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On the Depth and Hilbert Series of the Fiber Cone

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Abstract Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with infinite residue field, I an $\mathfrak{m}$-primary ideal and K an ideal containing I. Let J be a minimal reduction of I such that, for some positive integer k, $KI^n \cap J = JKI^{n-1}$ for $n \le k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$. We show that if depth $G(I) \ge d-2$, then such fiber cones have almost maximal depth. We also compute, in this case, the Hilbert series of $F_K(I)$ assuming that depth $G(I) \ge d-1$.

Keywords Cohen-Macaulay local ring; fiber cone; depth; Hilbert series; associated graded ring; multiplicity.

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1. Introduction and preliminaries

Throughout the paper, let $(R,\mathfrak{m})$ be a Cohen-Macaulay (abbreviated to CM) local ring of dimension d with infinite residue field, I an $\mathfrak{m}$-primary ideal and K an ideal containing I. Let J be a minimal reduction of I such that, for some positive integer k, $KI^n \cap J = JKI^{n-1}$ for $n \le k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$.

Let $F_K(I) = \bigoplus_{n\ge 0} I^n/KI^n$ be the fiber cone of I with respect to K. For $K = I$, $F_K(I) = G(I)$, the associated graded ring of I. The Hilbert series of $F_K(I)$ is defined by $\sum_{n\ge 0} H_K(I,n)t^n$. In order to state the theorem of this paper we recall the necessary definition first. Let $\lambda(\cdot)$ denote length, it was proved in [1] that for $n \gg 0$, the function $H_K(I,n) := \lambda(R/KI^n)$ is given by a polynomial $P_K(I,n)$ of degree d. This polynomial can be written in the following way:

$$P_K(I,n) = g_0\binom{n+d-1}{d} - g_1\binom{n+d-2}{d-1} + \cdots + (-1)^d g_d.$$

Then $g_0 = e_0(I)$, where $e_0(I)$ is the multiplicity of I.

For $x \in I$, let x^* and x^0 denote the initial form in degree one component of $G(I)$ and $F_K(I)$ respectively. x^* is said to be superficial in $G(I)$ if there exists an integer $c > 0$ such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. Similarly, x^0 is said to be superficial in $F_K(I)$ if there exists an integer $c > 0$ such that $(KI^n : x) \cap I^c = KI^{n-1}$ for all $n > c$. Superficial sequences are defined inductively.

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We say that the ideal $J \subseteq I$ is a reduction of I if $I^{n+1} = JI^n$ for some $n \geq 0$. If J is the smallest such ideal, then it is called a minimal reduction of I. As R is CM and I is an $\mathfrak{m}$-primary ideal, any minimal reduction J of I is generated by d elements, and $e_0(I) = \lambda(R/J)$. The integers $r_J(I) = \min\{n | I^{n+1} = JI^n\}$ and $r_J^K(I) = \min\{n | KI^{n+1} = JKI^n\}$ are called the reduction number of I with respect to J and the K-reduction number of I with respect to J, respectively.

In this paper, we are interested in the depth and the Hilbert series of $F_K(I)$.

Jayanthan and Verma [9] showed that if, for some positive integer k, $KI^n \cap J = JKI^{n-1}$ for $n \leq k-1$ and $KI^k = JKI^{k-1}$, then $F_K(I)$ is CM if and only if depth $G(I) \geq d-1$. It is natural to consider the class of $\mathfrak{m}$-primary ideals I such that $KI^n \cap J = JKI^{n-1}$ for $n \leq k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$. When $k = 1$, i.e., I has almost minimal multiplicity with respect to K, Jayanthan and Verma showed that if depth $G(I) \geq d-2$, then depth $F_K(I) \geq d-1$. They also characterized, in this case, the Hilbert series of $F_K(I)$ and obtained that if depth $G(I) \geq d-1$ and $r = r_J^K(I)$, then $\sum_{n\geq 0} H_K(I,n)t^n = \frac{\lambda(\frac{R}{K})+[e_0(I)-\lambda(\frac{R}{K})-1)]t+t^{r+1}}{(1-t)^{d+1}}$.

