Lie Triple Derivations on Upper Triangular Matrices over a Commutative Ring

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Abstract  Let \( T(n, R) \) be the Lie algebra consisting of all \( n \times n \) upper triangular matrices over a commutative ring \( R \) with identity 1 and \( \mathcal{M} \) be a 2-torsion free unital \( T(n, R) \)-bimodule. In this paper, we prove that every Lie triple derivation \( d : T(n, R) \to \mathcal{M} \) is the sum of a Jordan derivation and a central Lie triple derivation.

Keywords  Jordan derivation; Lie triple derivation; upper triangular matrices.

1. Introduction

Throughout the paper, let \( R \) be a commutative ring with identity 1 and \( T(n, R) \) be the Lie algebra consisting of all \( n \times n \) upper triangular matrices over \( R \) and with the bracket operation \([x, y] = xy - yx\). We denote by \( E \) the identity matrix in \( T(n, R) \) and \( E_{ij} \) the matrix in \( T(n, R) \) whose sole nonzero entry 1 is in the \((i, j)\) position.

Let \( A \) be an associative algebra over \( R \) and \( \mathcal{M} \) an \( A \)-bimodule. Recall that an \( R \)-linear mapping \( d : A \to \mathcal{M} \) is called a Jordan derivation if
\[
d(xy + yx) = d(x)y + xd(y) + d(y)x + yd(x)
\]
for all \( x, y \in A \). Let \([m, x] = mx - xm\) for all \( x \in A \) and \( m \in \mathcal{M} \). The \( d : A \to \mathcal{M} \) is called a Lie triple derivation if it satisfies
\[
d([x, y], z) = [[d(x), y], z] + [x, [d(y), z]] + [[x, y], d(z)]
\]
for all \( x, y, z \in A \).

Since
\[
[x, y], z = x(yz + zy) + (yz + zy)x - y(xz + zx) - (xz + zx)y,
\]
every Jordan derivation is also a Lie triple derivation, while the converse may not be true.

Recently, significant work has been done in studying the Lie derivations, Jordan derivations and Lie triple derivations on the general linear Lie algebra and its subalgebras over a commutative ring, such as Lie derivations on the subalgebra consisting of all \( n \times n \) upper triangular matrices...
Let $m$ be a torsion free if $2m = 0$. It was shown in [8] that every Jordan derivation from $T(n, R)$ into a 2-torsion free unital $T(n, R)$-bimodule $M$ is the sum of a derivation and an antiderivation. It was shown in [9] that every Lie derivation from $T(n, R)$ into a 2-torsion free unital $T(n, R)$-bimodule $M$ is the sum of a derivation and a central Lie derivation. However, the study on the Lie triple derivations from a Lie algebra into its arbitrary bimodule has not been reported. In this paper, we give a decomposition of a Lie triple derivation $d : T(n, R) \to M$ and then show the relationship between Lie triple derivations and Jordan derivations.

2. Lie triple derivations from $T(n, R)$ into $M$

Let $d : T(n, R) \to M$ be a Lie triple derivation and $M$ be a 2-torsion free unital $T(n, R)$-bimodule and $Z^2(M) = \{m \in M||m, [x, y]| = 0, \text{ for all } x, y \in T(n, R)\}$. We say that $M$ is 2-torsion free if $2m = 0$, then $m = 0$.

**Lemma 2.1** Let $d : T(n, R) \to M$ be a Lie triple derivation. Then for all $1 \leq i < j \leq n$, we have

$$d(e_{ii}) = \sum_{k \neq i} e_{ii}d(e_{ii})e_{kk} + \sum_{k \neq i} e_{kk}d(e_{ii})e_{ii} + \sum_{k=1}^{n} e_{kk}d(e_{ii})e_{kk}, \quad (2.1)$$

$$d(e_{jj}) = d(e_{ii})e_{ij} - e_{jj}d(e_{ij})e_{jj} + e_{ii}d(e_{ij})e_{ii} + e_{ij}d(e_{jj})e_{ii} - e_{ii}d(e_{jj})e_{ij}. \quad (2.2)$$

**Proof** Let $i \neq j, j \neq k, i \neq k$. Then

$$0 = d([e_{ii}, e_{jj}], e_{kk}) = [[d(e_{ii}), e_{jj}], e_{kk}] + [e_{ii}, d(e_{jj})], e_{kk}]$$

$$= [d(e_{ii}), e_{jj} - e_{jj}d(e_{ii})], e_{kk}] + [e_{ii}d(e_{jj}) - d(e_{jj})e_{ii}], e_{kk}]$$

$$= -e_{jj}d(e_{ii})e_{kk} - e_{kk}d(e_{ii})e_{jj} + e_{ii}d(e_{jj})e_{kk} + e_{kk}d(e_{jj})e_{ii}.$$

