v_0$.

Then A has a fixed point in $[u_0, v_0]$.

Definition 1.6 *A function $\alpha \in S[I, E]$ is called a generalized lower solution of PBVP (1.1)–(1.2) if α satisfies*

$$\alpha'(t) \leq H(t, \alpha(t), (K\alpha)(t), (T\alpha)(t)), \quad \text{a.e. } t \in I, \tag{1.4}$$

$$\alpha(0) \leq \alpha(2\pi). \tag{1.5}$$

Similarly, $\beta \in S[I, E]$ is called a generalized upper solution of PBVP (1.1)–(1.2) if β satisfies

$$\beta'(t) \geq H(t, \beta(t), (K\beta)(t), (T\beta)(t)), \quad \text{a.e. } t \in I, \tag{1.6}$$

$$\beta(0) \geq \beta(2\pi). \tag{1.7}$$

Throughout this paper, all solutions are used in generalized sense.

Let us list the following assumptions for convenience:

(A_1) (i) α, β are the lower and upper solutions for PBVP (1.1)–(1.2) respectively and satisfy $\alpha(t) \leq \beta(t)$, $t \in I$.

(ii) α, β are the lower and upper solutions for PBVP (1.1)–(1.2) respectively and satisfy $\beta(t) \leq \alpha(t)$, $t \in I$.

(A_2) $H(t, u, v, w)$ satisfies the Carathéodory's conditions, that is, $H(\cdot, u, v, w)$ is measurable for each (u, v, w), and $H(t, \cdot, \cdot, \cdot)$ is continuous for a.e. $t \in I$.

(A_3) For every $A > 0$ which satisfies $\| u \| \leq A$, $\| v \| \leq A$, $\| w \| \leq A$ $((u, v, w) \in E \times E \times E)$, there is a function $h_A \in L^1(I)$ such that $\| H(t, u, v, w) \| \leq h_A(t)$, a.e. $t \in I$.

(A_4) The kernel $k : I \times I \longrightarrow R_+$ satisfies Carathéodory's conditions, that is, $k(t, \cdot)$ is measurable for each $t \in I$, and $k(\cdot, s)$ is continuous for a.e. $s \in I$. And $k(t, s) \leq f(s)$, a.e. $t \in I$, $f \in L^1(I)$.

(A_5) (i)

$$\begin{aligned} &H(t, u(t), v(t), w(t)) - H(t, \bar{u}(t), \bar{v}(t), \bar{w}(t)) \\ &\quad \geq -M(t)(u(t) - \bar{u}(t)) - N(t)(v(t) - \bar{v}(t)) - G(t)(w(t) - \bar{w}(t)), \text{ a.e. } t \in I, \end{aligned} \tag{1.8}$$

whenever $\alpha(t) \leq \bar{u}(t) \leq u(t) \leq \beta(t)$, $(K\alpha)(t) \leq \bar{v}(t) \leq v(t) \leq (K\beta)(t)$, $(T\alpha)(t) \leq \bar{w}(t) \leq w(t) \leq (T\beta)(t)$, $t \in I$, where $M(t) > 0$, $N(t)$, $G(t) \geq 0$, a.e. $t \in I$, M, N, $G \in L^1(I)$ that satisfy

$$\int_0^{2\pi} \Big[M(t) + N(t) \int_0^t k(t, s)\mathrm{d}s + G(t) \int_0^{2\pi} h(t, s)\mathrm{d}s\Big]\mathrm{d}t < 1. \tag{1.9}$$

(ii)

$$\begin{aligned} &H(t, u(t), v(t), w(t)) - H(t, \bar{u}(t), \bar{v}(t), \bar{w}(t)) \leq \\ &\quad M(t)(u(t) - \bar{u}(t)) + N(t)(v(t) - \bar{v}(t)) + G(t)(w(t) - \bar{w}(t)), \text{ a.e. } t \in I, \end{aligned}$$

whenever $\beta(t) \leq \bar{u}(t) \leq u(t) \leq \alpha(t)$, $(K\beta)(t) \leq \bar{v}(t) \leq v(t) \leq (K\alpha)(t)$, $(T\beta)(t) \leq \bar{w}(t) \leq w(t) \leq (T\alpha)(t)$, $t \in I$, where $M(t) > 0$, $N(t)$, $G(t) \geq 0$, a.e. $t \in I$, M, N, $G \in L^1(I)$ that satisfy

$$\int_0^{2\pi} \Big[M(t) + N(t) \int_0^t k(t, s)\mathrm{d}s + G(t) \int_0^{2\pi} h(t, s)\mathrm{d}s\Big]\mathrm{d}t \leq 1/2. \tag{1.10}$$

