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Uniqueness of Meromorphic Functions Sharing One or Two Small Functions IM

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Abstract In this paper, the uniqueness problems of two meromorphic functions sharing one or two small functions IM are studied. By imposing certain conditions on the counting functions of four or three other small functions and constructing auxiliary function, some general uniqueness results are obtained.

Keywords meromorphic function; uniqueness; small function; sharing IM.

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1. Introduction and main results

We assume that the reader is familiar with the basic results and standard notations of Nevanlinna's value distribution theory such as $T(r,f), m(r,f), s(r,f), \ldots$ in [1–3]. Let E denote a set of finite linear measure, and I denote a set of infinite linear measure, not necessarily the same at each occurrence. We denote by $s(r,f)$ any function satisfying $s(r,f)=o\{T(r,f)\}$, as $r\to\infty, r\notin E$.

A meromorphic function $a(z)(\not\equiv\infty)$ is called a small function with respect to $f(z)$, provided that $T(r,a)=s(r,f)$, and let $s(f)$ denote the set of $a(z)$, where $a(z)\equiv\infty$ or $a(z)$ is a small function of $f(z)$. Moreover, $\overline{N}_0(r,a,f,g)$ denotes the reduced counting function of those common zeros of $f(z)-a(z)$ and $g(z)-a(z)$. $\overline{N}_{12}(r,a,f,g)$ denotes the reduced counting function of those non-common zeros of $f(z)-a(z)$ and $g(z)-a(z)$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM(IM) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicity (ignoring multiplicity).

In 1929, Nevanlinna [3, 4] proved the famous five values theorem. A natural question is whether the five values theorem can be extended to small functions? Some partial results were obtained in this direction [5–8].

In 1999, Li and Qiao solved this problem completely and proved the following result:

Theorem A ([9]) *Let five distinct meromorphic functions $a_j(z)$ $(j=\overline{1,5})$ (one of them may*

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equal ∞ identically) be small functions of nonconstant meromorphic functions $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j=\overline{1,5})$ IM, then $f(z)\equiv g(z)$.

In 2001, Yi and Li improved the above results and obtained the following two theorems.

Theorem B ([10]) *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and $a_j(z)$ $(j=\overline{1,5})$ be five distinct functions in $s(f)\bigcap s(g)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j=\overline{1,4})$ IM and satisfy $\overline{N}_0(r,a_5,f,g)\neq s(r,f)+s(r,g)$, then $f(z)\equiv g(z)$.*

Theorem C ([10]) *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ $(j=\overline{1,5})$ be five distinct functions in $s(f)\bigcap s(g)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j=\overline{1,4})$ IM and satisfy $\overline{N}_{12}(r,a_5,f,g)\leq \mu T(r,f)+\nu T(r,g)+s(r,f)+s(r,g)$ for I, where $0\leq\mu<\frac{2}{9}$ and $0\leq\nu<\frac{2}{9}$, then $f(z)\equiv g(z)$.*

In 2007, Deng and Yao [11] obtained two results below.

Theorem D *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ $(j=\overline{1,5})$ be five distinct functions in $s(f)\bigcap s(g)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j=\overline{1,3})$ IM and satisfy*

$$\sum_{j=4}^{5}\overline{N}_0(r,a_j,f,g)\geq\lambda\sum_{j=4}^{5}[\overline{N}(r,a_j,f)+\overline{N}(r,a_j,g)]+s(r,f)+s(r,g)$$

for I, where $\lambda>\frac{1}{3}$, then $f(z)\equiv g(z)$.

Theorem E *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ $(j=\overline{1,5})$ be five distinct functions in $s(f)\bigcap s(g)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j=\overline{1,3})$ IM and satisfy*

$$\sum_{j=4}^{5}\overline{N}_{12}(r,a_j,f,g)\leq\mu T(r,f)+\nu T(r,g)+s(r,f)+s(r,g)$$

for I, where $0\leq\mu<\frac{1}{7}$ and $0\leq\nu<\frac{1}{7}$, then $f(z)\equiv g(z)$.

