Several Classes of Additively Non-Regular Semirings

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Abstract In this paper, we introduce Green’s *-relations on semirings and define [left, right] adequate semirings to explore additively non-regular semirings. We characterize the semirings which are strong b-lattices of [left, right] skew-halfrings. Also, as further generalization, the semirings are described which are subdirect products of an additively commutative idempotent semiring and a [left, right] skew-halfring. We extend results of constructions of generalized Clifford semirings (given by M. K. Sen, S. K. Maity, K. P. Shum, 2005) and the semirings which are subdirect products of a distributive lattice and a ring (given by S. Ghosh, 1999) to additively non-regular semirings.

Keywords Green’s *-relations; subdirect product; adequate semiring; skew-halfring.

A semiring S is an algebraic structure (S, +, ·) consisting of a non-empty set S together with two binary operations + and · on S such that (S, +) and (S, ·) are semigroups connected by distributivity, that is, a(b + c) = ab + ac and (b + c)a = ba + ca for all a, b, c ∈ S. The additive identity (if it exists) of a semiring S is called zero and denoted by 0S. An additively commutative semiring S with a zero satisfying 0 · x = x · 0 = 0 for all x ∈ S, is called a hemiring. A halfring is a hemiring whose additive reduct is a cancellative monoid. A skew-ring (S, +, ·) [8] is a semiring whose additive reduct (S, +) is a group, not necessarily an abelian group. To explore additively non-regular semiring, we introduce the concept of [left, right] skew-halfring which is a semiring whose additive reduct is an additively [left, right] cancellative monoid, not necessarily to be additively commutative. A semiring (S, +, ·) is said to be a b-lattice [14] if its additive reduct (S, +) is a semilattice and its multiplication reduct (S, ·) is a band. A b-lattice is said to be commutative if the multiplication in (S, ·) commutes. A semiring (S, +, ·) is called an additive idempotent semiring if its additive reduct (S, +) is a band.

Throughout this paper we denote the set of all additive idempotents [if they exist] of a semiring S by E+(S). By a subdirect product of two semirings S₁ and S₂ we mean a semiring

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which is isomorphism to a subsemiring $T$ of the semiring $S$ such that the projection maps of $T$ into both $S_1$ and $S_2$ are surjective, where $S$ is the direct product of $S_1$ and $S_2$.

Since semirings are generalizations of distributive lattices, b-lattices, additive commutative idempotent semirings, rings, skew-rings and [left, right] skew-half rings, it is interesting to use those semirings to establish constructions of some semirings. Bandelt and Petrich in [2] introduced Bandelt-Petrich Construction in semirings and described the semirings with regular addition which is a subdirect products of a distributive lattice and a ring. Ghosh in [5] established the constructions “strong distributive lattice of semirings” which include the Bandelt-Petrich Construction, and characterized all semirings which are subdirect products of a distributive lattice and a ring. In particular the author introduced the semirings, Clifford semirings [additively commutative inverse semirings such that the set of its additive idempotents is a distributive sublattice as well as a k-ideal], and verified that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of rings, and if and only if it is an inverse subdirect product of a distributive lattice and a ring. Later, in [14], Sen, Maity and Shum defined the Clifford semiring which is a completely regular and an additively inverse semiring such that the set of its additive idempotents is a distributive sublattice as well as a k-ideal [without assuming that its additive reduct is commutative] and proved that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings. Moreover, they introduced generalized Clifford semirings which are completely regular and inverse semirings such that its additive idempotent set is a k-ideal, and obtained that a semiring is a generalized Clifford semiring if and only if it is a strong b-lattice of skew-rings, and if and only if it is an additively inverse semiring and is a subdirect product of a b-lattice and a skew-ring. All semirings studied in [2], [5] and [14] are additively regular.

Let $(S, +, \cdot)$ be a semiring. We denote the Green’s $\mathcal{H}$-relation on the additive reduct $(S, +)$ by $\mathcal{H}^+$. By Theorems 1.4, 2.5 and 3.3 in [14] we know that $\mathcal{H}^+$ is a congruence on both Clifford semirings and generalized Clifford semirings. McAlister [10] and Pastijn [13] introduced Green’s $*$-relations on semigroups. In this paper we introduce Green’s $*$-relations on semirings and consider non-regular semirings, [left, right] adequate semirings on which Green’s $*$-relation $\mathcal{L}^+$, $\mathcal{R}^+$] $\mathcal{H}^*$ is a congruence. Our purpose is to extend results of generalized Clifford semirings in [14] and the semirings which are subdirect products of a distributive lattice and a ring in [5] to non-regular semirings.

To study the [left, right] adequate semirings in Section 1, we first introduce Green’s $*$-relations on semirings and then recall some results of right adequate semigroups and adequate semigroups. In Section 2 we introduce [left, right] adequate semirings and discuss adequate semirings on which Green’s $*$-relation $\mathcal{H}^*$ is a congruence. Section 3 explores right adequate semirings on which Green’s $*$-relation $\mathcal{L}^+$ is a congruence. We do not discuss the left adequate semiring on which Green’s $*$-relation $\mathcal{R}^*$ is a congruence since symmetrically we can obtain the results of left adequate semirings. In Section 4 we characterize the semirings which are a strong b-lattice of [left] skew-half rings. Section 5 describes the semirings which are subdirect products
of an additively commutative idempotent semiring and a [left] skew-halfring.

In this paper, we refer to [3], [4], [6] and [9] for the undefined notions and notations about semigroups.

1. Preliminaries

Let \((S, +, \cdot)\) be any semiring. We denote the Green’s relations \(L, R, H\) on additive reduct \((S, +)\) by \(\overset{+}{L}, \overset{+}{R}, \overset{+}{H}\), respectively. These are also equivalence relations on semiring \((S, +, \cdot)\). Now, we introduce Green’s \(*\)-relations \(\overset{*}{L}, \overset{*}{R}, \overset{*}{H}\) on the semiring \(S\) which are given by: for \(a, b \in S\),

\[
a \overset{+}{L} b \iff \text{for all } x, y \in S^1, a + x = a + y \text{ if and only if } b + x = b + y
\]

\[
a \overset{+}{R} b \iff \text{for all } x, y \in S^1, x + a = y + a \text{ if and only if } x + b = y + b
\]

\[
\overset{*}{H} = \overset{*}{L} \cap \overset{*}{R}.
\]

It is clear that \(\overset{+}{L} \subseteq \overset{*}{L}, \overset{+}{R} \subseteq \overset{*}{R}\) on \((S, +, \cdot)\). In particular, if \(S\) is an additively regular semiring, \(\overset{+}{L} = \overset{*}{L}, \overset{+}{R} = \overset{*}{R}\) (see [3]). In general, since the Green’s equivalence relations \(L, R\) and \(H\) are not congruences on a semigroup [6], Green’s \(*\)-equivalence relations \(\overset{*}{L}, \overset{*}{R}\) and \(\overset{*}{H}\) are not congruences on \((S, +, \cdot)\).

