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Oscillation of High Order Neutral Delay Difference Equations with Continuous Arguments

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Abstract In this paper, we study oscillation of solutions for a class of high order neutral delay difference equations with variable coefficients

$$\Delta_\tau^m[x(t)-c(t)x(t-\tau)]=(-1)^m p(t)x(t-\sigma),\quad t\ge t_0>0.$$

Some sufficient conditions are obtained for bounded oscillation of the solutions.

Keywords delay difference equation; bounded solution; oscillation; nonoscillation.

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1. Introduction

Recently, a lot of literature focus on the oscillatory theory for delay difference equations, and there have been some results on the second order neutral difference equations [1–8]. However, there are relatively scarce papers related to oscillation for second order neutral difference equations with continuous arguments and even order neutral delay difference equations with continuous arguments [9, 10].

Consider the following m-order neutral delay difference equation with variable coefficients

$$\Delta_\tau^m[x(t)-c(t)x(t-\tau)]=(-1)^m p(t)x(t-\sigma),\quad t\ge t_0>0,\tag{1}$$

where (H): τ, t_0 are fixed real numbers with $\tau\ge 0$, $t_0\ge 0$, m is a positive integer, $\sigma=k\tau$ and k is some positive integer, $p(t)\in C([t_0,+\infty),R^+)$, $p(t)\not\equiv 0$, $\Delta_\tau x(t)=x(t+\tau)-x(t)$, $\Delta_\tau^2 x(t)=\Delta_\tau(\Delta_\tau x(t))$.

By a solution of (1) we mean a continuous function $x(t)$ which is defined for $-\rho\le t\le 0$, where $\rho=\max\{\tau,\sigma\}$, and satisfies Eq.(1) for $t\ge t_0$. Clearly, if

$$x(t)=\Phi(t),\quad -\rho\le t\le 0\tag{2}$$

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are given, where $\Phi(t) \in C([-\rho, 0], R)$, and $c(t)$, $p(t)$ of (1) are continuous functions, then Eq.(1) has a unique solution satisfying the initial conditions (2).

As is customary, a solution $x(t)$ of (1) is said to be eventually positive if $x(t) > 0$ for all large t, and eventually negative if $x(t) < 0$ for all large t. A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative.

2. Lemma

In this paper, we will establish the bounded oscillation criteria for Eq.(1). To prove our results, we need the following Lemma.

Lemma 2.1 *Assume that there exists a constant* α $(0 \le \alpha < 1)$ *such that* $\alpha \le c(t) < 1$. *If* $x(t)$ *is an eventually positive bounded solution of (1), set* $y(t) = x(t) - c(t)x(t-\tau)$, *then we have eventually* $(-1)^m \Delta_\tau^m y(t) \ge 0$, $(-1)^m \Delta_\tau^{m-1} y(t) < 0$, $(-1)^m \Delta_\tau^{m-2} y(t) > 0, \ldots, \Delta_\tau y(t) < 0, y(t) > 0$.

Proof Assume that $x(t)$ is an eventually positive bounded solution of Eq.(1) such that

$$x(t) > 0, x(t-\tau) > 0, x(t-\sigma) > 0, \quad t \ge t_1 \ge t_0.$$

Set

$$y(t) = x(t) - c(t)x(t-\tau), \tag{3}$$

then

$$(-1)^m \Delta_\tau^m y(t) = (-1)^{2m} p(t) x(t-\sigma) \ge 0,$$

that is

$$(-1)^m \Delta_\tau^{m-1} y(t+\tau) \ge (-1)^m \Delta_\tau^{m-1} y(t), \quad t \ge t_1. \tag{4}$$