In this paper, by constructing auxiliary functions, we prove some results of two meromorphic functions sharing one or two small functions as follows:

Theorem 1 *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ $(j=\overline{1,5})$ be five distinct functions in $s(f)\bigcap s(g)$. If $f(z)$ and $g(z)$ share $a_j(z)$ $(j=1,2)$ IM and satisfy*

$$\sum_{j=3}^{5}\overline{N}_0(r,a_j,f,g)\geq\lambda\sum_{j=4}^{5}[\overline{N}(r,a_j,f)+\overline{N}(r,a_j,g)]+s(r,f)+s(r,g)$$

for I, where $\lambda>\frac{2}{5}$, then $f(z)\equiv g(z)$.

Theorem 2 *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ $(j=\overline{1,5})$ be five distinct functions in $s(f)\bigcap s(g)$. If $f(z)$ and $g(z)$ share $a_1(z)$ IM and satisfy*

$$\sum_{j=2}^{5}\overline{N}_0(r,a_j,f,g)\geq\lambda\sum_{j=2}^{5}[\overline{N}(r,a_j,f)+\overline{N}(r,a_j,g)]+s(r,f)+s(r,g)\tag{1.1}$$

for I, where $\lambda>\frac{3}{7}$, then $f(z)\equiv g(z)$.

Theorem 3 *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $a_j(z)$ $(j=\overline{1,5})$ be five distinct functions in $s(f)\bigcap s(g)$. If $f(z)$ and $g(z)$ share $a_1(z)$ IM and satisfy*

$$\sum_{j=2}^{5}\overline{N}_{12}(r,a_j,f,g)\le \mu T(r,f)+\nu T(r,g)+s(r,f)+s(r,g) \tag{1.2}$$

for I, where $0\le\mu<\frac{1}{3}$ and $0\le\nu<\frac{1}{3}$, then $f(z)\equiv g(z)$.

Remark Suppose that $f(z)$ and $g(z)$ share $a_j(z)$ $(j=1,2)$ IM, and satisfy

$$\sum_{j=3}^{5}\overline{N}_{12}(r,a_j,f,g)\le \mu T(r,f)+\nu T(r,g)+s(r,f)+s(r,g).$$

Thus $\overline{N}_0(r,a_2,f,g)=\overline{N}(r,a_2,f)$, from this we know $\overline{N}_{12}(r,a_2,f,g)=s(r,f)+s(r,g)$, hence (1.2) still holds. Namely, if $f(z)$ and $g(z)$ share $a_j(z)$ $(j=1,2)$ IM, the conclusion of Theorem 3 is still valid.

Similarly, Theorems C and E are the corollary of Theorem 3.

2. Some lemmas

In order to prove our results, we need the following lemmas:

Lemma 1 ([2]) *Let $f(z)$ be a nonconstant meromorphic function, and $a_j(z)$ $(j=\overline{1,3})$ be three distinct functions in $s(f)$. Then*

$$T(r,f)\le\sum_{j=1}^{3}\overline{N}(r,a_j,f)+s(r,f).$$

Lemma 2 ([12]) *Let $f(z)$ be a nonconstant meromorphic function, and $a_j(z)$ $(j=\overline{1,5})$ be three distinct functions in $s(f)$. Then*

$$(3-\varepsilon)T(r,f)\le\sum_{j=1}^{5}\overline{N}(r,a_j,f)+s(r,f).$$

Lemma 3 *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $0,1,\infty,a(z)$ and $b(z)$ be five distinct functions in $s(f)\bigcap s(g)$. If one of the following conditions holds, then $f(z)\equiv g(z)$.*

(i) $\frac{f'(a'g-ag')}{(f-1)(g-a)}\equiv\frac{g'(a'f-af')}{(g-1)(f-a)}$ *and* $N_0(r,b,f,g)\ne s(r,f)+s(r,g)$;

(ii) $\frac{(b'f-bf')(a'g-ag')}{(f-1)(g-a)}\equiv\frac{(b'g-bg')(a'f-af')}{(g-1)(f-a)}$ *and* $N_0(r,1,f,g)\ne s(r,f)+s(r,g)$;

(iii) $\frac{[b'(f-1)-(b-1)f'][a'(g-1)-(a-1)g']}{(f-b)(g-a)}\equiv\frac{[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']}{(g-b)(f-a)}$ *and* $N_0(r,0,f,g)\ne s(r,f)+s(r,g)$.