2. Some auxiliary lemmas

Lemma 2.1 *Assume that $m(t) \in S[I, R]$, and it satisfies*

$$m'(t) \leq -M(t)m(t) - N(t) \int_0^t k(t, s)m(s)\mathrm{d}s - G(t) \int_0^{2\pi} h(t, s)m(s)\mathrm{d}s, \text{ a.e. } t \in I, \tag{2.1}$$

where $M(t)$, $N(t)$, $G(t) \in L^1(I)$, $M(t) > 0$, $N(t)$, $G(t) \geq 0$, a.e. $t \in I$ and k satisfies (A_4). Suppose further that

$$\int_0^{2\pi} \Big[M(t) + N(t)\int_0^t k(t,s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)\mathrm{d}s\Big]\mathrm{d}t \leq 1. \tag{2.2}$$

Then either $m(0) \leq 0$ or $m(0) \leq m(2\pi)$ implies that $m(t) \leq 0, t \in I$.

We leave out the proof of this lemma because it can be completed similarly to the proof of Theorem 3.1 in [4].

Lemma 2.2 *Assume that $m(t) \in S[I,R]$, and it satisfies*

$$m'(t) \geq M(t)m(t) + N(t)\int_0^t k(t,s)m(s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)m(s)\mathrm{d}s, \text{ a.e. } t \in I, \tag{2.3}$$

where $M(t)$, $N(t)$, $G(t)$, $k(t,s)$ satisfy the same assumptions as in Lemma 2.1, except for

$$\int_0^{2\pi} \Big[M(t) + N(t)\int_0^t k(t,s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)\mathrm{d}s\Big]\mathrm{d}t \leq 1/2 \tag{2.4}$$

replacing (2.2). Then $m(0) \geq m(2\pi)$ implies that $m(t) \leq 0, t \in I$.

The proof of this lemma can be completed similarly to the proof of Theorem 3.2 in [4].

Corollary 2.3 *Suppose that the assumptions of Lemma 2.2 hold, but $m(2\pi) \leq 0$ replace $m(0) \geq m(2\pi)$. Then $m(0) \leq 0$.*

Theorem 2.4 *Suppose E is reflexive space and P is a normal cone in E. Assume that (A_1)(i), (A_2)–(A_4), (A_5)(i) hold. Then the following linear PBVP*

$$u'(t) = H_\eta(t) - M(t)u(t) - N(t)\int_0^t k(t,s)u(s)\mathrm{d}s - G(t)\int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \text{ a.e. } t \in I, \tag{2.5}$$

$$u(0) = u(2\pi) \tag{2.6}$$

has a unique solution $u(t)$ such that $u(t) \in [\alpha,\beta] = \{v \in C[I,E] : \alpha(t) \leq v(t) \leq \beta(t), t \in I\}$, where $\eta \in [\alpha,\beta]$, $H_\eta(t) = H(t,\eta(t),(K\eta)(t),(T\eta)(t)) + M(t)\eta(t) + N(t)\int_0^t k(t,s)\eta(s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)\eta(s)\mathrm{d}s$.

Proof First, we consider the following linear initial value problem (IVP)

$$u'(t) = H_\eta(t) - M(t)u(t) - N(t)\int_0^t k(t,s)u(s)\mathrm{d}s - G(t)\int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \text{ a.e. } t \in I, \tag{2.7}$$

$$u(0) = u_0, \tag{2.8}$$

where $u_0 \in [\alpha(0),\beta(0)]$. It is equivalent to the following operator equation $u(t) = Qu(t)$, $(Qu)(t) = u_0 + \int_0^t [H_\eta(r) - M(r)u(r) - N(r)\int_0^r k(r,s)u(s)\mathrm{d}s - G(r)\int_0^{2\pi} h(t,s)u(s)\mathrm{d}s]\mathrm{d}r$. We will show that Q is a contractive operator in Banach space $C[I,E]$. Indeed, $\forall u$, $\bar{u} \in C[I,E]$, we have

$$\| Qu - Q\bar{u} \| = \max_{t\in I} \| (Qu)(t) - (Q\bar{u})(t) \|$$

$$\leq \int_0^{2\pi} \Big[M(t) + N(t)\int_0^t k(t,s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)\mathrm{d}s\Big]\mathrm{d}t \| u - \bar{u} \| .$$

By (1.9), the above initial value problem has a unique solution $u(t) = u(t; u_0)$.