Before starting our discussion of several classes of non-regular semirings, we need some concepts for semigroups and some results of [left, right] adequate semigroups. Let \((S, +)\) be a semigroup. We call \((S, +)\) a [right, a left] adequate semigroup if its idempotents commute and every \(\overset{+}{L}\)-class, \(\overset{+}{R}\)-class and \(\overset{+}{H}\)-class contain an idempotent (which is, in fact, unique [4]). For an element \(a\) of such a semigroup, the idempotent in the \(\overset{+}{L}\)-class \([\overset{+}{R}\]-class\) containing \(a\) is denoted by \(a^*[a^+]\). A [right, left] adequate semigroup \(S\) is called [right, left] type A if \([e + a = a + (e + a)^*], a + e = (a + e)^* + a\) for all \(a \in S\) and \(e \in E(S)\). In [9] an equivalent relation \([\sigma_r, \sigma_l]\sigma\) on a [right, left] type A semigroup \((S, +)\) is defined by: for \(a, b \in S\), \([a \sigma_r b, a \sigma_l b][ab]\) if and only if there exists an idempotent \(e\) in \(S\) such that \([a + e = b + e, a + e = b + e]\). A [right, left] type A semigroup \(S\) is said to be proper if and only if \([\sigma_l \cap \overset{+}{L} = \iota, \sigma_r \cap \overset{+}{R} = \iota] \sigma \cap \overset{+}{L} = \sigma \cap \overset{+}{R} = \iota\), where \(\iota\) is the identity mapping on \(S\).

**Lemma 1.1**

(1) Let \((S, +)\) be a right adequate semigroup with semilattice of idempotents \(E\). If \(L^*\) is a congruence on \(S\) and \(S/L^*\) is a semilattice, then \((S/L^*, +) \cong (E, +)\).

(2) Let \((S, +)\) be an adequate semigroup with semilattice of idempotents \(E\). If \(H^*\) is a congruence on \(S\) and \(S/H^*\) is a semilattice, then every \(H^*\)-class contains an idempotent and \((S/H^*, +) \cong (E, +)\).

**Proof**

The conclusion follows from Lemma 2.5 and Corollary 2.6 in [4]. □

**Lemma 1.2** (Lemma 1.12 [4]) Let \(H^*\) be an \(H^*\)-class of a semigroup \(S\). If \(H^*\) contains an
idempotent, then $H^*$ is a cancellative subsemigroup of $S$ with an identity.

Armstrong [1] defined two partial orders $\leq_l$ and $\leq_r$ on an adequate semigroup as follows:

\[
a \leq_l b \text{ if and only if } b = a + e \\
a \leq_r b \text{ if and only if } b = e + a
\]

for some idempotent $e$. The partial order $\leq = \leq_l \cap \leq_r$ is called the natural partial order on an adequate semigroup.

2. Adequate semirings on which $H^*$ is a congruence

In this section we first introduce the concepts of [right, left] adequate semirings, then consider adequate semirings on which $H^*$ is a congruence.

**Definition 2.1** Let $(S, +, \cdot)$ be a semiring.

1. $(S, +, \cdot)$ is an [a right, a left] adequate if its additive reduct $(S, +)$ is an [a right, a left] adequate semigroup.

2. $(S, +, \cdot)$ is [right, left] type A if its additive reduct $(S, +)$ is a [right, left] type A semigroup.

3. $(S, +, \cdot)$ is [right, left] proper type A if its additive reduct $(S, +)$ is a [right, left] proper type A semigroup.

For an element $a$ of a semiring $(S, +, \cdot)$, the additive idempotent in the $L^*$-class containing $a$ will be denoted by $a^*$ and the additive idempotent in the $R^*$-class containing $a$ will be denoted by $a^\dagger$. For an adequate semiring $(S, +, \cdot)$, $\leq$ denotes the partial order determined by $\leq$ on the additive reduct $(S, +)$.

We now give two examples of adequate semirings, one is an adequate semiring on which $H^*$ is a multiplication congruence and the other on which $H^*$ is not.

**Example 2.1** Let $(A, +)$ be the infinite cyclic monoid generated by $a$ with identity $1$ and let $(B, +)$ be the free monoid generated by $b, c$ with identity $0$. The mapping $\alpha : \{b, c, 0\} \rightarrow A$ given by $\alpha(b) = \alpha(c) = a$ and $\alpha(0) = 1$, extends uniquely to a homomorphism $\alpha : B \rightarrow A$. Let $S = A \cup B$ with additive operation: for $x \in A, y \in B$, $x + y = x + \alpha(y) = \alpha(y) + x = y + x$. Thus we have an addition on $S$ which extends those on $A$ and $B$. By Example 2.3 in [9], $S$ is an adequate semigroup with $H^* = L^* = R^*$.

1. Define a multiplication on $S$ as follows:

\[
xy = \begin{cases} 
0 & \text{for } x, y \in B, \text{ or } x \in A, y \in B, \text{ or } y \in A, x \in B, \\
1 & \text{for } x, y \in A.
\end{cases}
\]

Then, $(S, +, \cdot)$ is an adequate semiring on which $H^*$ is a semiring congruence on $S$.

2. Define a multiplication on $S$ as follows:

\[
xy = \begin{cases} 
0 & \text{for any } x, y \in S, x = 0 \text{ or } y = 0, \\
1 & \text{for any } x, y \in S, x \neq 0, y \neq 0.
\end{cases}
\]
Then $S$ is an adequate semiring with $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^*$. It is easy to check that $A$ and $B$ are the $\mathcal{L}^*$-classes. Since $x^* = 1$, $y^* = 0$ for $x \in A$, $y \in B$, $x^*y^* = 1 \cdot 0 = 0$. But $(xy)^* = 1$ and $(xy)^* \neq x^*y^*$. Hence, $\mathcal{H}^*$ is not a multiplication congruence on $S$.

Let $(S, +, \cdot)$ be a semiring. We denote the largest congruence on $(S, +, \cdot)$ contained in $\mathcal{L}^*[\mathcal{R}^*]$ by $\mu_{\mathcal{L}^*}$, $\mu_{\mathcal{R}^*}$, and the largest semiring congruence contained in $\mathcal{H}^*$ is denoted by $\mu^*$. A congruence $\delta$ on a semiring $(S, +, \cdot)$ is said to be a [left, right] skew-halfring congruence if the additive reduct $(S/\delta, +)$ is a [left, right] cancellative monoid.