Multiplying this identity from the left by $e_{kk}$ and from the right by $e_{jj}$, we obtain

$$e_{kk}d(e_{ii})e_{jj} = 0 \quad \text{for all } i \neq j, j \neq k, i \neq k. \quad (2.3)$$

Let $i \neq j$. Then

$$0 = d([e_{ii}, e_{jj}], e_{ii}) = [[d(e_{ii}), e_{jj}], e_{ii}] + [e_{ii}, d(e_{jj})], e_{ii}]$$

$$= [d(e_{ii}), e_{jj} - e_{jj}d(e_{ii})], e_{ii}] + [e_{ii}d(e_{jj}) - d(e_{jj})e_{ii}], e_{ii}]$$

$$= -e_{jj}d(e_{ii})e_{ii} - e_{ii}d(e_{ii})e_{jj} + e_{ii}d(e_{jj})e_{ii} - d(e_{jj})e_{ii} - e_{ii}d(e_{jj})e_{ii} + e_{ii}d(e_{jj})e_{ii}.$$

Multiplying this identity from the left by $e_{jj}$ and from the right by $e_{ii}$ gives

$$e_{jj}d(e_{jj})e_{ii} + e_{jj}d(e_{ii})e_{ii} = 0 \quad \text{for all } i \neq j. \quad (2.4)$$

Using $d(e_{ii}) = Ed(e_{ii}) = \sum_{j, k=1}^{n} e_{jj}d(e_{ii})e_{kk}$ in (2.3), we obtain (2.1).

From (2.1) and (2.4), we see that

$$e_{jj}d(e_{ii})e_{ij} + d(e_{jj})e_{ij} = e_{jj}d(e_{ii})e_{ij} + e_{jj}d(e_{jj})e_{ij} + e_{ii}d(e_{jj})e_{ij}$$

$$= (e_{jj}d(e_{ii})e_{ii} + e_{jj}d(e_{jj})e_{ii})e_{ij} + e_{ii}d(e_{jj})e_{ij}.$$
We only need to prove $0 = \tau$.

**Case 1**

Put Lemma 2.2.

Multiplying this identity from the left by $e$ gives

$$
0 = \tau(x, y, z) = \tau([x, [y, z]]) = [x, [y, d(z)]] = [x, [y, d(z)]] \quad \text{for all } x, y, z \in \mathcal{T}(n, R).
$$

**Lemma 2.2** Put $\tau(e_{ii}) = \sum_{k=1}^{n} e_{kk}d(e_{ii})e_{kk}$. Then $\tau : \mathcal{T}(n, R) \rightarrow \mathcal{Z}^{2}(\mathcal{M})$ is a central Lie triple derivation.

**Proof** It is obvious that for all $x, y, z \in \mathcal{T}(n, R)$, $\tau([x, [y, z]]) = 0$. Firstly, we prove that

$$0 = \tau(x, y, z).$$

We only need to prove $0 = [\tau(e_{ii}), e_{kl}]$ for all $1 \leq k < l \leq n$. We discuss $k, l$ in the following cases, respectively.

**Case 1** $k \neq i, l \neq i$

We have

$$0 = d([e_{ii}, [e_{kk}, e_{kl}]] = [d(e_{ii}), [e_{kk}, e_{kl}]] + [e_{ii}, [d(e_{kk}), e_{kl}]] + [e_{ii}, [e_{kk}, d(e_{kl})]]$$

$$= [d(e_{ii}), e_{kl}] + [e_{ii}, d(e_{kk})e_{kl} - e_{kl}d(e_{kk})] + [e_{ii}, e_{kk}d(e_{kl}) - d(e_{kl})e_{kk}]$$

$$= d(e_{ii})e_{kl} - e_{kl}d(e_{ii}) + e_{ii}d(e_{kk})e_{kl} + e_{kl}d(e_{kk})e_{ii} - e_{ii}d(e_{kl})e_{kk} - e_{kk}d(e_{kl})e_{ii}.$$  