Next, we will show that $u(t) = u(t; u_0) \in [\alpha, \beta]$. For any $\phi \in P^*$, let $m(t) = \phi(\alpha(t) - u(t))$. In view of (A_5)(i), we have $m(0) \le 0$ and

$$\begin{aligned} m'(t) =& \phi(\alpha'(t) - u'(t)) \\ \le& \phi(H(t, \alpha(t), (K\alpha)(t), (T\alpha)(t)) - H(t, \eta(t), (K\eta)(t), (T\eta)(t) - M(t)\eta(t) - \\ & N(t) \int_0^t k(t,s)\eta(s)ds - G(t) \int_0^{2\pi} h(t,s)\eta(s)\mathrm{d}s + M(t)u(t) + \\ & N(t) \int_0^t k(t,s)u(s)\mathrm{d}s + G(t) \int_0^{2\pi} h(t,s)u(s)\mathrm{d}s) \\ \le& -M(t)m(t) - N(t) \int_0^t k(t,s)m(s)\mathrm{d}s - G(t) \int_0^{2\pi} h(t,s)m(s)\mathrm{d}s, \text{ a.e. } t \in I. \end{aligned}$$

By Lemma 2.1, we have $m(t) \le 0, t \in I$. Because $\phi \in P^*$ is arbitrary, $\alpha(t) \le u(t)$, $t \in I$. Similarly, we can prove $u(t) \le \beta(t)$, $t \in I$.

Further, we will show that there is $u_0^* \in [\alpha(0), \beta(0)]$ such that the solution $u(t; u_0^*)$ satisfies $u(0) = u_0^* = u(2\pi)$, which indicates that PBVP (2.5)–(2.6) has a solution. In fact, $[\alpha(2\pi), \beta(2\pi)] \subset [\alpha(0), \beta(0)]$. Hence, for each $u_0 \in [\alpha(2\pi), \beta(2\pi)]$, the IVP (2.7)–(2.8) has a unique solution $u(t; u_0)$ such that $u(2\pi; u_0) \in [\alpha(2\pi), \beta(2\pi)]$. Therefore, the Poincaré operator $P_{2\pi} : u_0 \to u(2\pi; u_0)$ with $u_0 \in [\alpha(2\pi), \beta(2\pi)]$ maps interval $[\alpha(2\pi), \beta(2\pi)]$ into $[\alpha(2\pi), \beta(2\pi)]$. Let $u_1, u_2 \in [\alpha(2\pi), \beta(2\pi)]$ and $u_1 \le u_2$. Suppose further that $\bar{u}_i(t) = u(t; u_i)$, $i = 1, 2$, is the solution of the following IVP

$$u'(t) = H_\eta(t) - M(t)u(t) - N(t) \int_0^t k(t,s)u(s)\mathrm{d}s - G(t) \int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \text{ a.e. } t \in I, \quad (2.9)$$

$$u(0) = u_i. \quad (2.10)$$

For any $\phi \in P^*$, let $m(t) = \phi(\bar{u}_1(t) - \bar{u}_2(t))$. Then $m(0) = \phi(\bar{u}_1(0) - \bar{u}_2(0)) = \phi(u_1 - u_2) \le 0$,

$$\begin{aligned} m'(t) &= \phi(\bar{u}_1'(t) - \bar{u}_2'(t)) \\ &= -M(t)m(t) - N(t) \int_0^t k(t,s)m(s)\mathrm{d}s - G(t) \int_0^{2\pi} h(t,s)m(s)\mathrm{d}s, \text{ a.e. } t \in I. \end{aligned}$$

By Lemma 2.1, we have $m(t) \le 0$, $t \in I$. So $\bar{u}_1(t) \le \bar{u}_2(t)$, $t \in I$ and $P_{2\pi}u_1 \le P_{2\pi}u_2$. We get $P_{2\pi}$ is monotone increasing operator.