The semiring $(S, +, \cdot)$ may satisfy some of the following axioms:

(A₁) $a + e = e + a$ for $e \in E^+(S)$ and $a \in S$;
(A₂) $(ab)^* + a^*b^* = a^*b^*$ for $a, b \in S$;
(A₃) $aa^* = a^*$ for $a \in S$;
(A₄) $ab^* = b^*a$ for $a, b \in S$;
(A₅) $a + a^*b = a$ for $a, b \in S$;
(A₆) if $a^* = b^*$ and $a + e = b + e$ for $a, b \in S$ and some $e \in E$, then $a = b$;
(A₆′) if there exists $c$ in $S$ for $a, b \in S$ such that $a \leq c, b \leq c$ and $a^* = b^*$, then $a = b$;
(B₁) $(a + e, e + a) \in \mathcal{L}^*$ for all $a \in S, e \in E^+(S)$;
(B₁′) $(a + e, e + a) \in \mathcal{R}^*$ for all $a \in S, e \in E^+(S)$.

Lemma 2.1 Let $(S, +, \cdot)$ be a type A semiring. Then,

(1) The equivalent relation $\sigma$ is the minimum cancellative monoid congruence on the additive reduce $(S, +)$;
(2) For $a, b \in S$, $a \sigma b$ if and only if there exists an element $c$ in $S$ such that $a \leq c$ and $b \leq c$.
(3) $\sigma$ is the minimum skew-halfring congruence on $(S, +, \cdot)$.

Proof
(1) It is from Proposition 1.7 in [9].
(2) It is from the second paragraph in [9, p283].
(3) If $a \sigma b$ for $a, b \in S$, there exists $e$ in $E^+(S)$ such that $a + e = b + e$.

Similarly, from the seventh paragraph in [9, p283] we have Lemma 2.2.

Lemma 2.2 Let $(S, +, \cdot)$ be a right type A semiring. Then,

(1) The equivalent relation $\sigma_r$ is the minimum left cancellative monoid congruence on the additive reduce $(S, +)$;
(2) $\sigma_r$ is the minimum left skew-halfring congruence on $(S, +, \cdot)$.

Lemma 2.3 Let $(S, +, \cdot)$ be an adequate [a right adequate] semiring. If $S$ satisfies conditions (A₁) and (A₂), then $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^*[\mathcal{L}^*]$ is a congruence on $(S, +, \cdot)$.

Proof If adequate semiring $(S, +, \cdot)$ satisfies (A₁), by Proposition 2.7 and Proposition 2.9 in
[4], $H^+ = L^+ = R^+$ is a semilattice congruence on $(S, +)$. To obtain that $H^+ = L^+ = R^+$ is a multiplication congruence on $S$, we only prove that, for any $a, b \in S$, $(ab)^* = a^*b = ab^* = a^*b^*$.

For any $a, b \in S$, we have

$$(ab)^* = [(a + a^*)(b + b^*)]^* = (ab + a^*b + a*b^* + a^*b^*)^* = (ab)^* + a^*b + ab^* + a*b^* \quad \text{(by \ } a^*b, ab^*, a^*b^* \in E^+ (S)).$$

So $(ab)^* + a^*b^* = (ab)^*$. On the other hand, if $S$ satisfies $(A_2)$, that is, $(ab)^* + a^*b^* = a^*b^*$, then $(ab)^* = a^*b^*$. We also have $(ab)^* + a^*b = (ab)^* a^*b + a^*b = a^*b$, which means that $(ab)^* = a^*b$. Similarly, $(ab)^* = ab^*$. Consequently, $(ab)^* = a^*b = ab^* = a^*b^*$.

Similarly, we can prove the case that $(S, +, \cdot)$ is a right adequate semiring. □

Proposition 2.1 The following conditions are equivalent on an adequate semiring $(S, +, \cdot)$ with $E = E^+(S)$:

1. $S$ satisfies $(A_1)$, $(A_2)$;
2. $\mu^+$ is the largest semiring congruence contained in $H^+$ (in this case $H^+ = L^+ = R^+ = \mu^+$);
3. $(S/\mu^+, +, \cdot) \cong (E, +, \cdot)$.

Proof (1) $\Rightarrow$ (2). It follows from Lemma 2.3.

(2) $\Rightarrow$ (3). Define a mapping by $\theta: (S/\mu^+, +, \cdot) \to (E, +, \cdot), a\mu^+ \to a^*$. It is routine to check that $\theta$ is an isomorphism.

(3) $\Rightarrow$ (1). Assume that $S/\mu^+ \cong E$. By Proposition 2.9 in [4], we have $\mu^+ = H^+$, and then $H^+$ is a semiring congruence and $E$ is central on $(S, +)$. Of course, $S$ satisfies $(A_1)$ and $(ab)^* = a^*b^*$, and then $(ab)^* + a^*b^* = a^*b^*$. Therefore $(A_1)$ and $(A_2)$ hold.

If $(S, +, \cdot)$ is a right adequate semiring, the proof is in the same way. □

Theorem 2.1 Let $(S, +, \cdot)$ be an adequate semiring. The following statements are true:

1. $S/H^+$ is a semiring with semilattice additive reduct if and only if $S$ satisfies $(A_1)$, $(A_2)$.
2. $S/H^+$ is a b-lattice if and only if $S$ satisfies $(A_1)$–$(A_3)$.
3. $S/H^+$ is a commutative b-lattice if and only if $S$ satisfies $(A_1)$–$(A_4)$.
4. $S/H^+$ is a distributive lattice if and only if $S$ satisfies $(A_1)$–$(A_5)$.

Proof (1) If $S/H^+$ is a semiring with semilattice additive reduct, by Lemma 1.1, $(S/H^+, +) \cong (E^+(S), +)$. Since $E^+(S)$ is an ideal of $S$, it is clear that $(S/H^+, +, \cdot) \cong (E^+(S), +, \cdot)$ under a mapping $\theta: aH^+ \to a^*$. According to Proposition 2.1, $S$ satisfies $(A_1)$, $(A_2)$.

Conversely, the conclusion follows from Proposition 2.1.

(2) Suppose that $S/H^+$ is a b-lattice. Then $(A_1)$ and $(A_2)$ hold. Since $aa^* \in E^+(S)$, $aa^*H^+a^*a^* = a^*$ implies that $aa^* = a^*$, that is, $(A_3)$ holds.
Conversely, let \( S \) satisfy (A_1)–(A_3). Since \((ab)^* = a^*b^* \) and \( aa^* = a^* \), we have
\[
aH^+a^* = aa^*H^+ a^*.
\]
By (1), \( S/H^+ \) is an idempotent semiring with semilattice additive reduct and band multiplicative reduct. Then \( S/H^+ \) is a b-lattice.