Proof The detailed proof of (i) can be found in [10]. The proof of (ii) is similar to the one for (iii). Therefore, we only prove (iii).

Suppose that $[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']\equiv 0$. Thus $f=c_1(a-1)+1$ or $g=c_2(b-1)+1$, where c_1 and c_2 are two nonzero constants. This contradicts that $a(\not\equiv\infty)$ and $b(\not\equiv\infty)$ are two distinct functions in $s(f)\bigcap s(g)$. Hence $[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']\not\equiv$

0. It is clear that $(f-b)(g-a)\not\equiv 0$. Thus we deduce

$$\frac{[b'(f-1)-(b-1)f'][a'(g-1)-(a-1)g']}{[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']}-1\equiv\frac{(f-b)(g-a)}{(g-b)(f-a)}-1.$$

By a simple computation, we get

$$\frac{[(f-1)g'-f'(g-1)][a'(b-1)-(a-1)b']}{[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']}\equiv\frac{(f-g)(b-a)}{(g-b)(f-a)}.$$

If $f\not\equiv g$, and noting $[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']\not\equiv 0$, we have

$$\begin{aligned}&[-(g-1)\frac{f'-g'}{f-g}+g'][(b-1)a'-(a-1)b']\\ &\equiv\frac{(b-a)[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']}{(g-b)(f-a)}.\end{aligned}\tag{2.1}$$

From $\overline{N}_0(r,0,f,g)\neq s(r,f)+s(r,g)$, we know that there exists at least a point z_0 which is a common zero of $f(z)$ and $g(z)$, and is not the zero and pole of $a(z),b(z),a'(z),b'(z),a(z)-1,b(z)-1$ and $(b(z)-1)a'(z)-(a(z)-1)b'(z)$. By computing, we know that z_0 is a pole of order 1 of the left-hand side of (2.1), but not the pole of the right-hand side of (2.1). This is a contradiction. Hence $f\equiv g$. This completes the proof of Lemma 3. □

Lemma 4 *Let $f(z)$ $(\neq\infty)$ be a nonconstant meromorphic function, and let $a(z)$ $(\neq\infty)$ and $b(z)$ $(\neq\infty)$ be two distinct small meromorphic functions with respect to $f(z)$. Let $L(f,a,b)$ be defined by*

$$L(f,a,b)=\begin{vmatrix} f & f' & 1\\ a & a' & 1\\ b & b' & 1\end{vmatrix}.$$

Then $m(r,\frac{L(f,a,b)f^k}{(f-a)(g-b)})=s(r,f)$, where $k=1,2$. By using the logarithmic derivative lemma, we can prove Lemma 4 easily.

3. Proofs of Theorems 1 and 2

Proof Since the proofs of Theorems 1 and 2 are very similar, we prove only Theorem 2 here. In the sequel, we only consider $\frac{3}{7}<\lambda<\frac{1}{2}$ in Theorem 2 since the conclusion of Theorem 2 is naturally valid for $\lambda\geq\frac{1}{2}$. We may assme that

$$\sum_{j=2}^{5}\overline{N}_0(r,a_j,f,g)\neq s(r,f)+s(r,g).\tag{3.1}$$

Otherwise, from Lemma 2 and (1.1), we deduce $T(r,f)=s(r,f)$. This is a contradiction. Without loss of generality, we may assume that

$$a_1(z)=\infty,\ a_2(z)=0,\ a_3(z)=1,\ a_4(z)=a(z),\ a_5(z)=b(z).$$

Otherwise, a quasi-fractional linear transformation will do. Put

$$H_1=\frac{L(f,0,1)(f-g)L(g,0,a)}{f(f-1)g(g-a)}-\frac{L(g,0,1)(f-g)L(f,0,a)}{f(f-a)g(g-1)},\tag{3.2}$$

$$H_2=\frac{L(f,0,1)(f-g)L(g,0,b)}{f(f-1)g(g-b)}-\frac{L(g,0,1)(f-g)L(f,0,b)}{f(f-b)g(g-1)},\tag{3.3}$$

$$H_3=\frac{L(f,0,b)(f-g)L(g,0,a)}{f(f-b)g(g-a)}-\frac{L(g,0,b)(f-g)L(f,0,a)}{f(f-a)g(g-b)},\tag{3.4}$$

$$H_4=\frac{L(f,1,b)(f-g)L(g,1,a)}{(f-1)(f-b)(g-1)(g-a)}-\frac{L(g,1,b)(f-g)L(f,1,a)}{(f-1)(f-a)(g-1)(g-b)}.\tag{3.5}$$