From (H), we have $\Delta_\tau^m y(t) \not\equiv 0$, which means that either eventually $(-1)^m \Delta_\tau^{m-1} y(t) > 0$ or eventually $(-1)^m \Delta_\tau^{m-1} y(t) < 0$. Furthermore, we have $(-1)^m \Delta_\tau^{m-2} y(t), \ldots, \Delta_\tau y(t)$ are either eventually positive or eventually negative. If $(-1)^m \Delta_\tau^{m-1} y(t) > 0$, then there exists $\tilde{t_0} \ge t_1$ such that $(-1)^m \Delta_\tau^{m-1} y(\tilde{t_0}) > 0$. Thus owing to (4), we have

$$(-1)^m \Delta_\tau^{m-1} y(\tilde{t_0} + i\tau) \ge (-1)^m \Delta_\tau^{m-1} y(\tilde{t_0}), \quad i \ge 1.$$

Summing up the above inequality from $i = 1$ to $i = n$, where n is a positive integer, we get

$$(-1)^m \Delta_\tau^{m-2} y[\tilde{t_0} + (n+1)\tau] - (-1)^m \Delta_\tau^{m-2} y(\tilde{t_0} + \tau) \ge n \cdot (-1)^m \Delta_\tau^{m-1} y(\tilde{t_0}).$$

Hence

$$\lim_{n\to\infty} (-1)^m \Delta_\tau^{m-2} y[\tilde{t_0} + (n+1)\tau] = +\infty,$$

which contradicts the fact that $x(t)$ is eventually bounded, and so $(-1)^m \Delta_\tau^{m-1} y(t) < 0$ eventually holds. By the same argument as in the above proof, we have eventually $(-1)^m \Delta_\tau^{m-2} y(t) > 0, \ldots, \Delta_\tau y(t) < 0$. It follows that $y(t) > 0$ eventually holds. Otherwise, eventually $y(t) < 0$, then there must exist constants $\tilde{t_1} \ge t_1$ and $\mu > 0$ such that

$$y(\tilde{t_1}) \le -\mu.$$

Thus

$$y(\tilde{t_1}+i\tau) \le -\mu, \quad i=1,2,\dots .$$

From (3) and the fact that $\alpha \le c(t) < 1$, we have

$$x(\tilde{t_1}+i\tau) < x[\tilde{t_1}+(i-1)\tau] - \mu, \quad i=1,2,\dots .$$

Summing up the above inequality from $i=1$ to $i=n$, we obtain

$$x(\tilde{t_1}+n\tau) < x(\tilde{t_1}) - n\cdot\mu.$$

Therefore

$$\lim_{n\to\infty} x(\tilde{t_1}+n\tau) = -\infty,$$

which contradicts the fact that $x(t)$ is an eventually positive bounded solution, so eventually $y(t)>0$. The proof is thus completed. □

3. Results and Proofs

Theorem 3.1 *Assume that (H) holds and there exists a constant α $(0 \le \alpha < 1)$ such that $\alpha \le c(t) < 1$. For $t \ge t_0$, $k > 1$, if*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} C_{i-(n-k)+m-1}^{m-1} p(t+i\tau) > 1-\alpha, \tag{5}$$

then every bounded solution of Eq.(1) oscillates.

Proof Assume for the sake of contradiction that Eq.(1) has an eventually positive bounded solution $x(t)$ such that

$$x(t-\tau)>0,\ x(t-\sigma)>0, \quad t \ge t_1 \ge t_0.$$

Set

$$y(t) = x(t) - c(t)x(t-\tau). \tag{6}$$

Then by Lemma, there exists $t_2 \ge t_1$ such that

$$(-1)^m \Delta_\tau^m y(t) \ge 0, (-1)^m \Delta_\tau^{m-1} y(t) < 0, \dots, \Delta_\tau y(t) < 0, y(t) > 0, t \ge t_2.$$