(3) If \( S/H^+ \) is a commutative b-lattice, then \( a^*b^* = b^*a^* \), and so \( ab^* = a^*b^* = ba^* \). Thus, \( A_4 \) holds. Conversely, assume that \( S \) satisfies (A_1)–(A_4). Since
\[
(aH^+)(bH^+) = (aH^+)(bH^+)^* = (aH^+)(bH^+)
\]
\[
= (ab)^*H^* = (b^*a)H^* = (bH^*)(aH^*)
\]
\[
= (b^*H^*)(a^*H^*),
\]
\( S/H^+ \) is a commutative b-lattice.

(4) If \( S/H^+ \) is a distributive lattice, then \( S \) satisfies (A_1)–(A_4). Moreover, for \( a, b \in S \),
\[
\begin{align*}
a + a^*b &= a + a^*b^* = (a + a^*) + a^*b \\
&= a + (a^* + a^*b^*) = a + a^* \quad \text{(by} a^*b^* \leq a^*) \\
&= a
\end{align*}
\]
which proves (A_5).

Conversely, assume that \( S \) satisfies (A_1)–(A_5). Since \( a + a^*b = a \) implies \((a + a^*b)^* = (a)^* \), by Proposition 1.6 (2) in [4] and \( a^*b = a^*b^* \) we have \( a^* + a^*b^* = a^* + a^*b = (a + a^*b)^* = a^* \) and \( b^*a^* + a^* = a^*b^* + a^* = a^* + a^*b^* = a^* \). Consequently, \( S/H^+ \) is a distributive lattice. □

**Definition 2.2** ([14]) A congruence \( \rho \) on a semiring \( S \) is called a b-lattice [distributive lattice] congruence if \( S/\rho \) is a b-lattice [distributive lattice]. A semiring \( S \) is called a b-lattice [distributive lattice] \( Y \) of semirings \( \{S_\alpha : \alpha \in Y\} \) if \( S \) admits a b-lattice [distributive lattice] congruence \( \rho \) on \( S \) such that \( Y = S/\rho \) and each \( S_\alpha \) is a \( \rho \)-class.

The following corollary follows from Lemma 1.12 [4], Lemma 1.1, Proposition 2.1 and Theorem 2.1.

**Corollary** For an adequate semiring \( S \) satisfying (A_1) and (A_2),

1. If \( S \) satisfies (A_3), then \( S \) is a b-lattice of skew-halfrings;
2. If \( S \) satisfies (A_3) and (A_4), then \( S \) is a commutative b-lattice of skew-halfrings;
3. If \( S \) satisfies (A_3)–(A_5), then \( S \) is a distributive lattice of skew-halfrings.

### 3. Right adequate semirings on which \( L^* \) is a congruence

Let \((S, +, \cdot)\) be a right adequate semiring. In this section we can obtain the results on right adequate semirings analogous to those on adequate semigroups on which \( H^+ \) is a congruence.
Proposition 3.1 The following conditions are equivalent for a semiring \((S, +, \cdot)\) with \(E = E^+(S)\):

1. \(S\) is right adequate and satisfies \((A_1)\) and \((A_2)\);
2. \(S\) is right type \(A\), and \(\mathcal{L}^* = \mu_{\mathcal{L}^*}^+\);
3. \(S\) is right type \(A\), and \((S/\mu_{\mathcal{L}^*}, +, \cdot) \cong (E, +, \cdot)\).

Proof (1) \(\Rightarrow\) (2). If a right adequate semiring \((S, +, \cdot)\) satisfies \((A_1)\) and \((A_2)\), then
\[
a + (e + a)^* = a + (a + e)^* = a + a^* + e = a + e = e + a.
\]
Hence, \(S\) is right type \(A\) and by Lemma 2.3, \(\mathcal{L}^* = \mu_{\mathcal{L}^*}^+\).

(2) \(\Rightarrow\) (3). If \(S\) is right type \(A\) and \(\mathcal{L}^* = \mu_{\mathcal{L}^*}^+\), then by Lemma 1.1 \((S/\mu_{\mathcal{L}^*}, +) \cong (E, +)\), and by Proposition 2.1 \((S/\mu_{\mathcal{L}^*}, +, \cdot) \cong (E, +, \cdot)\).

(3) \(\Rightarrow\) (1). If \(S\) is right type \(A\) and \((S/\mu_{\mathcal{L}^*}, +, \cdot) \cong (E, +, \cdot)\), then \((S, +)\) is a right type \(A\) semigroup. According to Corollary 2.8 in [4], \(\mu_{\mathcal{L}^*}^+ = \mathcal{L}^*\) and \(E\) is central in \((S, +)\), which means that \(S\) satisfies \((A_1)\) and \((ab)^* = a^*b^*\) since \(\mathcal{L}^*\) is a congruence on the semiring \(S\). We deduce \((ab)^* + a^*b^* = a^*b^*\). Therefore, \(S\) satisfies \((A_1)\) and \((A_2)\). \(\Box\)

Following example shows that the condition that \(S\) is right type \(A\) cannot be deleted in Proposition 3.1 (2) and (3).

Example 3.1 Let \(N\) denote the set of natural numbers and put \(I = N \times N\). On \(S = N \cup I\) define operations \(\oplus\) and \(\cdot\) as follows:

For all \(m, n, h, k \in N\): \(m \oplus n = m + n; m \oplus (h, k) = (m + h, k); (h, k) \oplus m = (h, k + m); (h, k) \oplus (m, n) = (h, k + m + n)\). For all \(x, y \in S\), \(xy = 0\). Then, \(S\) is a semiring whose additive idempotents are \(\{0, (0, 0)\}\). \(\mathcal{L}^*\)-classes of \(S\) are \(\{N, I\}\) so that \(S\) is right adequate. Notice that \((0, 0) \oplus m \neq m \oplus (0, 0)\) as \(m \neq 0\), and so \(S\) does not satisfy \((A_1)\). Since \((0, 0) \oplus k = (0, k)\) whereas \(k \oplus ((0, 0) \oplus k)^* = k \oplus (0, 0) = (k, 0)\), we see that \(S\) is not right type \(A\), while \(\mathcal{L}^*\) is a semiring congruence on \(S\).

By Proposition 3.1, the proof of following Theorem 3.1 can be proved similarly to Theorem 2.1.

