Block-Transitive 2-(v, k, 1) Designs and Groups $E_6(q)$

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Abstract This article is a contribution to the study of block-transitive automorphism groups of 2-(v, k, 1) block designs. Let $D$ be a 2-(v, k, 1) design admitting a block-transitive, point-primitive but not flag-transitive automorphism group $G$. Let $k_r = (k, v - 1)$ and $q = p^f$ for prime $p$. In this paper we prove that if $G$ and $D$ are as above and $q > (3(k, k - k_r + 1)f)^{1/3}$, then $G$ does not admit a simple group $E_6(q)$ as its socle.

Keywords block design; block-transitive; point-primitive; automorphism group.

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1. Introduction

A 2-(v, k, 1) design $D = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of $v$ points and a collection $\mathcal{B}$ of $k$-subsets of $\mathcal{P}$, called blocks, such that each 2-subset of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2 < k < v$.

Recall that an automorphism of a 2-(v, k, 1) design $D$ is a permutation of the set $\mathcal{P}$ of points which maps blocks to blocks. The set of all automorphisms is called the automorphism group Aut($D$) of $D$, a subgroup of Sym($\mathcal{P}$). Let $G \leq$ Aut($D$). Then $G$ is said to be block transitive on $D$ if $G$ is transitive on $\mathcal{B}$, and is said to be point transitive (point primitive) on $D$ if $G$ is transitive (primitive) on $\mathcal{P}$. A flag of $D$ is a pair consisting of a point and a block through that point. Then $G$ is flag transitive on $D$ if $G$ is transitive on the set of flags.

In 1990, a six-person team [4] classified the pairs $(G, D)$ where $G$ is a flag-transitive automorphism group of $D$, with the exception of those in which $G$ is a one-dimensional affine group. In this paper we contribute to the classification of designs which have an automorphism group transitive on blocks. It follows from a result of Block [2] that a block-transitive automorphism group of a 2-(v, k, 1) design is transitive on points. In [7] it was shown that the study of block-transitive 2-(v, k, 1) designs can be reduced to three cases, distinguishable by properties of the action of $G$ on the point set $\mathcal{P}$: that in which $G$ is of affine type in the sense that it has an

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elementary abelian transitive normal subgroup; that in which $G$ is almost simple, in the sense that $G$ has a simple nonabelian transitive normal subgroup $T$ whose centralizer is trivial, so that $T \leq G \leq \text{Aut}T$; and that in which $G$ has an intransitive minimal normal subgroup. Much work is needed to achieve this classification [5, 7, 9]. Liu et al have studied the special case where $G = T := \text{Soc}(G)$ is any finite group of Lie type of Lie rank 1 acting block-transitively on a design in [15–18]. Here we focus on the second case, that is, classifying $2-(v, k, 1)$ designs with a block-transitive automorphism group of almost simple type under the conditions that $G$ is point-primitive but not flag-transitive.

Our paper is organized as follows: In Section 2 we collect some preliminary results and in Section 3 we use them to prove Theorem 1.1.

2. Preliminary results

Let $\mathcal{D}$ be a 2-$(v, k, 1)$ design defined on the point set $\mathcal{P}$, and suppose that $G$ is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a 2-$(v, k, 1)$ design, as usual, $b$ denotes the number of blocks and $r$ denotes the number of blocks through a given point. If $B$ is a block, $G_B$ denotes the setwise stabilizer of $B$ in $G$ and $G(B)$ is the pointwise stabilizer of $B$ in $G$. Also, $G_B^*$ denotes the permutation group induced by the action of $G_B$ on the points of $B$, and so $G_B^* \cong G_B/G(B)$.

For the basic notions and results of design theory and finite permutation groups, the reader is referred to [1, 19]. We will follow the notations of [10] for simple groups $T = E_6(q)$. Let $W$ be the Weyl group associated with the simple group $T$, $N$ the monomial subgroup of $T$, and $H$ the diagonal subgroup of $T$. From [10, Theorem 7.2.2], it is well known that there exists a homomorphism $\phi : N \to W$ such that $N/H \cong W$. Let $\Phi$ be the root system corresponding to $T$ with fundamental system $\Pi$, also let $\Phi^+$ ($\Phi^-$) be the set of positive (negative) roots in $\Phi$. If $J$ is a subset of the set $\Pi$ of fundamental roots and $V_J$ is the subspace of $V$ spanned by $J$, then $\Phi_J$ denotes the set of roots of $\Phi$ lying in the subspace $V_J$. We use the standard labelling for Dynkin diagram with fundamental roots $\alpha_i$ as in [3, pp.250-275].