Next, we will show that $P_{2\pi}$ is continuous operator. Let $u_n \in [\alpha(2\pi), \beta(2\pi)]$, $n = 0, 1, 2, \ldots$, $\| u_n - u_0 \| \to 0$, $n \to \infty$. Suppose further that $u(t; u_n)$ is the solution of the following IVP

$$u'(t) = H_\eta(t) - M(t)u(t) - N(t) \int_0^t k(t,s)u(s)\mathrm{d}s - G(t) \int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \text{ a.e. } t \in I,$$

$$u(0) = u_n.$$

For any $t \in I$, we have

$$\| u(t; u_n) - u(t; u_0) \| \le \| u_n - u_0 \| + \| \int_0^t [-M(r)u(r; u_n) -$$

$$
\begin{aligned}
&N(r)\int_0^r k(r,s)u(s;u_n)\mathrm{d}s - G(r)\int_0^{2\pi} h(r,s)u(s;u_n)\mathrm{d}s +\\
&M(r)u(r;u_0) + N(r)\int_0^r k(r,s)u(s;u_0)\mathrm{d}s\\
&G(r)\int_0^{2\pi} h(r,s)u(s;u_0)\mathrm{d}s]\mathrm{d}r \parallel\\
\le{}& \parallel u_n - u_0 \parallel + \int_0^{2\pi}[M(r) + N(r)\int_0^r k(r,s)\mathrm{d}s +\\
&G(r)\int_0^{2\pi} h(r,s)\mathrm{d}s]\mathrm{d}r \parallel u(t;u_n) - u(t;u_0) \parallel .
\end{aligned}
$$

So

$$\parallel u(t;u_n) - u(t;u_0) \parallel \le \frac{\parallel u_n - u_0 \parallel}{1 - \int_0^{2\pi}[M(r) + N(r)\int_0^r k(r,s)ds - G(r)\int_0^{2\pi} h(r,s)u(s)\mathrm{d}s]\mathrm{d}r}.$$

We get

$$\parallel u(t;u_n) - u(t;u_0) \parallel \to 0, \quad n \to \infty.$$

Then

$$\parallel u(2\pi;u_n) - u(2\pi;u_0) \parallel \to 0, \quad n \to \infty.$$

Therefore, $P_{2\pi}$ is continuous on $[\alpha(2\pi), \beta(2\pi)]$.

Further, we will show $P_{2\pi}\alpha(2\pi) \ge \alpha(2\pi)$. Since $\alpha(2\pi) \in [\alpha(0), \beta(0)]$, we get the unique solution $u(t;\alpha(2\pi)) \in [\alpha, \beta]$ of IVP (2.7)–(2.8). So $u(2\pi;\alpha(2\pi)) \in [\alpha(2\pi), \beta(2\pi)]$. Then $P_{2\pi}\alpha(2\pi) \ge \alpha(2\pi)$. Similarly we can get $P_{2\pi}\beta(2\pi) \le \beta(2\pi)$. By Theorems 1.4 and 1.5, $P_{2\pi}$ has a fixed point $u_0^* \in [\alpha(2\pi), \beta(2\pi)]$. So IVP (2.7)–(2.8) has a solution $u(t;u_0^*)$ and $u(0;u_0^*) = u_0^* = u(2\pi;u_0^*)$. We also know $u(t;u_0^*) \in [\alpha, \beta]$, so $u(t;u_0^*)$ is a solution of PBVP(2.5)–(2.6).

Solutions of PBVP(2.5)–(2.6) must be unique. If not, let $u_1(t)$, $u_2(t)$, $t \in I$, are solutions of PBVP(2.5)–(2.6). For any $\phi \in P^*$, let $m(t) = \phi(u_1(t) - u_2(t))$. Then $m(0) = 0$ and

$$
\begin{aligned}
m'(t) &= \phi(u_1'(t) - u_2'(t))\\
&= -M(t)m(t) - N(t)\int_0^t k(t,s)m(s)\mathrm{d}s - G(t)\int_0^{2\pi} h(t,s)m(s)\mathrm{d}s, \text{ a.e. } t \in I.
\end{aligned}
$$

By Lemma 2.1, $m(t) \le 0, t \in I$. So $u_1(t) \le u_2(t)$, $t \in I$. Similarly let $m(t) = \phi(u_2(t) - u_1(t))$. We get $u_2(t) \le u_1(t)$, $t \in I$. At last we have $u_1(t) = u_2(t)$, $t \in I$.

Theorem 2.5 *Suppose E is reflexive space and P is a normal cone in E. Assume that (A_1)(ii), (A_2)–(A_4), (A_5)(ii) hold. Then the following linear PBVP*

$$u'(t) = H_\eta(t) + M(t)u(t) + N(t)\int_0^t k(t,s)u(s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \text{ a.e. } t \in I, \quad (2.11)$$

$$u(0) = u(2\pi) \quad (2.12)$$

has a unique solution $u(t) \in [\beta, \alpha]$, where $H_\eta(t) = H(t, \eta(t), (K\eta)(t), (T\eta)(t)) - M(t)\eta(t) - N(t)\int_0^t k(t,s)\eta(s)\mathrm{d}s - G(t)\int_0^{2\pi} h(t,s)\eta(s)\mathrm{d}s$, $\eta \in [\beta, \alpha]$.