Theorem 3.1 Let \((S, +, \cdot)\) be a right type \(A\) adequate semiring on which \(\mathcal{L}^*\) is a congruence. Then the following statements are true:

1. \(S/\mathcal{L}^*\) is a b-lattice if and only if \(S\) satisfies \((A_3)\);
2. \(S/\mathcal{L}^*\) is a commutative b-lattice if and only if \(S\) satisfies \((A_3)\) and \((A_4)\);
3. \(S/\mathcal{L}^*\) is a distributive lattice if and only if \(S\) satisfies \((A_3)-(A_5)\).

Corollary 3.1 For a right adequate semiring \((S, +, \cdot)\) satisfying \((A_1)\) and \((A_2)\), the following statements are valid:

1. If \(S\) satisfies \((A_3)\), then \(S\) is right type \(A\) and is a b-lattice of left skew-halfrings.
2. If \(S\) satisfies \((A_3)\) and \((A_4)\), then \(S\) is right type \(A\) and is a commutative b-lattice of left skew-halfrings.
(3) If \( S \) satisfies (A\(_3\))–(A\(_5\)), then \( S \) is right type A and is a distributive lattice of left skew-halfrings.

Now, we consider another class of right adequate semirings on which \( \hat{L}^* \) is also a congruence.

**Proposition 3.2** For a right adequate semiring \((S, +, \cdot)\), the following statements are equivalent:

1. \( S \) satisfies (B\(_1\)) and (A\(_2\));
2. \( \hat{L}^* = \mu_{\hat{L}^*} \);
3. \( (S/\mu_{\hat{L}^*}, +, \cdot) \cong (E^+(S), +, \cdot) \).

**Proof** (1) \( \Rightarrow \) (2). If \((S, +, \cdot)\) satisfies (B\(_1\)), \( \hat{L}^* \) is the smallest semilattice congruence on \((S, +)\) by Proposition 2.7 in [4]. Since \( \hat{L}^* \) is a congruence on additive reduce \((S, +)\) and \( S \) satisfies (A\(_2\)), \( \hat{L}^* \) is also a multiplication congruence on \( S \). Therefore, \( \hat{L}^* = \mu_{\hat{L}^*} \).

(2) \( \Rightarrow \) (3). Clearly, \( (S/\mu_{\hat{L}^*}, +, \cdot) \cong (E^+(S), +, \cdot) \).

(3) \( \Rightarrow \) (1). If \( (S/\mu_{\hat{L}^*}, +, \cdot) \cong (E^+(S), +, \cdot) \), then \( \hat{L}^* \) is a semiring congruence on \( S \) implies \( S \) satisfies (A\(_2\)). By Proposition 2.7 in [4] we know that \((a + e, e + a) \in \hat{L}^* \) for \( e \in E^+(S) \) and \( a \in S \). Therefore, (B\(_1\)) and (A\(_2\)) hold on \( S \). \( \square \)

**Theorem 3.2** Let \((S, +, \cdot)\) be a right adequate semiring. Then the following statements are true.

1. If \( S \) satisfies (B\(_1\)) and (A\(_2\)), then \( S/\hat{L}^* \) is a semiring with semilattice additive reduct.
2. If \( S \) satisfies (B\(_1\)), (A\(_2\)) and (A\(_3\)), then \( S/\hat{L}^* \) is a b-lattice.
3. If \( S \) satisfies (B\(_1\)), (A\(_2\))–(A\(_4\)), then \( S/\hat{L}^* \) is a commutative b-lattice.
4. If \( S \) satisfies (B\(_1\)), (A\(_2\))–(A\(_5\)), then \( S/\hat{L}^* \) is a distributive lattice.

**Proof** The proof is the same as that of Theorem 2.1. \( \square \)

In the dual case, we can obtain the similar results on left adequate semirings which satisfy (B\(_1\)').

Now, let us consider the adequate semiring satisfying (B\(_1\)) and (B\(_1\)').

**Proposition 3.3** An adequate semiring \((S, +, \cdot)\) satisfies (B\(_1\)) and (B\(_1\)') if and only if \( S \) satisfies (A\(_1\)).

**Proof** Suppose that the adequate semiring \( S \) satisfies conditions (B\(_1\)) and (B\(_1\)'). Then

\[
(a + e, e + a) \in \hat{H}^* = \hat{L}^* \cap \hat{R}^* \text{ for } e \in E^+(S) \text{ and } a \in S
\]

\[
\Rightarrow (a + a^*, a^* + a) \in \hat{H}^* \text{ and } (a + a^*, a^* + a) \in \hat{H}^*
\]

\[
\Rightarrow (a + a^*)\hat{L}^*(a^* + a), (a^* + a)\hat{R}^*(a^* + a)
\]

\[
(a^* + a)\hat{L}^*(a^* + a), (a + a^*)\hat{R}^*(a^* + a)
\]
\[ \Rightarrow a^* L^* a L^*(a^* + a^+) \text{ and } a^+ R^* a R^*(a^* + a^+) \]
\[ \Rightarrow a^* = a^* + a^+ = a^+. \]

From Proposition 2.9 in [4], \( a + e = e + a \) for \( e \in E^+(S) \) and \( a \in S \). Therefore, \( S \) satisfies \((A_1)\).

Conversely, if \( S \) is an adequate semiring satisfying \((A_1)\), then we have \( a + e = e + a \in H^* = L^* \cap R^* \) for \( e \in E^+(S) \) and \( a \in S \). Hence, \( S \) satisfies \((B_1)\) and \((B'_1)\). 

\[ \square \]

4. Strong b-lattices of [left] skew-halfrings

In this section, we want to extend results of inverse semirings in [2], [5] and [14] to adequate semirings and right adequate semirings.

Lemma 4.1 ([14]) If \( T \) is a b-lattice, then \( \alpha \beta \leq \alpha + \beta \) for all \( \alpha, \beta \in T \).

Proof Since \( \alpha + \beta = (\alpha + \beta)(\alpha + \beta) = \alpha(\alpha + \beta) + \beta(\alpha + \beta) = \alpha + \alpha \beta + \beta \alpha + \beta \alpha \leq \alpha + \beta \).

Let \( T \) be a b-lattice. From Lemma 4.1 we know that if, for \( \alpha, \beta, \gamma \in T \), \( \alpha + \beta \leq \gamma \), then \( \alpha + \beta + \alpha \beta \leq \gamma \).

Now, let us cite the construction which was introduced in [14].