The main result regarding the maximal subgroups of $E_6(q)$ to be used is the following

**Lemma 2.1** (Liebeck and Saxl [14]) Suppose that $T := \text{Soc}(G) \cong E_6(q)$ is a simple exceptional group of Lie type over $GF(q)$, where $q = p^f$ for a prime $p$ and positive integer $f$. Let $M$ be a
maximal subgroup of $G$ not containing $T$. Then one of the following holds:

(a) $|M| < q^3 |G : T|$;

(b) $T \cap M$ is a parabolic subgroup of $T$;

(c) $T \cap M$ is isomorphic to one of (i) $F_4(q)$; (ii) $E_6(q^2)$ with $q$ square; (iii) $E_7(q^2)$ with $q$ square; (iv) $(D_5(q) \circ (q-1)/e+1).f_+1$ where $e+1 = (q-1,3)$ and $f_+1 = (q-1,4)$; or (v) $(SL_2(q) \circ A_5(q)).d_1$ where $d_1 = (2,q-1)$.

3. Proof of Theorem 1.1

We assume throughout this section that $q = p^f$ for $p$ a prime integer and $f$ a positive integer. If $n$ is a positive integer, then $|n|_p$ denotes the $p$-part of $n$ and $|n|_{p'}$ denotes the $p'$-part of $n$. In other words, $|n|_p = p^f$ where $p^f \mid n$ but $p^{f+1} \nmid n$, and $|n|_{p'} = n/|n|_p$.

Following Fang and Li [11], we shall use the following parameters of $2-(v,k,1)$ designs:

$$k_v = (k,v), \quad k_r = (k,r) = (k,v-1), \quad b_v = (b,v), \quad b_r = (b,r) = (b,v-1).$$

It is easy to check that $k = k_v k_r$, $b = b_v b_r$, $v = k_v b_v$ and $r = k_r b_r$.

**Proposition 3.1** Let $D$ and $G$ satisfy the conditions of Theorem 1.1, $T = Soc(G)$ and $T_\alpha = T \cap G_\alpha$, where $\alpha \in \mathcal{P}$. Then we have the following properties:

(P1) $v = 1 + k_v (k-1)b_r$;

(P2) $\frac{v}{x} < (k_v k_r + 1)|G : T|$ or $\frac{v-1}{x} \leq (k_v k_r + 1)|G : T|$, where $x$ is the size of a $T_\alpha$-orbit in $\mathcal{P} \setminus \{\alpha\}$;

(P3) $\frac{|T|}{|T_\alpha|} < \left\lfloor \frac{k_v k_r + 1}{2} \right\rfloor$, where $\left\lfloor \frac{k_v k_r + 1}{2} \right\rfloor$ is the ceiling of $\frac{k_v k_r + 1}{2}$;

(P4) If $(v-1,q) = 1$, then there exists a $T_\alpha$-orbit with size $y$ in $\mathcal{P} \setminus \{\alpha\}$ such that $y || T_\alpha $$p'$;

(P5) $b_r || G_\alpha$.

**Proof** (P1) Since $kb = lr$ and $r = \frac{v-1}{b_v}$, we have $k(k-1)b = v(v-1)$, which is the same as $k_r(k-1)b_r = v - 1$. Hence $v = 1 + k_r(k-1)b_r$.

(P2) Let $\Delta$ be any $T_\alpha$-orbit in $\mathcal{P} \setminus \{\alpha\}$ with size $x$, and $\Gamma$ a nontrivial suborbit of $G_\alpha$ such that $\Delta \subseteq \Gamma$. Since $\frac{|G|}{|G_\alpha|} = \frac{|T|}{|T_\alpha|}$, we have

$$|G : T| = |G_\alpha : T_\alpha|,$$

and

$$|\Gamma| = \frac{|G_\alpha|}{|G_{\alpha\beta}|} \leq \frac{|G_\alpha|}{|T_{\alpha\beta}|} = \frac{|T_\alpha|}{|T_{\alpha\beta}|} = x|G : T|,$$

where $\beta \in \Delta$. Since $v = 1 + k_r(k-1)b_r$, we have $\frac{v}{b_v} < 1 + k_r(k-1)$.