From Lemma 4, we know $m(r,H_1)=s(r,f)+s(r,g)$. Now we consider the poles of H_1 in the following cases:

Case 1 Since $f(z)$ and $g(z)$ share ∞ IM, we suppose that z_0 is a pole of $f(z)$ with multiplicity p and the pole of $g(z)$ with multiplicity q. We may assume that $1\le p\le q$. From (3.2), we deduce

$$H_1=\frac{f-g}{fg}\cdot\frac{a'fg'(g-a)(f-1)-a'f'g(f-a)(g-1)+af'g'(f-g)(a-1)}{(f-1)(f-a)(g-1)(g-a)}.\tag{3.6}$$

From (3.6) and by computing, we know that z_0 is the pole of the numerator of (3.6) with multiplicity $2p+3q+1$, and the pole of the denominator of (3.6) with multiplicity $3p+3q$. Hence z_0 is not the pole of H_1.

Case 2 The case of the zeros of $f-1$ and $g-1$ is similar to the zeros of f and g, and in the sequel, we only consider the zeros of f and g.

Firstly, suppose that z_0 is a zero of $f(z)$ with multiplicity p and a zero of $g(z)$ with multiplicity q, which is neither the zero nor the pole of $a(z)$ and $a(z)-1$. We may assume that $1\le p\le q$. From (3.6), we see that z_0 is the zero of the numerator of (3.6) with multiplicity $2p+q-1$, and the zero of the denominator of (3.6) with multiplicity $p+q$. Hence z_0 is not the pole of H_1. Namely, the common zeros of $f(z)$ and $g(z)$ are not the poles of H_1.

Secondly, suppose that z_0 is a zero of $f(z)$ with multiplicity p, which is not the zero of $g(z)$. From (3.6), we know that z_0 is the zero of the numerator and denominator of (3.6) with multiplicity $p-1$ and p, respectively. Hence z_0 is the pole of H_1 with order 1.

Case 3 The zeros of $f-a$ and $g-a$, now (3.2) can be rewritten as

$$H_1=\frac{f'(f-g)[a(g'-a')-a'(g-a)]}{f(f-1)g(g-a)}-\frac{g'(f-g)[a(f'-a')-a'(f-a)]}{f(f-a)g(g-1)}.\tag{3.7}$$

Combining the above three cases, we have

$$N(r,H_1)\le\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+s(r,f)+s(r,g).$$

From this we get

$$T(r,H_1)\le\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+s(r,f)+s(r,g).\tag{3.8}$$

Similarly, we have

$$T(r,H_2)\le\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,b,f,g)+s(r,f)+s(r,g),\tag{3.9}$$

$$T(r,H_3)\le\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)+s(r,f)+s(r,g),\tag{3.10}$$

$$T(r,H_4)\le\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)+s(r,f)+s(r,g).\tag{3.11}$$

We first suppose that $H_j\not\equiv 0$ $(j=\overline{1,4})$. From (3.2), we know that the common zeros of $f(z)-b(z)$ and $g(z)-b(z)$ are the zeros of H_1. From this and combining (3.8), we get

$$\overline{N}_0(r,b,f,g)\le\overline{N}(r,0,H_1)+s(r,f)+s(r,g)$$

$$\leq \overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+s(r,f)+s(r,g). \quad (3.12)$$

Similarly, we have

$$\begin{aligned}\overline{N}_0(r,a,f,g) &\leq \overline{N}(r,0,H_2)+s(r,f)+s(r,g)\\ &\leq \overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,b,f,g)+s(r,f)+s(r,g), \quad (3.13)\\ \overline{N}_0(r,1,f,g) &\leq \overline{N}(r,0,H_3)+s(r,f)+s(r,g)\\ &\leq \overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)+s(r,f)+s(r,g), \quad (3.14)\\ \overline{N}_0(r,0,f,g) &\leq \overline{N}(r,0,H_4)+s(r,f)+s(r,g)\\ &\leq \overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)+s(r,f)+s(r,g). \quad (3.15)\end{aligned}$$