Definition 4.1 (Definition 2.3 in [14]) Let \( T \) be a b-lattice and \( \{ S_\alpha : \alpha \in T \} \) be a family of pairwise disjoint semirings which are indexed by the elements of \( T \). For each \( \alpha \leq \beta \) in \( T \), we now embed \( S_\alpha \) in \( S_\beta \) via a semiring monomorphism \( \phi_{\alpha, \beta} \) satisfying the following conditions:

1. \( \phi_{\alpha, \alpha} = I_{S_\alpha} \), the identity mapping on \( S_\alpha \).
2. \( \phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma} \) if \( \alpha \leq \beta \leq \gamma \).
3. \( S_\alpha \phi_{\alpha, \gamma} S_\beta \phi_{\beta, \gamma} \subseteq S_{\alpha \beta} \phi_{\alpha \beta, \gamma} \) if \( \alpha + \beta \leq \gamma \), i.e., \( \alpha + \beta + \alpha \beta \leq \gamma \).

On \( S = \bigcup_{\alpha \in T} S_\alpha \), we define addition \( + \) and multiplication \( \cdot \) for \( a \in S_\alpha, b \in S_\beta \), as follows:

\[ a + b = a \phi_{\alpha, \alpha + \beta} + b \phi_{\beta, \alpha + \beta} \]

and

\[ a \cdot b = c \in S_{\alpha \beta} \text{ such that } c \phi_{\alpha \beta, \alpha + \beta} = a \phi_{\alpha, \alpha + \beta} \cdot b \phi_{\beta, \alpha + \beta}. \]

We denote the above system by \( S = \langle T, S_\alpha, \phi_{\alpha, \alpha + \beta} \rangle \) and call it the strong b-lattice \( T \) of the semirings \( \{ S_\alpha : \alpha \in T \} \).

In an obvious way, we may replace “b-lattice \( T \)” in the above definition by “distributive lattice \( D \)”, \( S = \langle D, S_\alpha, \phi_{\alpha, \alpha + \beta} \rangle \) and call it the strong distributive lattice \( D \) of the semirings \( \{ S_\alpha : \alpha \in T \} \).

Lemma 4.2 (Theorem 2.4 in [14]) The system \( S = \langle T, S_\alpha, \phi_{\alpha, \alpha + \beta} \rangle \) defined above is a semiring.

Ghosh [5] considered the concept of full subdirect products of a distributive lattice and a ring, and proved that subdirect products of a distributive lattice and a ring are full. Now, we introduce the additively full subdirect products of an additive idempotent semiring \( T \) and a
Proof (1) ⇒ (2). Assume that $S$ is an additive full subdirect product of a b-lattice $T$ and a skew-halfring $R$. To prove that $S$ is a strong b-lattice of skew-halfring, put $R_\alpha = (\{\alpha\} \times R) \cap S$ for $\alpha \in T$. Then $R_\alpha$ is a skew-halfring for each $\alpha \in T$ and $S = \bigcup_{\alpha \in T} R_\alpha$. Now for each pair $\alpha, \beta \in T$ with $\alpha \leq + \beta$, we define a mapping $\phi_{\alpha, \beta} : R_\alpha \to R_\beta$ by $(\alpha, r) \phi_{\alpha, \beta} = (\alpha, r) + (\beta, 0_R) = (\beta, r)$. Clearly, $\phi_{\alpha, \beta}$ is a monomorphism satisfying the condition $\phi_{\alpha, \alpha} = I_{R_\alpha}$ and $\phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma}$ if $\alpha \leq + \beta \leq \gamma$ for $\alpha, \beta, \gamma \in T$.

Let $\alpha + \beta \leq \gamma$, $a = (\alpha, r) \in R_\alpha$ and $b = (\beta, r') \in R_\beta$. Then we have

$$a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') \in R_{\alpha + \beta}$$

and

$$ab = (\alpha, r)(\beta, r') = (\alpha \beta, rr') \in R_{\alpha \beta}.$$ 

Now, $(a \phi_{\alpha, \gamma})(b \phi_{\beta, \gamma}) = (\gamma, r)(\gamma, r') = (\gamma, rr') = (\alpha \beta, rr') \phi_{\alpha \beta, \gamma} = (ab) \phi_{\alpha \beta, \gamma}$ if $\alpha + \beta \leq + \gamma$. Also,

$$a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') = (\alpha + \beta, r) + (\alpha + \beta, r') = a \phi_{\alpha, \alpha + \beta} + b \phi_{\beta, \alpha + \beta}$$
Therefore, $S$ is a strong b-lattice of skew-halfrings $R_\alpha$, i.e., $S = (T, R_\alpha, \phi_{a_\alpha a_\beta})$.

(2) $\Rightarrow$ (3). Suppose that $S$ is a strong b-lattice $T$ of skew-halfrings $\{R_\alpha : \alpha \in T\}$. First, we prove that the additive reduce $(S, +)$ is adequate. If $a + x = a + y$ for $a \in R_\alpha$, $x \in R_\beta$ and $y \in R_\gamma$, then $a\phi_{a_\alpha a_\beta} + x\phi_{a_\alpha a_\beta} = a\phi_{a_\alpha a_\beta} + y\phi_{a_\alpha a_\beta}$. Since $R_\alpha a_\beta = R_\alpha a_\gamma$ and $R_\alpha a_\beta$ is cancellative, we deduce that $x\phi_{a_\alpha a_\beta} = y\phi_{a_\alpha a_\beta}$ and so $0_\alpha a_\alpha a_\beta + x\phi_{a_\alpha a_\beta} = 0_\alpha a_\alpha a_\beta + y\phi_{a_\alpha a_\beta}$, that is, $0_\alpha + x = 0_\alpha + y$. Similarly, $0_\alpha + x = 0_\alpha + y$ implies $a + x = a + y$, which shows that $a\mathcal{L}^*0_\alpha$.

Therefore, every $\mathcal{L}^*$-class contains an idempotent. Symmetrically, every $\mathcal{R}^*$-class contains an idempotent. Hence, the additive reduce $(S, +)$ is adequate. Since $(ab)^* S_\alpha$ and $b \in S_\beta$, we have $(ab)^* = 0_\alpha b = 0_\alpha b^*$. By $aa^* S_\alpha \cap E^+(S)$ we have $aa^* = a^*$. Finally, if $a^* = b^*$ and $a +_\gamma b +_\gamma c$, then there exist $\gamma, \delta$ in $T$ such that $a +_\gamma b +_\gamma 0_\delta$, where $a +_\gamma c$ and $a +_\gamma b$. Hence, $a + (0_\gamma +_\delta) = b + (0_\gamma +_\delta)$. Since $0_\gamma +_\delta = 0_\gamma +_\delta$, immediately $a = b$ by $(A_6)$.