Multiplying this identity from the left by $e_{kk}$ and from the right by $e_{ll}$ gives

$$e_{kk}d(e_{ii})e_{kl} - e_{kl}d(e_{ii})e_{ll} = 0.$$

Then

$$\tau(e_{ii}), e_{kl}] = \tau(e_{ii})e_{kl} - e_{kl}\tau(e_{ii}) = e_{kk}d(e_{ii})e_{kl} - e_{kl}d(e_{ii})e_{ll} = 0.$$

**Case 2** $k = i, l \neq i$

We have

$$d(e_{ii}) = d([e_{ii}, [e_{ii}, e_{il}]] = [d(e_{ii}), [e_{ii}, e_{il}]] = [e_{ii}, [d(e_{ii}), e_{il}]] + [e_{ii}, [e_{ii}, d(e_{il})]]$$

$$= [d(e_{ii}), e_{il}] + [e_{ii}, d(e_{ii})e_{il} - e_{il}d(e_{ii})] + [e_{ii}, e_{il}d(e_{il}) - d(e_{il})e_{ii}]$$

$$= d(e_{ii})e_{il} - e_{il}d(e_{ii}) + e_{ii}d(e_{ii})e_{il} - e_{il}d(e_{ii}) + e_{il}d(e_{ii})e_{ii} + e_{ii}d(e_{il}) - 2e_{ii}d(e_{il})e_{ii} + d(e_{il})e_{ii}. $$

Multiplying this identity from the left by $e_{ii}$ and from the right by $e_{il}$ yields

$$2(e_{ii}d(e_{ii})e_{il} - e_{il}d(e_{ii})e_{ll}) = 0.$$

As $\mathcal{M}$ is 2-torsion free, we have $e_{ii}d(e_{ii})e_{il} - e_{il}d(e_{ii})e_{ll} = 0$. Then

$$[\tau(e_{ii}), e_{il}] = \tau(e_{ii})e_{il} - e_{il}\tau(e_{ii}) = e_{ii}d(e_{ii})e_{il} - e_{il}d(e_{ii})e_{ll} = 0.$$
Case 3  $k \neq i, \ l = i$

The proof is the same as in Case 2. Summing up, we have

$$0 = [\tau(x), [y, z]] \text{ for all } x, y, z \in T(n, R).$$

Secondly, we prove that $0 = [[\tau(x), y], z]$ for all $x, y, z \in T(n, R)$. We only need to prove

$$0 = [[\tau(e_{ii}), e_{kk}], e_{pq}], \ 1 \leq i, k \leq n, \ 1 \leq p \leq q \leq n.$$

For $1 \leq i, k \leq n, \ 1 \leq p \leq q \leq n$, we have

$$[[\tau(e_{ii}), e_{kk}], e_{pq}] = [\tau(e_{ii}) e_{kk} + e_{pq} \tau(e_{ii})]$$

We discuss $p, q$ in the following cases, respectively.

(1) $p = q$

$$\delta_{kp} \tau(e_{ii}) e_{kq} - e_{kk} \tau(e_{ii}) e_{pq} - e_{pq} \tau(e_{ii}) e_{kk} + \delta_{kq} e_{pk} \tau(e_{ii}).$$

(2) $k \neq p$

By the form of $\tau$, it is obvious that

$$\delta_{kp} \tau(e_{ii}) e_{kq} - e_{kk} \tau(e_{ii}) e_{pq} - e_{pq} \tau(e_{ii}) e_{kk} + \delta_{kq} e_{pk} \tau(e_{ii}) = 0.$$

(2) $k = p$

As $\tau(e_{ii}) = \sum_{k=1}^{n} e_{kk} d(e_{ii}) e_{kk}$, we get

$$\tau(e_{ii}) e_{mm} = e_{mm} \tau(e_{ii}) e_{mm} = e_{mm} \tau(e_{ii}) \text{ for all } 1 \leq m \leq n. \quad (2.5)$$

Then

$$\delta_{kp} \tau(e_{ii}) e_{kp} - e_{kk} \tau(e_{ii}) e_{pp} - e_{pp} \tau(e_{ii}) e_{kk} + \delta_{kq} e_{pk} \tau(e_{ii}) = 0.$$

(II) $p \neq q$

(1) $k = p$

By the form of $\tau$ and (2.5), we have

$$\delta_{kp} \tau(e_{ii}) e_{kq} - e_{kk} \tau(e_{ii}) e_{pq} - e_{pq} \tau(e_{ii}) e_{kk} + \delta_{kq} e_{pk} \tau(e_{ii})$$

$$= \tau(e_{ii}) e_{pp} - e_{pp} \tau(e_{ii}) e_{pp} - e_{pp} \tau(e_{ii}) e_{pp} + e_{pp} \tau(e_{ii}) = 0.$$