When $K = I$, Corso, Polini and Pinto [2], Elias [6] and Rossi [12] independently proved that if, for some positive integer k, $I^n \cap J = JI^{n-1}$ for $n \leq k-1$ and $\lambda(\frac{I^k}{JI^{k-1}}) = 1$, then depth $G(I) \geq d-1$. Furthermore, Rossi [12] gave the Hilbert series of $G(I)$, that is:

$$\sum_{n\geq 0} \lambda(\frac{I^n}{I^{n+1}})t^n = \frac{\sum_{n=0}^{k-2} \lambda(\frac{I^n}{I^{n+1}+JI^{n-1}})t^n + [\lambda(\frac{I^{k-1}}{JI^{k-2}}) - 1]t^{k-1} + t^s}{(1-t)^d},$$

where $s = r_J(I)$.

The main results of this paper are the following:

(1) Suppose $d \geq 1$, J is a minimal reduction of I such that $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$ for some $k > 0$, and let depth $G(I) \geq d-1$. Then $F_K(I)$ is CM if and only if $KI^n \cap J = JKI^{n-1}$ for all $n \leq k-1$, $KI^k \nsubseteq J$ and $KI^{k+1} = JKI^k$.

(2) Suppose $d \geq 2$, J is a minimal reduction of I such that $KI^n \cap J = JKI^{n-1}$ for $n \leq k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$.

(a) Suppose depth $G(I) \geq d-2$, then depth $F_K(I) \geq d-1$.

(b) Suppose depth $G(I) \geq d-1$ and $r = r_J^K(I)$, then

$$\sum_{n\geq 0} H_K(I,n)t^n = \frac{\lambda(\frac{R}{K}) + [\lambda(\frac{K}{J}) - \lambda(\frac{KI}{KJ})]t + \sum_{n=2}^{k}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n + t^{r+1}}{(1-t)^{d+1}}.$$

2. Depth of the fiber cone

In this section, we study the depth properties of the fiber cone of any $\mathfrak{m}$-primary ideal I for which there exists a positive integer k such that $KI^n \cap J = JKI^{n-1}$ for all $n \leq k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$.

Theorem 2.1 *Suppose $d \geq 1$, J is a minimal reduction of I such that $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$ for some $k > 0$, and let depth $G(I) \geq d-1$. Then $F_K(I)$ is CM if and only if $KI^n \cap J = JKI^{n-1}$ for all*

$n \leq k-1$, $KI^k \nsubseteq J$ *and* $KI^{k+1} = JKI^k$.

Proof Choose $J = (x_1, \ldots, x_d)$ such that $x_1^*, \ldots, x_{d-1}^*$ is a regular sequence in $G(I)$ and $x_1^0, \ldots, x_d^0$ is a superficial sequence in $F_K(I)$. Suppose that $F_K(I)$ is CM, then by Theorem 5.1 of [9] we have that $KI^n \cap J = JKI^{n-1}$ for all n. In particular, $KI^k \cap J = JKI^{k-1}$ and $KI^k \nsubseteq J$, as $JKI^{k-1} \subsetneq KI^k$. Moreover, from the fact that $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$, we have that $KI^{k+1} \subseteq \mathfrak{m}KI^k \subseteq JKI^{k-1} \subseteq J$. Hence $KI^{k+1} = KI^{k+1} \cap J = JKI^k$.

Conversely, note that $JKI^{k-1} \subseteq KI^k \cap J \subseteq KI^k$. From the fact that $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$ and $KI^k \neq KI^k \cap J$ (as $KI^k \nsubseteq J$), we have that $KI^k \cap J = JKI^{k-1}$. However, $KI^{k+1} = JKI^k$ implies that $KI^n \cap J = JKI^{n-1}$ for all $n \geq k+1$. Then by Theorem 5.1 of [9] we get that $F_K(I)$ is CM. □

Lemma 2.2 *Let k be a positive integer such that $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$. Then $\lambda(\frac{KI^n}{JKI^{n-1}}) = 1$ for any $k \leq n \leq r$, where $r = r_J^K(I)$.*

Proof Since $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$, there exist $a \in I$, $b \in KI^{k-1}$ such that $KI^k = JKI^{k-1} + (ab)$ with $\mathfrak{m}\mathfrak{a}\mathfrak{b} \in JKI^{k-1}$. Then it can easily be seen by induction that $KI^n = JKI^{n-1} + (a^{n-k+1}b)$ with $\mathfrak{m}a^{n-k+1}b \in JKI^{n-1}$. Hence $\lambda(\frac{KI^n}{JKI^{n-1}}) = 1$ for all $k \leq n \leq r$. □