(3) $\Rightarrow$ (4). We need only to prove that $S$ is proper type A. Since $S$ satisfies $(A_1)$, $S$ is type A with $E^+(S)$ as central on additive reduct. According to Lemma 2.3, $\mathcal{R}^* = \mathcal{L}^* = \mathcal{H}^*$. If $a(\sigma \cap \mathcal{H}^\ast)b$ for $a, b \in S$, then $a^* = b^*$ and there exists $e$ in $E^+(S)$ such that $a + e = b + e$. Put $c = a + e$. Then $a +_c^* c$ and $b +_c^* c$. According to $(A_2^\prime)$, $a = b$, which means that $\sigma \cap \mathcal{H}^* = \sigma \cap \mathcal{R}^* = \iota S$. Therefore, $S$ is proper type A.

(4) $\Rightarrow$ (5). If a semiring $S$ is proper type A with $E^+(S)$ the central of additive reduct, and satisfies $(A_2)$ and $(A_3)$, then $S$ is adequate and satisfies the condition $(A_1)$, $(A_2)$ and $(A_3)$. By Theorem 2.1 (2), $\mathcal{R}^* = \mathcal{L}^* = \mathcal{H}^*$. Since $S$ is proper type A, $\sigma \cap \mathcal{H}^* = \iota S$. In view of Lemma 2.1 (3), $\sigma$ is a congruence on the semiring $S$. Then $S$ can be embedded in direct product $S / \sigma \times S / H^\ast$ under the mapping $a \rightarrow (a\sigma, aH^\ast)$ and both projections from $S / \sigma$ and $S / H^\ast$ are onto. Consequently, $S$ is a subdirect product of a b-lattice and a skew-halfring. $\square$

Corollary 4.1 The following statements are equivalent on a semiring $S$:
(1) $S$ is an additive full subdirect product of a commutative $b$-lattice and a skew-halfring.
(2) $S$ is adequate and satisfies $(A_1)$–$(A_4)$ and $(A_6)$.
(3) $S$ is adequate and satisfies $(A_1)$–$(A_4)$ and $(A'_6)$.
(4) $S$ is proper type $A$ with $E^+(S)$ as central on additive reduct, and it satisfies $(A_2)$–$(A_4)$.

Proof The conclusions follow from Theorem 2.1 (3) and Theorem 4.1. □

Corollary 4.2 The following statements are equivalent on a semiring $S$:
(1) $S$ is an additive full subdirect product of a distributive lattice and a skew-halfring.
(2) $S$ is adequate and satisfies $(A_1)$–$(A_6)$.
(3) $S$ is adequate and satisfies $(A_1)$–$(A_6)$.
(4) $S$ is proper type $A$ with $E^+(S)$ as central on additive reduct, and it satisfies $(A_2)$–$(A_5)$.

Proof The conclusions follow from Theorem 2.1 (4) and Theorem 4.1. □

Let $S$ be a semiring. In [2], [5] and [14] that $E^+(S)$ is a k-ideal plays an important role in studying the structure of Clifford semirings and Generalized Clifford semirings. Recall that an ideal $I$ of a semiring $S$ is a k-ideal if $a \in I$ and either $a + x \in I$ or $x + a \in I$ for some $x \in S$ implies $x \in I$.

Proposition 4.1 Let $(S, +, \cdot)$ be a type $A$ semiring with $E^+(S)$ as central on additive reduct. If $(S, +, \cdot)$ is proper, then $E^+(S)$ is a k-ideal.

Proof Assume that $(S, +, \cdot)$ is proper type $A$ with $E^+(S)$ as central on additive reduct. It is clear that $E^+(S)$ is an ideal of $S$. If $a + x = e \in E^+(S)$ for $a \in S$ and $x \in E^+(S)$, then $a + (x + e) = a + a^* + (x + e) = a^* + (a + x) + e = a^* + e$ from which we get $a\sigma a^*$. But $a\mathcal{L}^*a^*$ together with the fact that $S$ is proper yields $a = a^* \in E^+(S)$. Hence, $E^+(S)$ is a k-ideal of $S$. □

Example 4.2 shows that the converse of Proposition 4.1 is not true. Hence, the condition of Theorem 4.1 (5) that type $A$ semiring $S$ is proper cannot be replaced by that $E^+(S)$ is a k-ideal.

Example 4.2 Let $S = A \cup B$ as Example 2.1. We give the new multiplication $\circ$ on $S$:

$$
 x \circ y = \begin{cases} 
 1 & \text{for } x, y \in A, \\
 0 & \text{for } x, y \in B, \\
 0 & \text{for } x \in A, y \in B \text{ or } y \in A, x \in B.
\end{cases}
$$

It is not difficult to check that $S$ is a type $A$ semiring with $E^+(S)$ as central on additive reduct and satisfies $(A_2)$ and $(A_3)$. Clearly, $E^+(S) = \{1, 0\}$ is an ideal of $S$. If $x + e \in E^+(S)$ for $e \in E^+(S)$, then $x \in E^+(S)$, that is, $E^+(S)$ is a k-ideal. On the other hand, since $a(\sigma b) = a = a(\sigma c)$ and $b\mathcal{L}^*c$ imply that $b(\sigma \cap \mathcal{L}^*)c$, we know that $S$ is not proper.

By using Lemmas 2.2 and 2.3 it is not difficult to prove the following results of right adequate semirings.

Theorem 4.2 The following conditions on a semiring $(S, +, \cdot)$ are equivalent:
(1) $S$ is an additive full subdirect product of a $b$-lattice and a left skew-halfring.
(2) $S$ is a strong b-lattice of left skew-halfrings.
(3) $S$ is right adequate and satisfies the conditions $(A_1)$–$(A_4)$ and $(A_6)$.
(4) $S$ is proper right type A with $E^+(S)$ as central on additive reduct, and satisfies $(A_2)$ and $(A_3)$.

**Corollary 4.3** The following statements are equivalent on a semiring $S$:
(1) $S$ is an additive full subdirect product of a commutative b-lattice and a left skew-halfring.
(2) $S$ is right adequate and satisfies $(A_1)$–$(A_4)$ and $(A_6)$.
(3) $S$ is proper right type A with $E^+(S)$ as central on additive reduct, and it satisfies $(A_2)$–$(A_4)$.

**Corollary 4.4** The following statements are equivalent on a semiring $S$:
(1) $S$ is an additive full subdirect product of a distributive lattice and a left skew-halfring.
(2) $S$ is right adequate and satisfies $(A_1)$–$(A_6)$.
(3) $S$ is proper right type A with $E^+(S)$ as central on additive reduct, and it satisfies $(A_2)$–$(A_5)$.

5. **Subdirect products of an additively commutative idempotent semiring and a [left] skew-halfring**

A semiring is called an additively commutative idempotent semiring if its additive reduct is a semilattice. In this section, we want to extend several results in section 5 to the full subdirect products of an additively commutative idempotent semiring and a [left] skew-halfring.