(2) $k = q$

By the form of $\tau$ and (2.5), we have

$$\delta_{kp} \tau(e_{ii}) e_{kq} - e_{kk} \tau(e_{ii}) e_{pq} - e_{pq} \tau(e_{ii}) e_{kk} + \delta_{kq} e_{pk} \tau(e_{ii})$$

$$= -e_{qq} \tau(e_{ii}) e_{pq} - e_{pq} \tau(e_{ii}) e_{qq} + e_{qp} \tau(e_{ii})$$

$$= -(e_{qq} \tau(e_{ii}) e_{pp} - e_{pp} (e_{qq} \tau(e_{ii}) e_{qq} - e_{qq} \tau(e_{ii}))) = 0.$$
(3) $k \neq p, k \neq q$

By the form of $\tau$, we have

$$
\delta_{kp}\tau(e_{ii})e_{kq} - e_{kk}\tau(e_{ii})e_{pq} - e_{pq}\tau(e_{ii})e_{kk} + \delta_{kq}e_{pk}\tau(e_{ii}) =
\begin{equation}
= -e_{kk}\tau(e_{ii})e_{pq} - e_{pq}\tau(e_{ii})e_{kk},
\end{equation}
\begin{equation}
= -(e_{kk}\tau(e_{ii})e_{pp}e_{pq} - e_{pq}(e_{qq}\tau(e_{ii})e_{kk})) = 0.
\end{equation}
$$

Summing up, we get $0 = [[\tau(x), y], z]$. Then

$$
0 = \tau([x, [y, z]]) = [\tau(x), [y, z]] = [x, [\tau(y), z]] = [x, [y, \tau(z)]],
$$

which completes the proof of Lemma 2.2. □

Define $\Delta : \mathcal{T}(n, R) \rightarrow \mathcal{M}$ by $\Delta = d - \tau$. Clearly, $\Delta$ is a Lie triple derivation and by (2.1) we get

$$
\Delta(e_{ii}) = \sum_{k \neq i} e_{ii}d(e_{ii})e_{kk} + \sum_{k \neq i} e_{kk}d(e_{ii})e_{ii} \quad \text{for all } 1 \leq i \leq n. \quad (6)
$$

Multiplying (6) from the the left and from the right by $e_{jj}$, we get

$$
e_{jj}\Delta(e_{ii})e_{jj} = 0 \quad \text{for all } 1 \leq i, j \leq n. \quad (7)
$$

As $\Delta$ is a Lie triple derivation, by Lemma 2.1 and (7) we get

$$
\Delta(e_{ij}) = \Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij})e_{jj} + e_{jj}\Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{jj}), \quad (8)
$$

for all $1 \leq i \leq j \leq n$.

**Lemma 2.3** $\Delta$ is a Jordan derivation.

**Proof** It suffices to show that

$$
\Delta(e_{ij}e_{kl} + e_{kl}e_{ij}) = \Delta(e_{ij})e_{kl} + e_{ij}\Delta(e_{kl}) + \Delta(e_{kl})e_{ij} + e_{kl}\Delta(e_{ij}), \quad (9)
$$

for all $1 \leq i \leq j \leq n$, $1 \leq k \leq l \leq n$. We discuss $i, j, k, l$ in the following cases, respectively.

**Case 1** $j = k, i = l$

Then $i = j = k = l$. It is obvious that $\Delta(e_{ii}) = \Delta(e_{ii})e_{ii} + e_{ii}\Delta(e_{ii})$, for all $1 \leq i \leq n$, which implies (9) holds.

**Case 2** $j \neq k, i \neq l$

It is obvious that $0 = \Delta(e_{ij}e_{kl} + e_{kl}e_{ij})$, and

$$
\Delta(e_{ij})e_{kl} + e_{ij}\Delta(e_{kl}) + \Delta(e_{kl})e_{ij} + e_{kl}\Delta(e_{ij}) =
\begin{equation}
= \delta_{ik}e_{ij}\Delta(e_{ij})e_{il} + \delta_{ik}e_{ij}\Delta(e_{kl})e_{kj} + e_{ij}\Delta(e_{jj})e_{kl} + e_{ij}\Delta(e_{kk})e_{kl} +
\end{equation}
\begin{equation}
\delta_{jl}e_{ii}\Delta(e_{kl})e_{kk} + \delta_{jl}e_{kk}\Delta(e_{ij})e_{ij} + e_{kl}\Delta(e_{il})e_{ij} + e_{kl}\Delta(e_{ii})e_{ij}.
\end{equation}
$$