Definition 2.3 *The Ratliff-Rush closure of I with respect to K is defined as the set of ideals $\{rr_K(I^n)\}_{n \geq 0}$ with $rr_K(I^n) = \bigcup_{k \geq 1}(KI^{n+k} : I^k)$.*

To simplify the notation, for $n \geq 0$, we let ν_n be the minimum number of generators of the R-module $\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}$, $\nu = \sum_{n\geq 0} \nu_n$, $\rho_n^K = \lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})})$, $\eta_n^K = \lambda(\frac{KI^n}{JKI^{n-1}})$ and $q = \min\{n | KI^{n+1} \subseteq Jrr_K(I^n)\}$.

Lemma 2.4 *Suppose $d = 2$, k is a positive integer and $J = (x, y)$ is a minimal reduction of I such that $KI^n \cap J = JKI^{n-1}$ for every $n \leq k-1$. Then*

(1) $KI^n : x = KI^n : y = KI^{n-1}$ *for all* $n \leq k-1$*;*

(2) $q \geq k-1$*;*

(3) $\lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}) = \rho_n^K - \eta_n^K$ *for all* $n \leq k-1$*. In particular,* $\nu_n \leq \rho_n^K - \eta_n^K$*.*

Proof (1) Apply induction on n. It is clear for $n = 1$. Suppose that $n > 1$ and let $xz \in KI^n \cap (x) \subseteq KI^n \cap J = JKI^{n-1}$. Then there exist $z_1, z_2 \in KI^{n-1}$ such that $x(z - z_1) = yz_2$. We get $z - z_1 = yb$ and $z_2 = xb$ since y, x is a regular sequence. So $b \in KI^{n-1} : x = KI^{n-2}$ by inductive hypothesis and $z = z_1 + yb \in KI^{n-1}$.

(2) If $q < k-1$, we have that $KI^{q+1} \subseteq J \cap KI^{q+1} = JKI^q$ because $KI^{q+1} \subseteq Jrr_K(I^q) \subseteq J$. But it contradicts the fact $r_J^K(I) \geq k$.

(3) Since, for any $n \leq k-1$, $JKI^{n-1} \subseteq Jrr_K(I^{n-1}) \cap KI^n \subseteq J \cap KI^n = JKI^{n-1}$, we get $Jrr_K(I^{n-1}) \cap KI^n = JKI^{n-1}$. It follows that

$$\lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}) = \lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})}) - \lambda(\frac{Jrr_K(I^{n-1})+KI^n}{Jrr_K(I^{n-1})})$$
$$= \lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})}) - \lambda(\frac{KI^n}{Jrr_K(I^{n-1}) \cap KI^n})$$

$$= \lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})}) - \lambda(\frac{KI^n}{JKI^{n-1}})$$
$$= \rho_n^K - \eta_n^K.$$

Lemma 2.5 *Suppose $d = 2$, k is a positive integer and $J = (x, y)$ is a minimal reduction of I such that $KI^n \cap J = JKI^{n-1}$ for $n \le k-1$. Let "$\bar{\ }$" denote the image modulo(x), $\bar{r} = r_{\bar{J}}^{\bar{K}}(\bar{I})$ and $\bar{r} > k-1$. Then*

(1) $g_1 = \sum_{n=1}^{k-1} \eta_n^K + (\bar{r} - k + 1) - \lambda(\frac{R}{K})$;

(2) $g_1 = \sum_{n=2}^{\infty} \rho_n^K + \lambda(\frac{rr_K(I)}{Jrr_K(I^0)}) - \lambda(\frac{R}{rr_K(I^0)})$.

Proof The second assertion is clear by Proposition 2.5 of [10]. As for the first one, by Lemma 3.5 and Theorem 5.3 of [9], we get that

$$\begin{aligned} g_1 = \bar{g_1} &= \sum_{n=1}^{\bar{r}} \lambda(\frac{\bar{K}\bar{I}^n}{\bar{J}\bar{K}\bar{I}^{n-1}}) - \lambda(\frac{\bar{R}}{\bar{K}}) \\ &= \sum_{n=1}^{k-1} \lambda(\frac{KI^n}{JKI^{n-1} + x(KI^n : x)}) + \sum_{n=k}^{\bar{r}} \lambda(\frac{\bar{K}\bar{I}^n}{\bar{J}\bar{K}\bar{I}^{n-1}}) - \lambda(\frac{R}{K}) \\ &= \sum_{n=1}^{k-1} \lambda(\frac{KI^n}{JKI^{n-1}}) + (\bar{r} - k + 1) - \lambda(\frac{R}{K}) \\ &= \sum_{n=1}^{k-1} \eta_n^K + (\bar{r} - k + 1) - \lambda(\frac{R}{K}). \end{aligned}$$