The proof of this theorem can be completed similarly to the proof of Theorem 2.4.

3. Statement of the main results

Theorem 3.1 *Suppose E is reflexive space, and P is a normal cone in E. For PBVP(1.1)–(1.2), (A_1)(i),(A_2)–(A_4) and (A_5)(i) hold. Then there are monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha(t)$, $\beta_0 = \beta(t)$ such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = \gamma(t)$ uniformly on I and ρ, γ are minimal and maximal solutions respectively between α and β for PBVP(1.1)–(1.2).*

We leave out the proof of this theorem because it can be completed similarly to the proof of the following theorem.

Theorem 3.2 *Suppose E is reflexive space, and P is a normal cone in E. For PBVP(1.1)–(1.2), (A_1)(ii),(A_2)–(A_4) and (A_5)(ii) hold. Then there are monotone sequences $\{\beta_n(t)\}$, $\{\alpha_n(t)\}$ with $\alpha_0 = \alpha(t)$, $\beta_0 = \beta(t)$ such that $\lim_{n\to\infty} \beta_n(t) = \rho(t)$, $\lim_{n\to\infty} \alpha_n(t) = \gamma(t)$ uniformly on I and ρ, γ are minimal and maximal solutions respectively between β and α for PBVP(1.1)–(1.2).*

Proof We define a mapping $A\eta = u$, $\eta \in [\beta, \alpha]$, where u is the unique solution of linear PBVP(2.11)–(2.12). We shall show that

(a) $\beta \le A\beta$, $A\alpha \le \alpha$.

(b) A possesses a monotone nondecreasing property on the segment $[\beta, \alpha]$.

To prove (a), let $m(t) = \phi(\beta(t) - \beta_1(t))$, where $\phi \in P^*$ and $\beta_1(t) = A\beta(t)$. By (A_5)(ii) we have

$$m'(t) \ge M(t)m(t) + N(t)\int_0^t k(t,s)m(s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)m(s)\mathrm{d}s, \text{ a.e. } t \in I,$$
$$m(0) \ge m(2\pi).$$

Hence, by Lemma 2.2, we get $m(t) \le 0$, $t \in I$. $\phi \in P^*$ is arbitrary, so we have $\beta \le A\beta$. Similarly, we can get $A\alpha \le \alpha$.

In order to prove (b), we denote $u_1 = A\eta_1$, $u_2 = A\eta_2$, where $\eta_1 \le \eta_2$. Let $m(t) = \phi(u_1(t) - u_2(t))$, $\phi \in P^*$. By (A_5)(ii), we get

$$m'(t) \ge M(t)m(t) + N(t)\int_0^t k(t,s)m(s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)m(s)\mathrm{d}s, \text{ a.e. } t \in I,$$
$$m(0) = m(2\pi).$$

Using Lemma 2.2 again, we get $m(t) \le 0$, $t \in I$, that is, $A\eta_1 \le A\eta_2$.

Now we can define the sequences $\{\beta_n\}$, $\{\alpha_n\}$ with $\beta_0 = \beta$, $\alpha_0 = \alpha$, $\beta_{n+1} = A\beta_n$, $\alpha_{n+1} = A\alpha_n$, $n = 0, 1, 2, \ldots$. From the properties of A, we have

$$\beta_0 \le \beta_1 \le \cdots \le \beta_n \le \cdots \le \alpha_n \le \cdots \le \alpha_2 \le \alpha_1 \le \alpha_0. \tag{3.1}$$

For any $t_0 \in I$, from Theorem 1.3, $\{\beta_n(t_0)\}$ has a subsequence $\beta_{n_k}(t_0)$ which is convergent.

As $\beta_n(t_0)$ is monotone, we can suppose $\beta_{n_k}(t_0) \le \beta_n(t_0) \le \beta_{n_{k+1}}(t_0)$. By the condition P is a normal cone, then we have $\| \beta_n(t_0) - \beta_{n_k}(t_0) \| \le L_1 \| \beta_{n_{k+1}}(t_0) - \beta_{n_k}(t_0) \|$. $\forall \varepsilon > 0$, when k is big enough, we get $\| \beta_{n_{k+1}}(t_0) - \beta_{n_k}(t_0) \| < \varepsilon$, and then $\| \beta_n(t_0) - \beta_{n_k}(t_0) \| \le L_1 \| \beta_{n_{k+1}}(t_0) - \beta_{n_k}(t_0) \| < L_1\varepsilon$. It follows that $\beta_n(t_0)$ is a convergent sequence in E.