(1) $i \neq k, j \neq l$

By (2.3), we get

$$
\Delta(e_{ij})e_{kl} + e_{ij}\Delta(e_{kl}) + \Delta(e_{kl})e_{ij} + e_{kl}\Delta(e_{ij})
$$
\[ e_{ij} \Delta(e_{jj}) e_{kl} + e_{ij} \Delta(e_{kk}) e_{kl} + e_{kl} \Delta(e_{il}) e_{ij} + e_{kl} \Delta(e_{ii}) e_{ij} = e_{ij} (e_{jj} \Delta(e_{jj}) e_{kk} + e_{jj} \Delta(e_{kk}) e_{kl} + e_{kl} (e_{il} \Delta(e_{il}) e_{ii} + e_{il} \Delta(e_{ii}) e_{ij} e_{ij} = 0. \]

(2) \( i = k, j = l \)

Since \( i \neq j \), we have

\[(2.10) e_{jj} \Delta(e_{ij}) e_{ij} = 0. \]
\[(2.11) e_{ij} \Delta(e_{ii}) e_{ii} = 0. \]

By (2.4), (2.10) and (2.11), we get

\[ \Delta(e_{ij}) e_{kl} + e_{ij} \Delta(e_{kl}) + \Delta(e_{kl}) e_{ij} + e_{kl} \Delta(e_{ij}) = 2(\Delta(e_{ij}) e_{ij} + e_{ij} \Delta(e_{ij})) = 2(e_{jj} \Delta(e_{ij}) e_{ij} + e_{ij} \Delta(e_{jj}) e_{ij} + e_{ij} \Delta(e_{ij}) e_{ii} + e_{ii} \Delta(e_{ij}) e_{il}) = 2(e_{jj} \Delta(e_{ij}) e_{ij} + e_{ij} \Delta(e_{ii}) e_{ii} + 2e_{ij} (e_{jj} \Delta(e_{ij}) e_{ii} + e_{jj} \Delta(e_{ii}) e_{ij} e_{ij} = 0. \]

(3) \( i = k, j \neq l \)

Then \( i \neq j, i \neq l \). We have

\[ 0 = \Delta([e_{il}, [e_{ij}, e_{jj}]] = [\Delta(e_{il}), e_{ij}] + [e_{il}, [\Delta(e_{ij}), e_{jj}]] + [e_{il}, [e_{ij}, \Delta(e_{jj})]] = \Delta(e_{il}) e_{ij} - e_{ij} \Delta(e_{il}) + e_{il} \Delta(e_{ij}) e_{ij} + e_{jj} \Delta(e_{ij}) e_{il} - e_{il} \Delta(e_{jj}) e_{ij} - e_{ij} \Delta(e_{jj}) e_{il}. \]

Multiplying this identity from the left by \( e_{jj} \) and from the right by \( e_{il} \), we get

\[ e_{jj} \Delta(e_{ij}) e_{il} = 0. \]

Based on the same argument, from

\[ 0 = \Delta([e_{ij}, [e_{ii}, e_{il}]]), \]

it follows \( e_{ii} \Delta(e_{ii}) e_{ij} = 0. \) Then

\[ \Delta(e_{ij}) e_{kl} + e_{ij} \Delta(e_{kl}) + \Delta(e_{kl}) e_{ij} + e_{kl} \Delta(e_{ij}) = e_{jj} \Delta(e_{ij}) e_{il} + e_{ij} \Delta(e_{jj}) e_{il} + e_{ij} \Delta(e_{ij}) e_{il} + e_{il} \Delta(e_{il}) e_{ij} + e_{il} \Delta(e_{il}) e_{ij} + e_{il} \Delta(e_{ii}) e_{ij} \]
= (e_{jj} \Delta(e_{ij})e_{il} + e_{il} \Delta(e_{il})e_{ij}) + e_{ij}(e_{jj} \Delta(e_{jj})e_{ii} + \\
e_{jj} \Delta(e_{ii})e_{ij} + e_{il} \Delta(e_{il})e_{ii} + e_{il} \Delta(e_{ii})e_{ij}) \\
= 0.

(4) \ i \neq k, \ j = l
The proof is the same as (3).