Let $h(t) = [\frac{t-t_2}{\tau}]$, where $[\cdot]$ denotes the greatest integer function. Then we have

$$\begin{aligned} x(t) =& y(t) + c(t)x(t-\tau) \ge y(t) + \alpha x(t-\tau) \\ =& y(t) + \alpha[y(t-\tau) + c(t-\tau)x(t-2\tau)] \\ \ge& y(t) + \alpha y(t-\tau) + \alpha^2 x(t-2\tau) \\ \ge& y(t) + \alpha y(t-\tau) + \alpha^2 y(t-2\tau) + \dots + \alpha^{h(t)-1} y[t-(h(t)-1)\tau] + \\ & \alpha^{h(t)} x(t-h(t)\tau), \quad t \ge t_2+\tau. \end{aligned}$$

From the fact that $\Delta_\tau y(t) < 0$, we get

$$x(t) \ge (1+\alpha+\alpha^2+\dots+\alpha^{h(t)-1}) y(t) = \frac{1-\alpha^{h(t)}}{1-\alpha} y(t), \quad t \ge t_2+\tau. \tag{7}$$

By means of (5), we know there exists a sufficiently small positive number ε $(0<\varepsilon<1)$ such that

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n} C_{i-(n-k)+m-1}^{m-1}p(t+i\tau)>\frac{1-\alpha}{1-\varepsilon}. \tag{8}$$

For this ε, since $h(t)\to\infty$ $(t\to\infty)$, $0\le\alpha<1$, there must exist $t_3\ge t_2+\tau$ such that

$$\frac{1-\alpha^{h(t)}}{1-\alpha}\ge\frac{1-\varepsilon}{1-\alpha},\quad t\ge t_3.$$

By (7), we get

$$x(t)\ge\frac{1-\alpha^{h(t)}}{1-\alpha}y(t)\ge\frac{1-\varepsilon}{1-\alpha}y(t),\quad t\ge t_3. \tag{9}$$

For sufficiently large $t_4>t_3$, let $t=t_4+i\tau$, where i is a positive integer. From (7) and the fact that $\sigma=k\tau$, we obtain

$$\Delta_\tau^m y(t_4+i\tau)=(-1)^m p(t_4+i\tau)x(t_4+i\tau-k\tau),$$

that is

$$\Delta_\tau^{m-1}y[t_4+(i+1)\tau]-\Delta_\tau^{m-1}y(t_4+i\tau)=(-1)^m p(t_4+i\tau)x[t_4+(i-k)\tau].$$

Summing up the last inequality from $i<n$ to n, we get

$$\Delta_\tau^{m-1}y[t_4+(n+1)\tau]-\Delta_\tau^{m-1}y(t_4+i\tau)=(-1)^m\sum_{j=i}^{n}p(t_4+j\tau)x[t_4+(j-k)\tau].$$

In view of the Lemma 2.1, it follows that

$$(-1)^m\Delta_\tau^{m-1}y(t_4+i\tau)\le-\sum_{j=i}^{n}p(t_4+j\tau)x[t_4+(j-k)\tau]. \tag{10}$$

Hence, summing up (10) from $i=n-k$ to $i=n$, we have

$$-(-1)^m\Delta_\tau^{m-2}y[t_4+(n-k)\tau]\le-\sum_{i=n-k}^{n}\sum_{j=i}^{n}p(t_4+j\tau)x[t_4+(j-k)\tau].$$

By repeating the same procedure $(m-1)$ times, we obtain

$$y[t_4+(n+1)\tau]-y[t_4+(n-k)\tau]\le-\sum_{v=n-k}^{n}\sum_{r=v}^{n}\cdots\sum_{i=t}^{n}\sum_{j=i}^{n}p(t_4+j\tau)x[t_4+(j-k)\tau].$$

Consequently, we get

$$\begin{aligned}
&y[t_4+(n+1)\tau]-y[t_4+(n-k)\tau]\le-\sum_{i=n-k}^{n}C_{i-(n-k)+m-1}^{m-1}p(t_4+i\tau)x[t_4+(i-k)\tau]\\
&\le-\sum_{i=n-k}^{n}C_{i-(n-k)+m-1}^{m-1}p(t_4+i\tau)\frac{1-\varepsilon}{1-\alpha}y[t_4+(i-k)\tau]\\
&\le-\frac{1-\varepsilon}{1-\alpha}y[t_4+(n-k)\tau]\sum_{i=n-k}^{n}C_{i-(n-k)+m-1}^{m-1}p(t_4+i\tau).
\end{aligned}$$