By Lemma 2.1 of [13], we have $b_r || |\Gamma|$, and $b_r \leq |\Gamma|$. Thus

$$\frac{v}{x|G : T|} \leq \frac{v}{|\Gamma|} \leq \frac{v}{b_r} < 1 + k_r(k-1),$$

and we have the first inequality. The proof of the other inequality is similar.

(P3) Let $B$ be a block of $D$. Then the following possibility of the type of $G^B$, the rank and
the subdegree of $G$ does not occur.

<table>
<thead>
<tr>
<th>Type of $G^B$</th>
<th>Rank</th>
<th>Subdegree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle 1 \rangle$</td>
<td>$1 + k_r(k-1)$</td>
<td>$1, b_r, \ldots, b_r$</td>
</tr>
</tbody>
</table>

In fact, since the order of $G^B$ is odd, the order of $G$ is odd. This contradicts the fact that $\text{Soc}(G)$ is $E_6(q)$. So the longest size of the suborbit of $G$ is not less than $2b_r$. Denote by $\theta$ and $\lambda$ the longest suborbits of $T$ and $G$, respectively. Then

$$\lambda \leq |G : T| \leq |G : T||T_{\alpha}|.$$  

We have

$$v = \frac{|T|}{|T_{\alpha}|} \leq \left[ \frac{v}{\lambda} \right] \leq \left[ \frac{v}{\lambda} \right] |G : T||T_{\alpha}|,$$

and

$$|T| \leq \left[ \frac{v}{\lambda} \right] |G : T||T_{\alpha}|^2.$$

Also since $\lambda \geq 2b_r$, we have

$$\frac{|T|}{|T_{\alpha}|^2} < \left[ \frac{k_rk - k_r + 1}{2} \right] |G : T|.$$  

(P4) Let $t$ be the size of any $T_{\alpha}$-orbit in $\mathcal{P} \setminus \{\alpha\}$. Suppose to the contrary that $t \not| |T_{\alpha}|$. Since $t | |T_{\alpha}|$, we have $pt$. Furthermore, since $\mathcal{P} \setminus \{\alpha\}$ is a union of $T_{\alpha}$-orbits, $p|v - 1$. Thus $p|(v - 1, q)$, which contradicts $(v - 1, q) = 1$.

(P5) This is a straightforward consequence of [13, Lemma 2.1].

**Proof of Theorem 1.1** Let $D$ and $G$ satisfy the hypotheses of Theorem 1.1. Suppose on the contrary that $T := \text{Soc}(G) \cong E_6(q)$, where $q = p^f > (3(k_rk - k_r + 1)f)^{1/3}$ and $p$ is prime.

Since $G$ is primitive on $\mathcal{P}$, $G_{\alpha}$ is a maximal subgroup of $G$ for any $\alpha \in \mathcal{P}$. Hence $M = G_{\alpha}$ satisfies one of the three cases in Lemma 2.1. We will rule out these cases one by one.

**Lemma 3.1** Case (a) in Lemma 2.1 does not occur.

**Proof** By (P3),

$$|T| < \left[ \frac{k_rk - k_r + 1}{2} \right]|T_{\alpha}|^2|G : T| < \left[ \frac{k_rk - k_r + 1}{2} \right]q^{74}|G : T|.$$  

Since $|E_6(q)| = q^{36}(q^{12} - 1)(q^9 - 1)(q^6 - 1)(q^5 - 1)(q^3 - 1)/d$, where $d = (3, q - 1)$, we have

$$\frac{|T|}{q^{74}} = \frac{(q^{12} - 1)(q^9 - 1)(q^6 - 1)(q^5 - 1)(q^3 - 1)}{dq^{38}} > \frac{q^{42} - q^{40} - q^{37} - (2^5 - 2)q^{36}}{dq^{38}} > \frac{q^{36} - q^4 - q^{30}}{dq^{2}} > \frac{k_rk - k_r + 1}{2}|G : T|,$$

contradicting (1).

**Lemma 3.2** Case (b) in Lemma 2.1 does not occur.
Proof Let \( \Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \) be the fundamental root system of \( E_6(q) \), let \( J_i = \Pi - \{\alpha_i\} \), and \( P_{J_i} \) be the parabolic subgroup of \( E_6(q) \) determined by \( J_i \).