**Case 3** \ j = k, \ i \neq l. Then
\[ \Delta(e_{il}) = \Delta(e_{ij}e_{jl} + e_{jl}e_{ij}), \]
and
\[ \Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl}) + \Delta(e_{jl})e_{ij} + e_{jl}\Delta(e_{ij}) = \Delta(e_{ii})e_{il} + e_{ii}\Delta(e_{ii})e_{il} + \]
\[ e_{ii}\Delta(e_{il})e_{il} + e_{il}\Delta(e_{il})e_{ii} + e_{il}\Delta(e_{ii})e_{il} + e_{il}\Delta(e_{ii})e_{il} = \Delta(e_{il}). \]

(2) \ i \neq j, \ j = l
By (2.4) and (2.7), we get
\[ \Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl}) + \Delta(e_{jl})e_{ij} + e_{jl}\Delta(e_{ij}) = \Delta(e_{ii})e_{il} + e_{ii}\Delta(e_{ii})e_{il} + \]
\[ e_{ii}\Delta(e_{il})e_{il} + e_{il}\Delta(e_{il})e_{ii} + e_{il}\Delta(e_{ii})e_{il} + e_{il}\Delta(e_{ii})e_{il} = \Delta(e_{il}). \]

We have
\[ \Delta(e_{il}) = \Delta([e_{ij}, e_{jl}], e_{il}) \]
\[ = [\Delta(e_{ij})e_{jl} - e_{jl}\Delta(e_{ij}), e_{il}] + [e_{ij}, \Delta(e_{jl}), e_{il}] + [e_{ij}, e_{jl}, \Delta(e_{il})] \]
\[ = \Delta(e_{ij})e_{jl} - e_{jl}\Delta(e_{ij}), e_{il}] + [e_{ij}, \Delta(e_{jl})e_{ij}, e_{il}] + e_{il}\Delta(e_{il}) - \Delta(e_{il})e_{il} \]
\[ = \Delta(e_{ij})e_{jl} - e_{jl}\Delta(e_{ij})e_{il} - e_{il}\Delta(e_{ij})e_{il} + e_{il}\Delta(e_{jl})e_{ij} + e_{il}\Delta(e_{il}) - \Delta(e_{il})e_{il}. \]
Multiplying this identity from the left by $e_{ii}$ and from the right by $e_{ll}$, we obtain
\[
e_{ii}\Delta(e_{ll})e_{ll} = e_{ii}\Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl})e_{ll}.
\]
(2.12)

Multiplying this identity from the left by $e_{ll}$ and from the right by $e_{ii}$, we obtain
\[
e_{ll}\Delta(e_{ll})e_{ii} = 0.
\]
(2.13)

By (2.4), (2.12) and (2.13), we get
\[
\Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl}) + \Delta(e_{jl})e_{ij} + e_{jl}\Delta(e_{ij}) = \Delta(e_{ii})e_{ll} + e_{ii}\Delta(e_{ll})e_{ij} + e_{ll}\Delta(e_{ii})e_{ij}
\]
\[
= \Delta(e_{ij})e_{ll} + e_{ii}\Delta(e_{ll})e_{ij} + e_{ll}\Delta(e_{ii})e_{ij} = \Delta(e_{ii})e_{ll} + e_{ii}\Delta(e_{ll})e_{ij} + e_{ll}\Delta(e_{ii})e_{ij} = \Delta(e_{ll}).
\]

Case 4 \quad j \neq k, i = l

The proof is the same as in Case 3. □

By Lemmas 2.1–2.3, we get the following Theorem 2.4.

**Theorem 2.4** Let $\mathcal{M}$ be a 2-torsion free unital $T(n, R)$-bimodule and $d : T(n, R) \rightarrow \mathcal{M}$ be a Lie triple derivation. Then $d = \Delta + \tau$, where $\Delta : T(n, R) \rightarrow \mathcal{M}$ is a Jordan derivation and $\tau : T(n, R) \rightarrow Z^2(\mathcal{M})$ is a central Lie triple derivation.

If $\mathcal{M} = T(n, R)$, since $e_{jj}\Delta(e_{ij})e_{ii} = 0$, for all $1 \leq i \leq j \leq n$, using the same proof as in Lemma 2.3, we get that $\Delta$ is a derivation. Then we can obtain the following Corollary 2.1.

**Corollary 2.1** Let $d : T(n, R) \rightarrow T(n, R)$ be a Lie triple derivation. Then $d = \Delta + \tau$, where $\Delta : T(n, R) \rightarrow T(n, R)$ is a derivation and $\tau : T(n, R) \rightarrow Z^2(T(n, R))$ is a central Lie triple derivation.

**References**