Thus

$$y[t_4+(n+1)\tau]+y[t_4+(n-k)\tau][\frac{1-\varepsilon}{1-\alpha}\sum_{i=n-k}^{n}C_{i-(n-k)+m-1}^{m-1}p(t_4+i\tau)-1]\le 0,$$

which contradicts (8). The proof is completed. □

Theorem 3.2 *Assume that (H) holds and there exists a constant α such that $\alpha \le c(t) < 0$. Set $k > 1$. If*

$$\limsup_{t\to\infty}[-c(t-\sigma)\frac{p(t)}{p(t-\tau)}]=\beta\in(0,+\infty), \tag{11}$$

and for $t \ge t_0$,

$$\limsup_{n\to\infty}\sum_{i=n+1-k}^{n}C_{i-(n+1-k)+m-1}^{m-1}p(t+i\tau)>1+\beta, \tag{12}$$

then every bounded solution of Eq.(1) oscillates.

Proof For the sake of contradiction, assume that Eq.(1) has an eventually positive bounded solution $x(t)$. Set $y(t)=x(t)-c(t)x(t-\tau)$. By Lemma 2.1, there exists $t_1\ge t_0$ such that

$$(-1)^m\Delta_\tau^m y(t)\ge 0,\ (-1)^m\Delta_\tau^{m-1}y(t)<0,\ldots,\Delta_\tau y(t)<0, y(t)>0,\quad t\ge t_1.$$

By means of (12), there must exist a constant $\mu>1$ such that

$$\limsup_{n\to\infty}\sum_{i=n+1-k}^{n}C_{i-(n+1-k)+m-1}^{m-1}p(t+i\tau)>1+\mu\beta. \tag{13}$$

For this μ, it follows from (11) that there must exist $t_2\ge t_1$ such that

$$-c(t-\sigma)\frac{p(t)}{p(t-\tau)}\le\mu\beta,\quad t\ge t_2. \tag{14}$$

In view of (1), we get

$$\begin{aligned}&\Delta_\tau^m y(t)-c(t-\sigma)\frac{p(t)}{p(t-\tau)}\Delta_\tau^m y(t-\tau)\\&=(-1)^m p(t)x(t-\sigma)-c(t-\sigma)\frac{p(t)}{p(t-\tau)}(-1)^m p(t-\tau)x(t-\tau-\sigma)\\&=(-1)^m p(t)[x(t-\sigma)-c(t-\sigma)x(t-\tau-\sigma)]\\&=(-1)^m p(t)y(t-\sigma),\quad t\ge t_2.\end{aligned}$$

From (14), we have $(-1)^m[\Delta_\tau^m y(t)+\mu\beta\Delta_\tau^m y(t-\tau)]\ge p(t)y(t-\sigma)$, $t\ge t_2$.

Set $z(t)=y(t)+\mu\beta y(t-\tau)$. Then

$$(-1)^m\Delta_\tau^m z(t)\ge p(t)y(t-\sigma),\quad t\ge t_2. \tag{15}$$

It is easy to see that

$$(-1)^m\Delta_\tau^m z(t)\ge 0,\ (-1)^m\Delta_\tau^{m-1}z(t)<0,\ldots,\Delta_\tau z(t)<0,\ z(t)>0,\ t\ge t_2.$$

On the other hand, on account of $\Delta_\tau y(t)<0$, we have

$$z(t)=y(t)+\mu\beta y(t-\tau)\le(1+\mu\beta)y(t-\tau),\quad t\ge t_2.$$

That is

$$y(t) \geq \frac{1}{1+\mu\beta} z(t+\tau), \quad t \geq t_2 + \tau = t_3.$$

Substituting the above inequality into (15) yields

$$(-1)^m \Delta_\tau^m z(t) \geq p(t) \frac{1}{1+\mu\beta} z(t-\sigma+\tau), \quad t \geq t_3 + \sigma. \tag{16}$$