**Theorem 5.1** The following statements are equivalent on a semiring $(S, +, \cdot)$:
(1) $S$ is an additive full subdirect product of an additively commutative idempotent semiring and a skew-halfring.
(2) $S$ is an adequate semiring and satisfies $(A_1)$, $(A_2)$, $(A_6)$.
(3) $S$ is an adequate semiring and satisfies $(A_1)$, $(A_2)$, $(A'_6)$.
(4) $S$ is proper type A with $E^+(S)$ as central on additive reduct, and satisfies $(A_2)$.

**Proof** $(1) \Rightarrow (2)$. Assume that $S$ is an additive full subdirect product of an additively commutative idempotent semiring $T$ and a skew-halfring $R$, then $E^+(S) = (T \times \{0_R\}) \cap S$. Since for each $a \in S$ there exist $\alpha$ in $T$ and $r$ in $R$ such that $a = (\alpha, r)$, we have $(\alpha, r) + (\beta, 0_R) = (\beta, 0_R) + (\alpha, r)$, where $(\beta, 0_R) \in E^+(S)$. Hence, $S$ satisfies $(A_1)$. Now, for $x = (\beta, r_1), y = (\beta, r_2) \in S^1$,

$$a + x = a + y \Leftrightarrow (\alpha + \beta, r + r_1) = (\alpha + \gamma, r + r_2)$$

$$\Leftrightarrow \alpha + \beta = \alpha + \gamma \text{ and } r_1 = r_2$$

$$\Leftrightarrow (\alpha, 0_R) + (\beta, r_1) = (\alpha, 0_R) + (\beta, r_2)$$

$$\Leftrightarrow (\alpha, 0_R) + x = (\alpha, 0_R) + y,$$

which means that $aL^+(\alpha, 0_R) = a^\ast$. Similarly, $aR^+(\alpha, 0_R) = a^\ast$. That is, $S$ is adequate. Now,
for any $a = (\alpha, r), b = (\beta, r') \in S$. $(ab)^* = (\alpha\beta, 0_R) = (\alpha, 0_R)(\beta, 0_R) = a^*b^*$. Hence, $(A_2)$ holds in $S$. Further, for $a = (\alpha, r), b = (\beta, r') \in S$, $e = (\gamma, 0_R) \in E^+(S)$,

$$a^* = b^*, \text{ and } a + e = b + e \Rightarrow (\alpha, r) + (\gamma, 0_R) = (\beta, r') + (\gamma, 0_R)$$

$$\Rightarrow \alpha = \beta \text{ and } r = r' \Rightarrow (\alpha, r) = (\beta, r') \Rightarrow a = b.$$  

Hence, $(A_6)$ holds in $S$.

(2) $\Rightarrow$ (3). It is easy to check that if $S$ is adequate and satisfies $(A_1)$, then $(A_6)$ is equivalent to $(A'_6)$.

(3) $\Rightarrow$ (4). Assume that $S$ is adequate and satisfies $(A_1), (A_2)$ and $(A'_6)$. Since $S$ satisfies $(A_1)$, $S$ is type A with $E^+(S)$ as central on additive reduct. According to Lemma 2.3, $\mathcal{R}^* = \mathcal{L}^* = \mathcal{H}^*$.

Now, assume that $S$ satisfies $(A'_6)$. If $a(\sigma \cap \mathcal{H}^*)b$ for $a, b \in S$, then $a^* = b^*$ and there exists $e$ in $E^+(S)$ such that $a + e = b + e$. Put $c = a + e$. Then $a^+ \leq c$ and $b^+ \leq c$. Hence, $a = b$. It means that $\sigma \cap \mathcal{L}^* = \sigma \cap \mathcal{R}^* = \iota_S$. Therefore, $S$ is proper type A.

(4) $\Rightarrow$ (1). Assume that $S$ is proper type A with $E^+(S)$ as central on additive reduct, and satisfies $(A_2)$. By Theorem 2.1 and Lemma 2.1, $S/\mathcal{H}^*$ is a semiring with semilattice additive reduct and $\sigma$ is the minimum skew-halfring congruence on $S$. Moreover, $\sigma \cap \mathcal{L}^* = \sigma \cap \mathcal{R}^* = \sigma \cap \mathcal{H}^* = \iota_S$. Now, we define a mapping $\Psi : S \to S/\sigma \times S/\mathcal{H}^*$ by $\Psi(a) = (a\sigma, a\mathcal{H}^*)$. It is a routine calculation that $\Psi$ is monomorphism and the corresponding projective mappings are surjective. It follows that $S$ is a subdirect product of the additively commutative idempotent semiring $S/\mathcal{H}^*$ and the skew-halfring $S/\sigma$. By Lemma 2.1 in [9] $E^+(S)$ is a $\sigma$-class, which shows that for any $a \in S$, $a\sigma(a+a)$ implies $a \in E^+(S)$. Moreover, by Proposition 2.9 in [3] every $\mathcal{H}^*$-class contains an idempotent. Then, for any additive idempotent $\alpha \in E^+(S/\sigma \times S/\mathcal{H}^*)$ there exists $a \in E^+(S)$ such that $\Psi(a) = \alpha = (a\sigma, a\mathcal{H}^*)$, which means that $\Psi(E^+(S)) \supseteq E^+(S/\sigma \times S/\mathcal{H}^*)$.

Therefore, $S$ is a full subdirect product of the additively commutative idempotent semiring $S/\mathcal{H}^*$ and the skew-halfring $S/\sigma$.

We complete the proof. $\square$

A semiring $(S, +, \cdot)$ can be a full subdirect product of an additively commutative idempotent semiring and a skew-halfring, but $E^+(S)$ may be not multiplicative idempotent.

Example 5.1 Assume that $T = \{e, f\}$ is a semiring with the addition $+$ and multiplication $\cdot$ as follows: $e + e = e, f + f = f, e + f = f + e = f, e \cdot e = f \cdot f = e \cdot f = f \cdot e = f$. Let $U$ be a skew-halfring. Additive idempotents $E^+(T \times U)$ of the direct product $T \times U$ is not multiplicative idempotent since $e \cdot e = f \neq e$.

Similarly, we can prove following theorem for right adequate semirings.

Theorem 5.2 The following statements are equivalent on a semiring $(S, +, \cdot)$:

(1) $S$ is an additive full subdirect product of an additively commutative idempotent semiring and a left skew-halfring.

(2) $S$ is a right adequate semiring and satisfies $(A_1), (A_2), (A_6)$. 

(3) $S$ is proper right type A with $E^+(S)$ as central on additive reduct, and satisfies $(A_2)$.

References