Next we will show $\{\beta_n\}$ is a convergent sequence in $C[I,E]$. If not, there are subsequences $\{n_i'\}$, $\{n_i''\}$ of $\mathbf{N}$, $\varepsilon_0>0$ and $t_i\in I$ $(i=1,2,\ldots)$ such that

$$\| \beta_{n_i'}(t_i)-\beta_{n_i''}(t_i)\| \geq \varepsilon_0. \tag{3.2}$$

We can assume that t_i converges to $t^*\in I$. Because β_n is the solution of linear PBVP(2.11)–(2.12), we can get

$$\begin{aligned}\beta_n(t)=&\frac{e^{\int_0^t M(\xi)\mathrm{d}\xi}}{e^{\int_0^{2\pi}-M(\xi)\mathrm{d}\xi}-1}\int_0^{2\pi} e^{\int_0^r -M(\xi)\mathrm{d}\xi}(H_{\beta_{n-1}}(r)+N(r)\int_0^r k(r,s)\beta_n(s)\mathrm{d}s+\\ &G(r)\int_0^{2\pi}h(r,s)\beta_n(s)\mathrm{d}s)\mathrm{d}r+e^{\int_0^t M(\xi)\mathrm{d}\xi}\int_0^t e^{\int_0^r -M(\xi)\mathrm{d}\xi}(H_{\beta_{n-1}}(r)+\\ &N(r)\int_0^r k(r,s)\beta_n(s)\mathrm{d}s+G(r)\int_0^{2\pi}h(r,s)\beta_n(s)\mathrm{d}s)\mathrm{d}r,\end{aligned}$$

where $H_{\beta_{n-1}}(t)=H(t,\beta_{n-1}(t),(K\beta_{n-1})(t),(T\beta_{n-1})(t))-M(t)\beta_{n-1}(t)-N(t)(K\beta_{n-1})(t)$ $-G(t)(T\beta_{n-1})(t)$. Let $t_1,t_2\in I$. We can assume $t_1\leq t_2$. Set

$$H_{\beta_{n-1}}(r)+N(r)\int_0^r k(r,s)\beta_n(s)\mathrm{d}s+G(r)\int_0^{2\pi}h(r,s)\beta_n(s)\mathrm{d}s=w(r).$$

Then we have

$$\begin{aligned}\| \beta_n(t_1)-\beta_n(t_2)\| \leq &\left|\frac{e^{\int_0^{t_1} M(\xi)\mathrm{d}\xi}-e^{\int_0^{t_2} M(\xi)\mathrm{d}\xi}}{e^{\int_0^{2\pi}-M(\xi)\mathrm{d}\xi}-1}\right|\int_0^{2\pi}\| w(r)\| \mathrm{d}r+\\ &\left|e^{\int_0^{t_1} M(\xi)\mathrm{d}\xi}-e^{\int_0^{t_2} M(\xi)\mathrm{d}\xi}\right|\int_0^{2\pi}\| w(r)\| \mathrm{d}r+e^{\int_0^{t_2} M(\xi)\mathrm{d}\xi}\int_{t_1}^{t_2}\| w(r)\| \mathrm{d}r.\end{aligned}$$

As P is a normal cone, $\{\int_0^t k(t,s)\beta_n(s)\mathrm{d}s\}$, $\{M(t)\beta_n(t)\}$, $\{N(t)\int_0^t k(t,s)\beta_n(s)\mathrm{d}s\}$, $\{G(t)\int_0^{2\pi}h(t,s)$ $\beta_n(s)\mathrm{d}s\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are all uniformly bounded. From

$$\begin{aligned}&H(t,\beta_n(t),(K\beta_n)(t),(T\beta_n)(t))\\ &\quad\leq H(t,\beta_0(t),(K\beta_0)(t),(T\beta_0)(t))+M(t)(\alpha_0(t)-\beta_0(t))+\\ &\qquad N(t)((k\alpha_0)(t)-(k\beta_0)(t))+G(t)((T\alpha_0)(t)-(T\beta_0)(t))\stackrel{\text{def}}{=}H^+(t),\\ &H(t,\beta_n(t),(K\beta_n)(t),(T\beta_n)(t))\\ &\quad\geq H(t,\alpha_0(t),(K\alpha_0)(t),(T\alpha_0)(t))-M(t)(\alpha_0(t)-\beta_0(t))-\\ &\qquad N(t)((k\alpha_0)(t)-(k\beta_0)(t))-G(t)((T\alpha_0)(t)-(T\beta_0)(t))\stackrel{\text{def}}{=}H^-(t).\end{aligned}$$