Proposition 2.6 *Suppose $d = 2$, k is a positive integer and $J = (x, y)$ is a minimal reduction of I, $KI^{k-1} \cap J = JKI^{k-2}$ and $\bar{r} > k-1$. Then $\nu \le \bar{r} - q$.*

Proof Firstly, we remark that we have $q \ge k$. In fact $KI^{k-1} \nsubseteq JKI^{k-2}$, otherwise $KI^{k-1} = KI^{k-1} \cap JKI^{k-2} \subseteq KI^{k-1} \cap J = JKI^{k-2}$. Thus $r_J^K(I) \le k-2$, which contradicts the assumption $k - 1 < \bar{r} \le r$.

By the definition of ν_n, we have that $\nu_n \le \lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1}) + KI^n})$. Thus, by Lemmas 2.4 and 2.5, we get that

$$\begin{aligned} \nu = \sum_{n \ge 0} \nu_n &= \sum_{n=1}^{k-1} \nu_n + \sum_{n=k}^{\infty} \nu_n + \lambda(\frac{rr_K(I^0)}{K}) \\ &\le \sum_{n=1}^{k-1} [\rho_n^K - \eta_n^K] + \sum_{n=k}^{\infty} \nu_n + \lambda(\frac{rr_K(I^0)}{K}) \\ &= \sum_{n=1}^{\infty} \rho_n^K - \sum_{n=k}^{\infty} \rho_n^K - \sum_{n=1}^{k-1} \eta_n^K + \sum_{n=k}^{\infty} \nu_n + \lambda(\frac{rr_K(I^0)}{K}) \\ &= \sum_{n=2}^{\infty} \rho_n^K + \lambda(\frac{rr_K(I)}{Jrr_K(I^0)}) + \lambda(\frac{rr_K(I^0)}{K}) - \sum_{n=k}^{\infty} \rho_n^K - \sum_{n=1}^{k-1} \eta_n^K + \sum_{n=k}^{\infty} \nu_n \\ &= g_1 + \lambda(\frac{R}{K}) - \sum_{n=k}^{\infty} \rho_n^K - \sum_{n=1}^{k-1} \eta_n^K + \sum_{n=k}^{\infty} \nu_n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{k-1}\eta_n^K + (\bar{r}-k+1) - \sum_{n=k}^{\infty}\rho_n^K - \sum_{n=1}^{k-1}\eta_n^K + \sum_{n=k}^{\infty}\nu_n \\
&= (\bar{r}-k+1) - \sum_{n=k}^{\infty}\rho_n^K + \sum_{n=k}^{\infty}\nu_n \\
&= (\bar{r}-k+1) + \sum_{n=k}^{q}[\nu_n - \rho_n^K] + \sum_{n=q+1}^{\infty}[\nu_n - \rho_n^K] \\
&\le (\bar{r}-k+1) + \sum_{n=k}^{q}[\nu_n - \rho_n^K] + \sum_{n=q+1}^{\infty}[\lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}) - \rho_n^K] \\
&\le (\bar{r}-k+1) + \sum_{n=k}^{q}[\lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}) - \rho_n^K] \\
&= (\bar{r}-k+1) - \sum_{n=k}^{q}\lambda(\frac{KI^n}{Jrr_K(I^{n-1})\cap KI^n}) \\
&\le \bar{r}-k+1-(q-k+1) = \bar{r}-q,
\end{aligned}
$$

where the last inequality holds because of $\lambda(\frac{KI^n}{Jrr_K(I^{n-1})\cap KI^n}) \le 1$ for all $n \ge k$, which comes from the fact $JKI^{n-1} \subseteq KI^n \cap Jrr_K(I^{n-1}) \subseteq KI^n$ and $\lambda(\frac{KI^n}{JKI^{n-1}}) \le 1$ for all $n \ge k$. □

Proposition 2.7 *Suppose $d=2$, k is a positive integer and $J=(x,y)$ is a minimal reduction of I, $KI^n \cap J = JKI^{n-1}$ for all $n \le k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}})=1$. Then*

(1) $r_J^K(I) \le \nu + q$;

(2) $r_J^K(I) \le g_1 + k - 1 + \lambda(\frac{R}{K}) - \sum_{n=1}^{k-1}\lambda(\frac{KI^n}{JKI^{n-1}})$.