The following Table 1 lists the order of \( T_\alpha \) and the value of \( v = |T|/|T_\alpha| \) in the corresponding subcases.

| \( T_\alpha \) | \( |T_\alpha| \) | \( v \) |
|----------------|----------------|--------|
| \( P_{J_1} \) | \( q^36(q - 1)(q^2 - 1)(q^4 - 1)(q^5 - 1)(q^6 - 1)/(q^8 - 1)/d \) | \( (q^{a_0-1})(q^{a_2-1})/(q-1)(q^d-1) \) |
| \( P_{J_2} \) | \( q^36\prod_{i=1}^6(q^i - 1)/d \) | \( (q^8-1)(q^9-1)/(q-1)(q^d-1) \) |
| \( P_{J_3} \) | \( q^36(q - 1)(q^2 - 1)(q^3 - 1)(q^4 - 1)(q^5 - 1)/d \) | \( (q-1)(q^4-1)(q^5-1)/(q^d-1) \) |
| \( P_{J_4} \) | \( q^36(q - 1)(q^2 - 1)(q^3 - 1)^2/(q^d-1) \) | \( (q-1)(q^4-1)(q^5-1)/(q^d-1) \) |
| \( P_{J_5} \) | \( q^36(q - 1)(q^2 - 1)(q^3 - 1)(q^4 - 1)(q^5 - 1)/d \) | \( (q-1)(q^4-1)(q^5-1)/(q^d-1) \) |
| \( P_{J_6} \) | \( q^36(q - 1)(q^2 - 1)(q^3 - 1)(q^4 - 1)(q^5 - 1)(q^6 - 1)/(q^8 - 1)/d \) | \( (q^{a_0-1})(q^{a_2-1})/(q-1)(q^d-1) \) |

Table 1 Subcases of Case (b)

Subcase 3.1 \( T_\alpha = P_{J_1} \). By [10, Theorem 7.2.2], there exists a homomorphism \( \phi : N \to W \) such that \( N/H \cong W \). Let \( \phi(n_1) = w_{\alpha_1} \), where \( n_1 \in N \), \( w_{\alpha_1} \) is the corresponding reflection of \( \alpha_1 \) in the Weyl group \( W \). Now we consider \( P_{J_1} \cap P_{J_2}^{a_1} \). Since \( P_{J_1} = \langle X_r, H|r \in \Phi^+ \cup \Phi_{J_1} \rangle \), we have

\[
P_{J_1}^{a_1} = \langle X_r, H|r \in (\Phi^+)^{a_1} \cup (\Phi_{J_1})^{n_1} \rangle
= \langle X_r, H|r \in (\Phi^+-\{\alpha_1\}) \cup \{-\alpha_1\} \cup \Phi_{w_{\alpha_1}(J_1)} \rangle.
\]

It follows that

\[
\langle X_r, H|r \in (\Phi^+-\{\alpha_1\}) \cup \Phi_{J'} \rangle \leq P_{J_1} \cap P_{J_2}^{a_1},
\]

where \( J' = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6\} \). Let

\[
P = \langle X_r, H|r \in (\Phi^+-\{\alpha_1\}) \cup \Phi_{J'} \rangle \quad \text{and} \quad \tilde{U} = \prod_{r \in (\Phi^+-\{\alpha_1\}) \cap \Phi_{J'}} X_r \leq U_{J'}.
\]

Then we have

\[
\tilde{P} = \tilde{U} L_{J'}, \quad |\tilde{P}| = \frac{1}{d} q^{35}(q - 1)^2(q^2 - 1)(q^3 - 1)(q^4 - 1)(q^5 - 1).
\]

Thus \( T_\alpha \) has an orbit of size

\[
x = \frac{|P_{J_1}|}{|P_{J_1} \cap P_{J_2}^{a_1}|} \leq \frac{|P_{J_1}|}{|P_{J_2}^{a_1}|} = \frac{q(q^6 - 1)(q^8 - 1)}{(q-1)(q^d-1)}.
\]

Therefore,

\[
\frac{v}{x} > q^4 > (k_r k - k_r + 1)|G:T|,
\]

where \( v \) is given in the first line of Table 1. This contradicts property \((P_2)\).