By the condition that P is a normal cone, we have

$$\| H(t,\beta_n(t),(K\beta_n)(t),(T\beta_n)(t))-H^-(t)\| \leq L_1\| H^+(t)-H^-(t)\| .$$

So

$$\| H(t,\beta_n(t),(K\beta_n)(t),(T\beta_n)(t))\| \leq \| H^-(t)\| +L_1\| H^+(t)-H^-(t)\| .$$

It follows that $\{H(t,\beta_n(t),(K\beta_n)(t),(T\beta_n)(t))\}$ is uniformly bounded.

From above we get $\int_0^{2\pi}\| w(r)\| \mathrm{d}r<+\infty$. If $|t_1-t_2|\to 0$, then

$$\| \beta_n(t_1)-\beta_n(t_2)\| \to 0.$$

It follows that there is $\delta > 0$ which has nothing to do with n, and whenever $t_1, t_2 \in I, |t_1 - t_2| < \delta$ we have

$$\| \beta_n(t_1) - \beta_n(t_2) \| \leq \frac{\varepsilon_0}{4}.$$

From $t_i \to t^*$, we know that there is i_0 such that when $i \geq i_0$, then $|t_i - t*| < \delta$ and

$$\| \beta_{n_i'}(t_i) - \beta_{n_i'}(t^*) \| \leq \frac{\varepsilon_0}{4}, \quad \| \beta_{n_i''}(t_i) - \beta_{n_i''}(t^*) \| \leq \frac{\varepsilon_0}{4}. \tag{3.3}$$

From (3.2), (3.3), we know if $i \geq i_0$, then

$$\| \beta_{n_i'}(t^*) - \beta_{n_i''}(t^*) \| \geq \frac{\varepsilon_0}{2}. \tag{3.4}$$

This is a contradiction to the result that $\{\beta_n(t^*)\}$ is a convergent sequence in E. So we get $\{\beta_n\}$ is a convergent sequence in $C[I, E]$. We may assume $\{\beta_n\}$ converges to ρ.

Because β_n is the solution of the following PBVP

$$\begin{aligned} &\beta_n'(t) = H_{\beta_{n-1}}(t) + M(t)\beta_n(t) + N(t)\int_0^t k(t,s)\beta_n(s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)\beta_n(s)\mathrm{d}s, \\ &\qquad\qquad \text{a.e. } \ t \in I, \\ &\beta_n(0) = \beta_n(2\pi), \end{aligned}$$

from this, we get $\rho(t)$ is a solution of PBVP(1.1)–(1.2). Similarly we can get $\{\alpha_n\}$ converges to γ in $C[I, E]$ and $\gamma(t)$ is a solution of PBVP(1.1)–(1.2).

Let $u(t)$ be a solution of PBVP(1.1)–(1.2), $u \in [\beta_0, \alpha_0]$. From the properties of A, we have $A\beta_0 \leq Au \leq A\alpha_0$, that is, $\beta_1 \leq u \leq \alpha_1$. So $\beta_n(t) \leq u(t) \leq \alpha_n(t), t \in I$. Let $n \to \infty$. We get $\rho(t) \leq u(t) \leq \gamma(t)$, $t \in I$. The proof of the theorem is completed. □

Example 3.1 l^2 is reflexive space,

$$P = \{a \in l^2 \mid a = (\xi_1, \xi_2, \xi_3, \ldots), \xi_i \geq 0, i \in \mathbf{N}\}$$

is a normal cone in l^2. $x_0 \in l^2$, and $x_0 \geq \theta$, $\| x_0 \| = 1$.

Consider the following PBVP

$$u'(t) = -g(t) \| u(t) \| x_0 - \int_0^t k(t,s)u(s)\mathrm{d}s - \int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \text{ a.e. } t \in I, \tag{3.5}$$

$$u(0) = u(2\pi), \tag{3.6}$$

where

$$g(t) = \begin{cases} te^{-7t}, & 0 \leq t \leq \pi; \\ \frac{e^{-6t}}{2}, & \pi < t \leq 2\pi, \end{cases}$$

$$k(t,s) = e^{-(6t+s)}, \ \ 0 \leq s \leq t \leq 2\pi,$$

$$h(t,s) = e^{-(6t+s)}, \ \ 0 \leq s \leq 2\pi, 0 \leq t \leq 2\pi.$$

It is easy to see that $\alpha(t) = -\sqrt{\pi}x_0$, $0 \leq t \leq 2\pi$, $\beta(t) \equiv \theta$, $0 \leq t \leq 2\pi$ are lower and upper solutions of PBVP (3.5)–(3.6), respectively. Indeed