Proof The first assertion is clear from Theorem 3.4 of [10]. As for the second one, by the definition of q, we have that

$$
\lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}) = \lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})}) - \lambda(\frac{Jrr_K(I^{n-1})+KI^n}{Jrr_K(I^{n-1})})
$$

$$
\le \begin{cases} \rho_n^K - 1 & : \quad n \le q \\ \rho_n^K & : \quad n \ge q+1. \end{cases}
$$

It follows that

$$
\begin{aligned}
r \le \nu + q &\le \sum_{n\ge 1}\lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}) + \lambda(\frac{rr_K(I^0)}{K}) + q \\
&\le \sum_{n=1}^{k-1}(\rho_n^K - \eta_n^K) + \sum_{n=k}^{\infty}\lambda(\frac{rr_K(I^n)}{Jrr_K(I^{n-1})+KI^n}) + \lambda(\frac{rr_K(I^0)}{K}) + q \\
&\le \sum_{n=1}^{k-1}(\rho_n^K - \eta_n^K) + \sum_{n=k}^{q}(\rho_n^K - 1) + \sum_{n=q+1}^{\infty}\rho_n^K + \lambda(\frac{rr_K(I^0)}{K}) + q \\
&\le \sum_{n=1}^{\infty}\rho_n^K - \sum_{n=1}^{k-1}\eta_n^K - (q-k+1) + \lambda(\frac{rr_K(I^0)}{K}) + q
\end{aligned}
$$

$$= \sum_{n=2}^{\infty} \rho_n^K + \lambda(\frac{rr_K(I)}{Jrr_K(I^0)}) + \lambda(\frac{rr_K(I^0)}{K}) - \sum_{n=1}^{k-1} \eta_n^K + k - 1$$
$$= g_1 + k - 1 + \lambda(\frac{R}{K}) - \sum_{n=1}^{k-1} \lambda(\frac{KI^n}{JKI^{n-1}}).$$

The last equality follows from Lemma 2.5 (2). □

Now, we can prove the main result of this section.

Theorem 2.8 *Suppose $d \geq 2$, k is a positive integer and J is a minimal reduction of I such that $KI^n \cap J = JKI^{n-1}$ for $n \leq k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$. Let depth $G(I) \geq d-2$. Then depth $F_K(I) \geq d-1$.*

Proof We apply induction on d. Let $d = 2$. Choose $J = (x, y)$ such that x^*, y^* is a superficial sequence in $G(I)$ and x^0, y^0 is a superficial sequence in $F_K(I)$. Let " ¯ " denote the image modulo (x), $r = r_J^K(I)$ and $\bar{r} = r_{\bar{J}}^{\bar{K}}(\bar{I})$. If $\bar{r} \leq k-1$, then $\bar{K}\bar{I}^n \cap \bar{J} = \bar{J}\bar{K}\bar{I}^{n-1}$ for all $n \geq 1$, thus $F_K(I)$ is CM by Lemma 2.7 of [9]. If $\bar{r} > k-1$, $\bar{r} = r$ by Propositions 2.6 and 2.7 (1). For $j \geq 0$, consider the following exact sequence:

$$0 \to \frac{KI^j : x}{KI^j : J} \xrightarrow{y} \frac{KI^{j+1} : x}{KI^j} \xrightarrow{x} \frac{KI^{j+1}}{JKI^j} \to \frac{\bar{K}\bar{I}^{j+1}}{\bar{J}\bar{K}\bar{I}^j} \to 0.$$

We have that $\lambda(\frac{KI^j:x}{KI^j:J}) = \lambda(\frac{KI^{j+1}:x}{KI^j})$ for all $j \geq k-1$.