Take a sufficiently large number $t_4 > t_3 + \sigma$ and set $t = t_4 + i\tau$, where i is a positive integer. Then from the fact that $\sigma = k\tau$, we get

$$(-1)^m \Delta_\tau^m z(t_4 + i\tau) \geq \frac{1}{1+\mu\beta} p(t_4 + i\tau) z[t_4 + (i+1-k)\tau].$$

Summing up the last inequality from i to n $(n \geq i)$, we have

$$(-1)^m \Delta_\tau^{m-1} z[t_4 + (n+1)\tau] - (-1)^m \Delta_\tau^{m-1} z(t_4 + i\tau) \geq \frac{1}{1+\mu\beta} \sum_{j=i}^{n} p(t_4 + j\tau) z[t_4 + (j+1-k)\tau].$$

Since $(-1)^m \Delta_\tau^{m-1} z[t_4 + (n+1)\tau] < 0$, we obtain

$$-(-1)^m \Delta_\tau^{m-1} z(t_4 + i\tau) \geq \frac{1}{1+\mu\beta} \sum_{j=i}^{n} p(t_4 + j\tau) z[t_4 + (j+1-k)\tau].$$

Summing up the above inequality from $i = n+1-k$ to $i = n$, we have

$$\begin{aligned} &-(-1)^m \Delta_\tau^{m-2} z[t_4 + (n+1)\tau] + (-1)^m \Delta_\tau^{m-2} z[t_4 + (n+1-k)\tau] \\ &\geq \frac{1}{1+\mu\beta} \sum_{i=n+1-k}^{n} \sum_{j=i}^{n} p(t_4 + j\tau) z[t_4 + (j+1-k)\tau]. \end{aligned}$$

Since $(-1)^m \Delta_\tau^{m-2} z[t_4 + (n+1)\tau] > 0$, we get

$$(-1)^m \Delta_\tau^{m-2} z[t_4 + (n+1-k)\tau] \geq \frac{1}{1+\mu\beta} \sum_{i=n+1-k}^{n} \sum_{j=i}^{n} p(t_4 + j\tau) z[t_4 + (j+1-k)\tau].$$

Repeating the same procedure $(m-1)$ times gives

$$\begin{aligned} &z[t_4 + (n+1-k)\tau] - z[t_4 + (n+1)\tau] \\ &\geq \frac{1}{1+\mu\beta} \sum_{v=n+1-k}^{n} \sum_{r=v}^{n} \cdots \sum_{i=t}^{n} \sum_{j=i}^{n} p(t_4 + j\tau) z[t_4 + (j+1-k)\tau] \\ &= \frac{1}{1+\mu\beta} \sum_{i=n+1-k}^{n} C_{i-(n+1-k)+m-1}^{m-1} p(t_4 + i\tau) z[t_4 + (i+1-k)\tau]. \end{aligned}$$

From the fact that $\Delta_\tau z(t) < 0$, we have

$$\begin{aligned} &z[t_4 + (n+1-k)\tau] - z[t_4 + (n+1)\tau] \\ &\geq \frac{1}{1+\mu\beta} z[t_4 + (n+1-k)\tau] \sum_{i=n+1-k}^{n} C_{i-(n+1-k)+m-1}^{m-1} p(t_4 + i\tau). \end{aligned}$$

Thus

$$z[t_4 + (n+1)\tau] + z[t_4 + (n+1-k)\tau] [\frac{1}{1+\mu\beta} \sum_{i=n+1-k}^{n} C_{i-(n+1-k)+m-1}^{m-1} p(t_4 + i\tau) - 1] \leq 0,$$

which contradicts (13). The proof is completed. □

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