Subcase 3.2 \( T_\alpha = P_{J_1} \). Let \( n_2 \) be the inverse image of \( w_{\alpha_2} \) under \( \phi \). Since

\[
P_{J_2} = \langle X_r, H|r \in \Phi^+ \cup \Phi_{J_2} \rangle,
\]
\[ P_{J_2}^{n_2} = (X_r, H| r \in (\Phi^+)^{n_2} \cup (\Phi_{J_2})^{n_2}) \]
\[ = (X_r, H| r \in (\Phi^+ - \{\alpha_2\}) \cup \{-\alpha_2\} \cup \Phi_{w_{\alpha_2}(J_2)}). \]

Then
\[ P_{J_2} \cap P_{J_2}^{n_2} \geq (X_r, H| r \in (\Phi^+ - \{\alpha_2\}) \cup \Phi_{J'}). \]
where \( J' = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}. \) Hence
\[ |P_{J_2} \cap P_{J_2}^{n_2}| > \frac{1}{d} q^{35}(q - 1)^2(q^2 - 1)^2(q^3 - 1)^2, \]
and \( T_{\alpha} \) has an orbit of size
\[ x = \frac{|P_{J_2}|}{|P_{J_2} \cap P_{J_2}^{n_2}|} \leq \frac{q(q^4 - 1)(q^5 - 1)(q^6 - 1)}{(q - 1)(q^2 - 1)(q^3 - 1)}. \]
It follows that
\[ \frac{v}{x} > q^{15} > (k_r k - k_r + 1)|G : T|, \]
contradicting \((P_2)\).

**Subcase 3.3** \( T_{\alpha} = P_{J_3} \). Let \( n_3 \) be the inverse image of \( w_{\alpha_3} \) under \( \phi \). Since
\[ P_{J_3} = (X_r, H| r \in \Phi^+ \cup \Phi_{J_3}), \]
\[ P_{J_3}^{n_3} = (X_r, H| r \in (\Phi^+)^{n_3} \cup (\Phi_{J_3})^{n_3}) \]
\[ = (X_r, H| r \in (\Phi^+ - \{\alpha_3\}) \cup \{-\alpha_3\} \cup \Phi_{w_{\alpha_3}(J_3)}), \]
then
\[ P_{J_3} \cap P_{J_3}^{n_3} \geq (X_r, H| r \in (\Phi^+ - \{\alpha_3\}) \cup \Phi_{J'}) \]
where \( J' = \{\alpha_2, \alpha_5, \alpha_6\} \), and
\[ |P_{J_3} \cap P_{J_3}^{n_3}| > \frac{1}{d} q^{35}(q - 1)^3(q^2 - 1)^2(q^3 - 1). \]
Thus \( T_{\alpha} \) has an orbit of size
\[ x = \frac{|P_{J_3}|}{|P_{J_3} \cap P_{J_3}^{n_3}|} \leq \frac{q(q^4 - 1)(q^5 - 1)}{(q - 1)^2}. \]
It follows that
\[ \frac{v}{x} > q^{15} > (k_r k - k_r + 1)|G : T|, \]
contradicting \((P_2)\).

**Subcase 3.4** \( T_{\alpha} = P_{J_4} \). Let \( n_4 \) be the inverse image of \( w_{\alpha_4} \) under \( \phi \). Since
\[ P_{J_4} = (X_r, H| r \in \Phi^+ \cup \Phi_{J_4}), \]
\[ P_{J_4}^{n_4} = (X_r, H| r \in (\Phi^+ - \{\alpha_4\}) \cup \{-\alpha_4\} \cup \Phi_{w_{\alpha_4}(J_4)}), \]
then
\[ P_{J_4} \cap P_{J_4}^{n_4} \geq (X_r, H| r \in (\Phi^+ - \{\alpha_4\}) \cup \Phi_{J'}) \]
where \( J' = \{\alpha_1, \alpha_6\} \), and
\[ |P_{J_4} \cap P_{J_4}^{n_4}| > \frac{1}{d} q^{35}(q - 1)^4(q^2 - 1)^2. \]
So \( T_{\alpha} \) has an orbit of size
\[ x = \frac{|P_{J_4}|}{|P_{J_4} \cap P_{J_4}^{n_4}|} \leq \frac{q(q^2 - 1)(q^3 - 1)^2}{(q - 1)^3}. \]
It follows that
\[ \frac{v}{x} > q^{20} > (k_r k - k_r + 1)|G : T|, \]
contradicting (P2).