Claim. For every $j \geq 0$, $KI^{j+1} : x = KI^j$. For all $j \leq k-1$, by Lemma 2.4, it is clear. Moreover, we have that $KI^{k-1} : J = (KI^{k-1} : x) \cap (KI^{k-1} : y) = KI^{k-2}$, thus $KI^k : x = KI^{k-1}$. Suppose that $j \geq k$ and that the claim is true for $j-1$. Then $KI^{j-1} \subseteq KI^j : J \subseteq KI^j : x = KI^{j-1}$, where the last equality follows by induction. Thus $KI^j : x = KI^j : J$. It follows from the equality $\lambda(\frac{KI^j:x}{KI^j:J}) = \lambda(\frac{KI^{j+1}:x}{KI^j})$ for all $j \geq k-1$ that $KI^{j+1} : x = KI^j$ for all $j \geq 0$. Hence x^0 is a regular element in $F_K(I)$ and depth $F_K(I) \geq 1$.

Let $d > 2$ and choose $J = (x_1, \ldots, x_d)$ such that $x_1^*, \ldots, x_{d-2}^*$ is a regular sequence in $G(I)$ and $x_1^0, \ldots, x_d^0$ is a superficial sequence in $F_K(I)$. Let " ¯ " denote the images modulo $(x_1, \ldots, x_{d-2})$. Then dim$\bar{R} = 2$, $\bar{K}\bar{I}^n \cap \bar{J} = \bar{J}\bar{K}\bar{I}^{n-1}$ for $n \leq k-1$ and $\lambda(\frac{\bar{K}\bar{I}^k}{\bar{J}\bar{K}\bar{I}^{k-1}}) \leq \lambda(\frac{KI^k}{JKI^{k-1}}) = 1$. If $\bar{r} \leq k-1$, then $\bar{K}\bar{I}^n \cap \bar{J} = \bar{J}\bar{K}\bar{I}^{n-1}$ for every $n \geq 1$, and $\bar{K}\bar{I}^k = \bar{J}\bar{K}\bar{I}^{k-1}$, it follows that depth $F_{\bar{K}}(\bar{I}) \geq 1$ from Lemma 4.6 of [4]. If $\bar{r} > k-1$, $\lambda(\frac{\bar{K}\bar{I}^k}{\bar{J}\bar{K}\bar{I}^{k-1}}) = 1$. Thus, by the first part, depth $F_{\bar{K}}(\bar{I}) \geq 1$. Since $x_1^*, \ldots, x_{d-2}^*$ is a regular sequence in $G(I)$, $F_{\bar{K}}(\bar{I}) \simeq \frac{F_K(I)}{(x_1^0, \ldots, x_{d-2}^0)F_K(I)}$, and hence depth $F_K(I) \geq d-1$ from Lemma 2.7 of [9]. □

3. The Hilbert series of the fiber cone

In this section, we compute the Hilbert series of the fiber cone $F_K(I)$ of any $\mathfrak{m}$-primary ideal I for which there exists a positive integer k such that $KI^n \cap J = JKI^{n-1}$ for all $n \leq k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$.

Lemma 3.1 ([13]) *Suppose that depth $G(I) \geq d-1$ and depth $F_K(I) \geq d-1$. Let J be a* *minimal reduction of I and $r = r_J^K(I)$. Then*

$$\sum_{n\geq 0} H_K(I,n)t^n = \frac{\lambda(\frac{R}{K}) + [\lambda(\frac{K}{J}) - \lambda(\frac{KI}{KJ})]t + \sum\limits_{n=2}^{r+1}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n}{(1-t)^{d+1}}.$$

Theorem 3.2 *Let $d \geq 1$, k a positive integer and J a minimal reduction of I such that $KI^n \cap J = JKI^{n-1}$ for all $n \leq k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$. Let depth $G(I) \geq d-1$ and $r = r_J^K(I)$. Then*

$$\sum_{n\geq 0} H_K(I,n)t^n = \frac{\lambda(\frac{R}{K}) + [\lambda(\frac{K}{J}) - \lambda(\frac{KI}{KJ})]t + \sum\limits_{n=2}^{k}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n + t^{r+1}}{(1-t)^{d+1}}.$$