$$\alpha'(t) \leq -g(t)\sqrt{\pi}\ x_0 + \int_0^t k(t,s)\sqrt{\pi}x_0\mathrm{d}s + \int_0^{2\pi} h(t,s)\sqrt{\pi}x_0\mathrm{d}s$$

$$= -g(t) \parallel \alpha(t) \parallel x_0 - \int_0^t k(t,s)\alpha(s)\mathrm{d}s - \int_0^{2\pi} h(t,s)\alpha(s)\mathrm{d}s.$$

Furthermore, let $M(t) = g(t)$, $N(t) = 1$, $G(t) = 1$. We can verify that

$$\int_0^{2\pi} \Big[M(t) + N(t)\int_0^t k(t,s)\mathrm{d}s + G(t)\int_0^{2\pi} h(t,s)\mathrm{d}s\Big]\mathrm{d}t < 1.$$

In order to verify PBVP (3.5)–(3.6) satisfies (1.8), we need to prove

$$\begin{aligned}
&- g(t) \parallel u_1(t) \parallel x_0 - \int_0^t k(t,s)u_1(s)\mathrm{d}s - \int_0^{2\pi} h(t,s)u_1(s)\mathrm{d}s - \\
&\Big[- g(t) \parallel u_2(t) \parallel x_0 - \int_0^t k(t,s)u_2(s)\mathrm{d}s - \int_0^{2\pi} h(t,s)u_2(s)\mathrm{d}s\Big] \\
&\geq -M(t)(u_1(t) - u_2(t)) - N(t)\Big[\int_0^t k(t,s)u_1(s)\mathrm{d}s - \int_0^t k(t,s)u_2(s)\mathrm{d}s] - \\
&G(t)\Big[\int_0^{2\pi} h(t,s)u_1(s)\mathrm{d}s - \int_0^{2\pi} h(t,s)u_2(s)\mathrm{d}s\Big],
\end{aligned}$$

where $\alpha(t) \leq u_2(t) \leq u_1(t) \leq \beta(t)$, $t \in I$.

Because $M(t) = g(t)$, $N(t) = 1$, $G(t) = 1$, we only need to prove

$$-g(t)(\parallel u_1(t) \parallel - \parallel u_2(t) \parallel)x_0 \geq -g(t)(u_1(t) - u_2(t)), \quad t \in I.$$

As P is the normal cone in l^2, we have

$$-u_2(t) \geq -u_1(t) \geq \theta \Rightarrow \parallel -u_2(t) \parallel \geq \parallel -u_1(t) \parallel \Rightarrow \parallel u_2(t) \parallel \geq \parallel u_1(t) \parallel, \quad t \in I.$$

Then

$$-g(t)(\parallel u_1(t) \parallel - \parallel u_2(t) \parallel)x_0 \geq \theta \geq -g(t)(u_1(t) - u_2(t)), \quad t \in I.$$

All assumptions of Theorem 3.1 with $M(t) = g(t)$, $N(t) = 1$, $G(t) = 1$ are satisfied. Then there is a solution $u(t)$ between $\alpha(t)$, $\beta(t)$ of PBVP (3.5)–(3.6) and it can be obtained by using monotone iterative method.

Example 3.2 Consider the following PBVP

$$u'(t) = -g(t) \parallel u(t) \parallel x_0 + \int_0^t k(t,s)u(s)\mathrm{d}s + \int_0^{2\pi} h(t,s)u(s)\mathrm{d}s, \text{ a.e. } t \in I, \tag{3.7}$$

$$u(0) = u(2\pi), \tag{3.8}$$

in l^2.

Suppose that all assumptions of Example 3.1 hold.

Then we can see that $\alpha(t) = \sqrt{\pi}x_0$, $0 \leq t \leq 2\pi$, $\beta(t) \equiv \theta$, $0 \leq t \leq 2\pi$ are lower and upper solutions of PBVP (3.7)–(3.8), respectively. Furthermore, one can verify that all assumptions of Theorem 3.2 with $M(t) = g(t)$, $N(t) = 1$, $G(t) = 1$ are satisfied. Then there is a solution $u(t)$ between $\beta(t)$, $\alpha(t)$ of PBVP (3.7)–(3.8) and it can be obtained by using monotone iterative method.

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