Combining (3.12)–(3.15), we obtain

$$\begin{aligned}&\overline{N}_0(r,0,f,g)+\overline{N}_0(r,1,f,g)+\overline{N}_0(r,a,f,g)+\overline{N}_0(r,b,f,g)\\ &\quad\leq 3[\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)]+s(r,f)+s(r,g).\end{aligned}$$

From this and noting that $\overline{N}(r,a,f)+\overline{N}(r,a,g)=\overline{N}_{12}(r,a,f,g)+2\overline{N}_0(r,a,f,g)$, we deduce

$$\begin{aligned}&\overline{N}_0(r,0,f,g)+\overline{N}_0(r,1,f,g)+\overline{N}_0(r,a,f,g)+\overline{N}_0(r,b,f,g)\\ &\quad\leq \frac{3}{7}[\overline{N}(r,0,f)+\overline{N}(r,0,g)+\overline{N}(r,1,f)+\overline{N}(r,1,g)+\overline{N}(r,a,f)+\\ &\qquad\overline{N}(r,a,g)+\overline{N}(r,b,f)+\overline{N}(r,b,g)]+s(r,f)+s(r,g). \quad (3.16)\end{aligned}$$

Combining (1.1) and (3.16) leads to a contradiction.

Hence, we may assume that $H_1\equiv 0$. Thus from (3.2), we obtain

$$f(z)\equiv g(z) \text{ or } \frac{f'(a'g-ag')}{(f-1)(g-a)}\equiv\frac{g'(a'f-af')}{(g-1)(f-a)}. \quad (*)$$

If $(*)$ holds, we consider two cases below.

1) $\overline{N}_0(r,b,f,g)\neq s(r,f)+s(r,g)$. From (i) of Lemma 3, we get $f(z)\equiv g(z)$.

2) $\overline{N}_0(r,b,f,g)=s(r,f)+s(r,g)$.

Suppose that $H_j\not\equiv 0 (j=\overline{2,4})$. Thus (3.13)–(3.15) hold, and in the sequel, we have

$$\begin{aligned}&\overline{N}_0(r,0,f,g)+\overline{N}_0(r,1,f,g)+\overline{N}_0(r,a,f,g)\\ &\quad\leq 2[\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)]+\\ &\qquad 3\overline{N}_{12}(r,b,f,g)+s(r,f)+s(r,g). \quad (3.17)\end{aligned}$$

On the other hand, from (1.1) and the fact that $\overline{N}(r,a_j,f)+\overline{N}(r,a_j,g)=\overline{N}_{12}(r,a_j,f,g)$ $+2\overline{N}_0(r,a_j,f,g)$ $(j=\overline{2,5})$, we deduce

$$\begin{aligned}&\overline{N}_0(r,0,f,g)+\overline{N}_0(r,1,f,g)+\overline{N}_0(r,a,f,g)+\frac{1}{1-2\lambda}\overline{N}_0(r,b,f,g)\\ &\geq\frac{\lambda}{1-2\lambda}[\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+\\ &\quad\overline{N}(r,b,f)+\overline{N}(r,b,g)]+s(r,f)+s(r,g).\end{aligned}$$

From which it follows

$$\overline{N}_0(r,0,f,g)+\overline{N}_0(r,1,f,g)+\overline{N}_0(r,a,f,g)$$

$$\geq \frac{\lambda}{1-2\lambda}[\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)]- \overline{N}_0(r,b,f,g)+s(r,f)+s(r,g). \tag{3.18}$$

From the assumption of $\frac{3}{7}<\lambda<\frac{1}{2}$, we have $\frac{\lambda}{1-2\lambda}>3$.

In the sequel, combining (3.17) and (3.18), and noting that $\overline{N}_0(r,b,f,g)=s(r,f)+s(r,g)$, we get a contradiction. Therefore, we may assume that $H_2\equiv 0$. Thus from (3.3), we obtain

$$f(z)\equiv g(z) \text{ or } \frac{f'(b'g-bg')}{(f-1)(g-b)}\equiv\frac{g'(b'f-bf')}{(g-1)(f-b)}. \tag{$**$}$$

If $(**)$ holds, we consider the following two cases again.