Proof By Lemma 2.2, we have $\lambda(\frac{KI^n}{JKI^{n-1}}) = 1$ for any $k \leq n \leq r$. It follows that

$$\begin{aligned}
&\sum_{n=2}^{r+1}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n = \sum_{n=2}^{r+1}\lambda(\frac{KI^{n-1}}{KJI^{n-2}})t^n - \sum_{n=2}^{r+1}\lambda(\frac{KI^n}{KJI^{n-1}})t^n \\
&= \sum_{n=2}^{r+1}\lambda(\frac{KI^{n-1}}{KJI^{n-2}})t^n - \sum_{n=3}^{r+1}\lambda(\frac{KI^{n-1}}{KJI^{n-2}})t^{n-1} \\
&= \lambda(\frac{KI}{KJ})t^2 - \sum_{n=2}^{r}\lambda(\frac{KI^n}{KJI^{n-1}})t^n(1-t) \\
&= \lambda(\frac{KI}{KJ})t^2 - \sum_{n=2}^{k-1}\lambda(\frac{KI^n}{KJI^{n-1}})t^n(1-t) - \sum_{n=k}^{r}\lambda(\frac{KI^n}{KJI^{n-1}})t^n(1-t) \\
&= \lambda(\frac{KI}{KJ})t^2 - \sum_{n=2}^{k-1}\lambda(\frac{KI^n}{KJI^{n-1}})t^n(1-t) - \sum_{n=k}^{r}(t^n - t^{n+1}) \\
&= \sum_{n=2}^{k-1}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n + \lambda(\frac{KI^{k-1}}{KJI^{k-2}})t^k - t^k + t^{r+1} \\
&= \sum_{n=2}^{k}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n + t^{r+1}.
\end{aligned}$$

On the other hand, by Theorem 2.8 we have depth $F_K(I) \geq d-1$. It follows that by Lemma 3.1

$$\begin{aligned}
\sum_{n\geq 0} H_K(I,n)t^n &= \frac{\lambda(\frac{R}{K}) + [\lambda(\frac{K}{J}) - \lambda(\frac{KI}{KJ})]t + \sum\limits_{n=2}^{r+1}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n}{(1-t)^{d+1}} \\
&= \frac{\lambda(\frac{R}{K}) + [\lambda(\frac{K}{J}) - \lambda(\frac{KI}{KJ})]t + \sum\limits_{n=2}^{k}[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})]t^n + t^{r+1}}{(1-t)^{d+1}}.
\end{aligned}$$

As consequences of the above result, we recover results of Rossi [12, Theorem 3.2 (2)], Jayanthan and Verma [10, Proposition 5.2].

Corollary 3.3 *Let $d \geq 1$, k a positive integer and J a minimal reduction of I such that*

$I^n \cap J = JI^{n-1}$ for all $n \le k-1$, $\lambda(\frac{I^k}{JI^{k-1}}) = 1$ and $r = r_J(I)$. Then

$$\sum_{n\ge 0} \lambda(\frac{I^n}{I^{n+1}})t^n = \frac{\sum\limits_{n=0}^{k-2} \lambda(\frac{I^n}{I^{n+1}+JI^{n-1}})t^n + [\lambda(\frac{I^{k-1}}{JI^{k-2}}) - 1]t^{k-1} + t^r}{(1-t)^d}.$$

Proof We recall that depth $G(I) \ge d-1$ by Theorem 3.2 of [12]. Put $K = I$ in Theorem 3.2, then we get

$$\begin{aligned}\sum_{n\ge 0} \lambda(\frac{R}{I^{n+1}})t^n &= \frac{\lambda(\frac{R}{I}) + [\lambda(\frac{I}{J}) - \lambda(\frac{I^2}{JI})]t + \sum_{n=2}^k [\lambda(\frac{I^n}{JI^{n-1}}) - \lambda(\frac{I^{n+1}}{JI^n})]t^n + t^r}{(1-t)^{d+1}} \\ &= \frac{\sum_{n=0}^k [\lambda(\frac{I^n}{JI^{n-1}}) - \lambda(\frac{I^{n+1}}{JI^n})]t^n + t^r}{(1-t)^{d+1}} \\ &= \frac{\sum_{n=0}^{k-1} [\lambda(\frac{I^n}{JI^{n-1}}) - \lambda(\frac{I^{n+1}}{JI^n})]t^n + t^r}{(1-t)^{d+1}}.\end{aligned}$$

Multiplying both sides by $(1-t)$, we get

$$\sum_{n\ge 0} \lambda(\frac{I^n}{I^{n+1}})t^n = \frac{\sum_{n=0}^{k-1} [\lambda(\frac{I^n}{JI^{n-1}}) - \lambda(\frac{I^{n+1}}{JI^n})]t^n + t^{r+1}}{(1-t)^d}.$$