**Subcase 3.5** $T_\alpha = P_{J_5}$. There exists an element $g \in \text{Aut}(T)$ such that $P_{J_5} = P_{J_5}^g$, a contradiction.

**Subcase 3.6** $T_\alpha = P_{J_6}$. There exists an element $g_1 \in \text{Aut}(T)$ such that $P_{J_6} = P_{J_6}^{g_1}$, a contradiction. \hfill \Box

**Lemma 3.3** Case (c) in Lemma 2.1 does not occur.

**Proof** Table 2 lists the order of $T_\alpha$ and the value of $v = |T|/|T_\alpha|$ in the corresponding subcases, where $d_4 = (4, q - 1)$, $d_5 = (6, q - 1)$.

| $T_\alpha$ | $|T_\alpha|$ | $v$ |
|---|---|---|
| (i) | $q^{12}(q^2 - 1)(q^6 - 1)(q^4 - 1)(q^{12} - 1)$ | $q^{12}(q^5 - 1)(q^3 - 1)/d$ |
| (ii) | $q^{18}(q - 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)/d_2$ | $d_2q^{18}(q + 1)(q^2 + 1)(q^3 + 1)$ |
| (iii) | $q^{18}(q - 1)(q^2 + 1)(q^4 - 1)(q^6 - 1)/d_3$ | $d_3q^{18}(q + 1)(q^2 - 1)(q^3 + 1)$ |
| (iv) | $\frac{3q^{18}(q - 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)}{d_4 + 1}$ | $\frac{d_4q^{18}(q + 1)(q^2 - 1)(q^3 + 1)}{d_4 + 1}$ |
| (v) | $q^{18}(q^2 - 1)^2(q^3 - 1)(q^4 - 1)(q^5 - 1)(q^6 - 1)/d_5$ | $\frac{d_5q^{20}(q^2 - 1)(q^3 - 1)(q^5 - 1)(q^6 - 1)}{d_5 + 1}$ |

Table 2 Subcases of Case (c)

**Subcase (i)** $T_\alpha = F_4(q)$.

By (P1), we have $(v, b_r) = 1$. Combining with (P5), then we have $b_r ||G_\alpha|_{v'}$, where $|G_\alpha|_{v'}$ denotes the $v'$-part of $|G_\alpha|$. Hence $b_r ||T_\alpha|_{v'} : |G : T|$. We have

$$|T_\alpha|_{v'} = (q^3 + 1)(q^5 + q^6 + \cdots + q + 1)(q^9 + q^6 + q^3 + 1)$$

and

$$v = 1 + k_r(k - 1)b_r \leq 1 + k_r(k - 1)|T_\alpha|_{v'} : |G : T|.$$ 

Then

$$\frac{1}{d}q^{25} < v < 1 + 2k_r(k - 1)q^{20}|G : T|. \quad (2)$$

When $q > (3(k_r k - k_r + 1)f)^{1/3}$, equation (2) leads to a contradiction.

**Subcase (ii)** $T_\alpha = E_6(q^2)$ where $q$ is square.

By (P4), $T_\alpha$ has an orbit of size $y$ such that

$$y \leq |T_\alpha|_{v'} = (q - 1)(q^5 - 1)(q^3 - 1)(q^4 - 1)(q^9 - 1)(q^{12} - 1)/d_2,$$
where \( d_2 = (3, q^{1/2} - 1) \). It follows that
\[
\frac{v}{y} > q^{14} > (k_r k - k_r + 1)|G : T|,
\]
contradicting (P2).

\textbf{Subcase (iii)} \( T_\alpha = 2E_6(q^{1/2}) \) where \( q \) is square.

By (P4), \( T_\alpha \) has an orbit of size \( y \) such that
\[
y \leq |T_\alpha|_{p'} = (q - 1)(q^{\frac{3}{2}} + 1)(q - 1)(q^{\frac{7}{2}} + 1)(q^{\frac{3}{2}} - 1)/(d_3),
\]
where \( d_3 = (3, q^{1/2} + 1) \). It follows that
\[
\frac{v}{y} > q^{13} > (k_r k - k_r + 1)|G : T|,
\]
contradicting (P2).

\textbf{Subcase (iv)} The proof is similar to subcase (ii).

\textbf{Subcase (v)} The proof is similar to subcase (ii).

By Lemmas 3.1–3.3, this completes the proof of Theorem 1.1.

\textbf{References}