2.1) $\overline{N}_0(r,a,f,g)\neq s(r,f)+s(r,g)$. From (i) of Lemma 3, we get $f(z)\equiv g(z)$.

2.2) $\overline{N}_0(r,a,f,g)=s(r,f)+s(r,g)$.

Suppose that $H_j\not\equiv 0$ $(j=3,4)$. Thus (3.14) and (3.15) hold, and in the sequel, we have

$$\overline{N}_0(r,0,f,g)+\overline{N}_0(r,1,f,g) \leq \overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+2[\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)]+s(r,f)+s(r,g).$$

On the other hand, similarly to the proof of Case 2, we deduce

$$\overline{N}_0(r,0,f,g)+\overline{N}_0(r,1,f,g) \geq \frac{\lambda}{1-2\lambda}[\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)]- \overline{N}_0(r,a,f,g)-\overline{N}_0(r,b,f,g)+s(r,f)+s(r,g). \tag{3.20}$$

From (3.19) and (3.20), and noting that $\frac{\lambda}{1-2\lambda}>3$ and $\overline{N}_0(r,a,f,g)=\overline{N}_0(r,b,f,g)=s(r,f)+s(r,g)$, we get a contradiction.

Hence, we may assume that $H_3\equiv 0$. Thus from (3.4), we obtain

$$f(z)\equiv g(z) \text{ or } \frac{(b'f-bf')(a'g-ag')}{(f-b)(g-a)}\equiv\frac{(b'g-bg')(a'f-af')'}{(f-a)(g-b)}. \tag{$***$}$$

If $(***)$ holds, we still distinguish the following two cases.

2.2.1) $\overline{N}_0(r,1,f,g)\neq s(r,f)+s(r,g)$. From (ii) of Lemma 3, we get $f(z)\equiv g(z)$.

2.2.2) $\overline{N}_0(r,1,f,g)=s(r,f)+s(r,g)$.

Suppose that $H_4\not\equiv 0$. Thus (3.15) holds.

On the other hand, similarly to the proof of Case 2, we deduce

$$\overline{N}_0(r,0,f,g)\geq\frac{\lambda}{1-2\lambda}[\overline{N}_{12}(r,0,f,g)+\overline{N}_{12}(r,1,f,g)+\overline{N}_{12}(r,a,f,g)+\overline{N}_{12}(r,b,f,g)]- \overline{N}_0(r,1,f,g)-\overline{N}_0(r,a,f,g)-\overline{N}_0(r,b,f,g)+s(r,f)+s(r,g). \tag{3.21}$$

From (3.15) and (3.21), and noting that $\frac{\lambda}{1-2\lambda}>3$ and

$$\overline{N}_0(r,1,f,g)=\overline{N}_0(r,a,f,g)=\overline{N}_0(r,b,f,g)=s(r,f)+s(r,g),$$

we still get a contradiction.

Therefore, we have $H_4\equiv 0$. Thus from (3.5), we deduce $f(z)\equiv g(z)$ or

$$\frac{[b'(f-1)-(b-1)f'][a'(g-1)-(a-1)g']}{(f-b)(g-a)}$$

$$\equiv \frac{[b'(g-1)-(b-1)g'][a'(f-1)-(a-1)f']}{(g-b)(f-a)}. \tag{****}$$

If (****) holds, from (3.1) and the fact that $\overline{N}_0(r,1,f,g) = \overline{N}_0(r,a,f,g) = \overline{N}_0(r,b,f,g) = s(r,f) + s(r,g)$, we get $\overline{N}_0(r,0,f,g) = s(r,f) + s(r,g)$. From this and (iii) of Lemma 3, we obtain $f(z) \equiv g(z)$. This completes the proof of Theorem 2. □

4. Proof of Theorem 3

Proof Suppose that $f(z) \not\equiv g(z)$. From Theorem 2, we get

$$\sum_{j=2}^{5} \overline{N}_0(r,a_j,f,g) \le \frac{3}{7}\sum_{j=2}^{5}[\overline{N}(r,a_j,f) + \overline{N}(r,a_j,g)] + s(r,f) + s(r,g). \tag{4.1}$$

Next we prove that

$$\overline{N}(r,a_1,f) \le \sum_{j=2}^{5} \overline{N}_{12}(r,a_j,f) + s(r,f) + s(r,g). \tag{4.2}$$

Without loss of generality, we may assume that

$$a_1(z) = a(z),\ a_2(z) = 0,\ a_3(z) = \infty,\ a_4(z) = 1,\ a_5(z) = b(z),$$

provided that $\overline{N}(r,a,f) \neq s(r,f) + s(r,g)$. Otherwise, (4.2) is naturally valid.