On the other hand, note that $JI^n \subseteq JI^{n-1} \cap I^{n+1} \subseteq J \cap I^{n+1} = JI^n$ for all $n \le k-2$, thus $JI^n = JI^{n-1} \cap I^{n+1}$. It follows that

$$\lambda(\frac{I^n}{I^{n+1} + JI^{n-1}}) = \lambda(\frac{I^n}{JI^{n-1}}) - \lambda(\frac{I^{n+1}}{I^{n+1} \cap JI^{n-1}}) = \lambda(\frac{I^n}{JI^{n-1}}) - \lambda(\frac{I^{n+1}}{JI^n}).$$

The proof is completed. □

Corollary 3.4 *Let I be an $\mathfrak{m}$-primary ideal with almost minimal multiplicity with respect to K such that depth $G(I) \ge d-1$ and $r = r_J^K(I)$. Then*

$$\sum_{n\ge 0} H_K(I,n)t^n = \frac{\lambda(\frac{R}{K}) + [e_0(I) - \lambda(\frac{R}{K}) - 1)]t + t^{r+1}}{(1-t)^{d+1}}.$$

Proof Put $k = 1$ in Theorem 3.2. □

Write $P_K(I,n) = g_0'\binom{n+d}{d} - g_1'\binom{n+d-1}{d-1} + \cdots + (-1)^d g_d'$. Then comparing with the earlier notation, we get $g_0' = g_0$ and $g_i' = g_i + g_{i-1}$, $i = 1, \ldots, d$.

Proposition 3.5 *Let $d \ge 1$, k a positive integer and J a minimal reduction of I such that $KI^n \cap J = JKI^{n-1}$ for all $n \le k-1$ and $\lambda(\frac{KI^k}{JKI^{k-1}}) = 1$, and let depth $G(I) \ge d-1$. Then $F_K(I)$ is CM if and only if $\lambda(\frac{KI^n + JKI^{n-1}}{JKI^{n-1}}) = 1$ for all $n = k, \ldots, r$.*

Proof From Theorem 3.2 and Proposition 4.19 of [1], we have that

$$\begin{aligned}g_1' &= \lambda(\frac{K}{J}) - \lambda(\frac{KI}{KJ}) + \sum_{n=2}^{k} n[\lambda(\frac{KI^{n-1}}{KJI^{n-2}}) - \lambda(\frac{KI^n}{KJI^{n-1}})] + r + 1 \\ &= \lambda(\frac{K}{J}) + \sum_{n=1}^{k-1} \lambda(\frac{KI^n}{KJI^{n-1}}) + (r-k+1).\end{aligned}$$

Thus $g_1 = g_1' - g_0 = \sum_{n=1}^{k-1} \lambda(\frac{KI^n}{KJI^{n-1}}) + (r-k+1) - \lambda(\frac{R}{K})$. From the proof of Lemma 2.2, it can easily be seen that $\lambda(\frac{KI^n+JI^{n-1}}{JI^{n-1}}) \le 1$ for all $n \ge k$.

On the other hand, note that $JKI^{n-1} \subseteq JI^{n-1} \cap KI^n \subseteq J \cap KI^n = JKI^{n-1}$ for all $n \le k-1$, thus $JKI^{n-1} = JI^{n-1} \cap KI^n$. As depth $G(I) \ge d-1$, by Theorem 4.3 of [9], $F_K(I)$ is CM if and only if

$$g_1 = \sum_{n\ge 1} \lambda(\frac{KI^n + JI^{n-1}}{JI^{n-1}}) - \lambda(\frac{R}{K}) = \sum_{n=1}^{k-1} \lambda(\frac{KI^n}{KJI^{n-1}}) + (r-k+1) - \lambda(\frac{R}{K})$$

if and only if

$$\sum_{n=1}^{k-1} \lambda(\frac{KI^n}{KJI^{n-1}}) + \sum_{n\ge k} \lambda(\frac{KI^n + JI^{n-1}}{JI^{n-1}}) - \lambda(\frac{R}{K}) = \sum_{n=1}^{k-1} \lambda(\frac{KI^n}{KJI^{n-1}}) + (r-k+1) - \lambda(\frac{R}{K})$$

if and only if $\lambda(\frac{KI^n+JKI^{n-1}}{JKI^{n-1}}) = 1$ for all $n = k, \dots, r$. $\square$

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