Since $f(z)$ and $g(z)$ share $a(z)$ IM, we get $\overline{N}_0(r,a,f,g) \neq s(r,f) + s(r,g)$. Set

$$H = \frac{f'(f-g)(b'g-bg')}{f(f-1)g(g-b)} - \frac{g'(f-g)(b'f-bf')}{f(f-b)g(g-1)}. \tag{4.3}$$

Suppose $H \equiv 0$. From (i) of Lemma 3 and noting that $\overline{N}_0(r,a,f,g) \neq s(r,f) + s(r,g)$, we get $f(z) \equiv g(z)$. This is a contradiction. Hence $H \not\equiv 0$. From Lemma 4, we know

$$m(r,H) = s(r,f) + s(r,g). \tag{4.4}$$

Similarly to the proof of Theorem 2, we have

$$N(r,H) \le \overline{N}_{12}(r,0,f,g) + \overline{N}_{12}(r,1,f,g) + \overline{N}_{12}(r,\infty,f,g) + \overline{N}_{12}(r,b,f,g) + s(r,f) + s(r,g). \tag{4.5}$$

Obviously, the common zeros of $f(z) - a(z)$ and $g(z) - a(z)$ are the zeros of H.

Combining (4.4) and (4.5), we get

$$\begin{aligned} N(r,a,f) = N_0(r,a,f,g) &\le \overline{N}(r,0,H) + s(r,f) + s(r,g) \\ &\le \overline{N}_{12}(r,0,f,g) + \overline{N}_{12}(r,1,f,g) + \overline{N}_{12}(r,\infty,f,g) + \\ &\quad \overline{N}_{12}(r,b,f,g) + s(r,f) + s(r,g). \end{aligned}$$

This implies that (4.2) holds. From Lemma 2, we obtain

$$(3-\varepsilon)T(r,f) \le \sum_{j=1}^{5} \overline{N}(r,a_j,f) + s(r,f), \tag{4.6}$$

$$(3-\varepsilon)T(r,g) \le \sum_{j=1}^{5} \overline{N}(r,a_j,g) + s(r,g). \tag{4.7}$$

Combining (4.2), (4.6) and (4.7), we deduce

$$\begin{aligned}&(3-\varepsilon)\{T(r,f)+T(r,g)\}\\ &\quad\le 2\sum_{j=2}^{5}\overline{N}_{12}(r,a_j,f,g)+\sum_{j=2}^{5}[\overline{N}(r,a_j,f)+\overline{N}(r,a_j,g)]+s(r,f)+s(r,g).\end{aligned}\tag{4.8}$$

It follows from (4.1)

$$\sum_{j=2}^{5}[\overline{N}(r,a_j,f)+\overline{N}(r,a_j,g)]\le 7\sum_{j=2}^{5}\overline{N}_{12}(r,a_j,f,g)+s(r,f)+s(r,g).\tag{4.9}$$

From (4.8) and (4.9), we obtain

$$T(r,f)+T(r,g)\le\frac{9}{3-\varepsilon}\sum_{j=2}^{5}\overline{N}_{12}(r,a_j,f,g)+s(r,f)+s(r,g).$$

From this and (1.2), we deduce

$$T(r,f)+T(r,g)\le\frac{9}{3-\varepsilon}\mu T(r,f)+\frac{9}{3-\varepsilon}\nu T(r,g)+s(r,f)+s(r,g).\tag{4.10}$$

Thus, for positive number ε, such that $\frac{9}{3-\varepsilon}\mu<1$ and $\frac{9}{3-\varepsilon}\nu<1$, combining (4.10), we obtain a contradiction. This completes the proof of Theorem 3. $\square$

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