Incompleteness of Complex Exponential System in $L^p_{\alpha}$ Space

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Abstract A necessary and sufficient condition is obtained for the incompleteness of complex exponential system in the weighted Banach space $L^p_{\alpha} = \{f : \int_{-\infty}^{\infty} |f(t)e^{-\alpha(t)}|^p dt < +\infty\}$, where $1 \leq p < +\infty$ and $\alpha(t)$ is a weight on $\mathbb{R}$.

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1. Introduction

A system $E = \{e_k : k = 1, 2, \ldots\}$ of elements of a Banach space $B$ is called incomplete if $\text{span} E$ does not coincide with the whole $B$, where $\text{span} E$ is the linear span of the system $E$ and the $\overline{\text{span}} E$ is the closure of $\text{span} E$ in $B$.

Suppose $\alpha(t)$ is a nonnegative continuous function, called a weight, on $\mathbb{R}$, and satisfies
\[
\lim_{t \to +\infty} t^{-1}\alpha(t) = +\infty, \quad a_0 = \limsup_{t \to -\infty} |t|^{-1}\alpha(t) < +\infty. \quad (1)
\]
Given a weight $\alpha(t)$, we take the weighted Banach space $C_\alpha$ consisting of complex continuous functions $f(t)$ defined on $\mathbb{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity and the norm $\|f\|_\alpha = \sup\{|f(t)e^{-\alpha(t)}| : t \in \mathbb{R}\}$. Suppose $L^p_{\alpha} = \{f : \|f\| = (\int_{-\infty}^{\infty} |f(t)e^{-\alpha(t)}|^p dt)^{1/p} < +\infty\}$, $1 \leq p < +\infty$. Then $L^p_{\alpha}$ is also a Banach space. Let $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ be a sequence of distinct complex numbers in the half plane $C_{a_0} = \{z = x + iy : x > a_0\}$ satisfying
\[
a_1 = \sup_{n} |\theta_n| < \frac{\pi}{2}. \quad (2)
\]
Let $M = \{m_n : n = 1, 2, \ldots\}$ be a sequence of positive integers and suppose that there exists an increasing positive function $q(r)$ on $[0, \infty)$ satisfying
\[
a_2 = \limsup_{r \to +\infty} q(r)r^{-1}\log r < +\infty; \quad (3)
\]
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\[ D(q) = \limsup_{r \to +\infty} \frac{n(r + q(r)) - n(r)}{q(r)}. \] (4)

where \( n(t) = \sum_{|\lambda_n| \leq t} m_n \). We denote the system of complex exponentials by
\[ E(\Lambda, M) = \{ t^{k-1} e^{\lambda_n t} : k = 1, 2, \ldots, m_n; n = 1, 2, \ldots \}. \]

The condition (1) guarantees that \( E(\Lambda, M) \) is a subset of \( C_0 \) and \( L_p^0 \). In the article [1], the author has obtained some results on incompleteness of \( E(\Lambda, M) \) in \( C_0 \). Now we ask whether \( E(\Lambda, E) \) is incomplete in \( L_p^0 \) in the norm \( \| \| \). The similar results were obtained in the articles [2], [3] and [4].

**Theorem A** ([(1)]) Let \( \alpha(t) \) be continuous on \( \mathbb{R} \) and convex on \( [t_0, \infty) \) for some constant \( t_0 \), and satisfy (1). Suppose that \( \Lambda = \{ \lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \ldots \} \) is a sequence of distinct complex numbers in \( C_{a_0} \) satisfying (2) and \( M = \{ m_n : n = 1, 2, \ldots \} \) is a sequence of positive integers.

If there exists a positive and increasing function \( q(r) \) on \( [0, \infty) \) such that (3) and (4) hold, then \( E(\Lambda, M) \) is incomplete in \( C_0 \) if and only if there exists \( a \in \mathbb{R} \) such that
\[ J(a) = \int_0^{+\infty} \frac{\alpha(\lambda(t) + a)}{1 + t^2} \, dt < +\infty, \] (5)

where
\[ \lambda(r) = \begin{cases} 2 \sum_{|\lambda| \leq r} \frac{m_n \cos \theta_n}{|\lambda_n|} \, dt, & \text{if } r \geq |\lambda_1|, \\ 0, & \text{otherwise}. \end{cases} \]

**Theorem 1** Assume \( \alpha(t) \), \( \Lambda \) and \( M \) satisfy the same conditions as Theorem A. If there exists a positive and increasing function \( q(r) \) on \( [0, \infty) \) such that (3) and (4) hold, then \( E(\Lambda, M) \) is incomplete in \( L_p^0 \) if and only if there exists \( a \in \mathbb{R} \) such that (5) holds.

2. Lemmas and Proof of Theorem 1

In order to prove Theorem 1, we need the following technical lemmas.

**Lemma 1** ([5]) Let \( \beta(x) \) be a convex function on \([0, \infty)\) and assume that
\[ \beta^*(t) = \sup\{ xt - \beta(x) : x \geq 0 \}, \quad t \in \mathbb{R} \] (7)

is the Legendre transform (or the Young dual function) ([6]) of \( \beta(x) \). Suppose that \( \lambda(r) \) is an increasing function on \([0, \infty)\) satisfying
\[ \lambda(R) - \lambda(r) \leq A(\log R - \log r + 1), \quad R > r > 1. \] (8)

Then there exists an analytic function \( f(z) \neq 0 \) in \( C_0 = \{ z = x + iy : x > 0 \} \) satisfying
\[ |f(z)| \leq A \exp\{Ax + \beta(x) - x\lambda(|z|)\}, \quad z = x + iy \in C_0, \] (9)

if and only if there exists \( a \in \mathbb{R} \) such that
\[ \int_1^{+\infty} \frac{\beta^*(\lambda(t) + a)}{1 + t^2} \, dt < +\infty. \] (10)
Remark We denote a positive constant by $A$, not necessarily the same at each occurrence.

Lemma 2 ([1]) Suppose that $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers satisfying (2) and $M = \{m_n : n = 1, 2, \ldots\}$ is a sequence of positive integers. If there exists a positive and increasing function $q(r)$ on $[0, \infty)$ such that (3) and (4) hold, then for each $b > 0$, the function

$$G_b(z) = Q_b(z) \prod_{\Re \lambda_n > b} \left(1 - \frac{z}{\lambda_n}\right) m_n \exp \left(\frac{2\pi m_n \cos \theta_n}{|\lambda_n|}\right)$$

(11)

is meromorphic and analytic in the half-plane $\mathcal{C}_{-b} = \{z = x + iy : x > -b\}$ with zeros of orders $m_n$ at each points $\lambda_n$ ($n = 1, 2, \ldots$) and satisfies the following inequality

$$|G_b(z)| \leq \exp\{|x|\lambda(2r) + A|x| + A\}, \quad z \in \mathcal{C}_{-b},$$

(12)

where

$$Q_b(z) = \prod_{\Re \lambda_n \leq b} \left(\frac{z - \lambda_n}{z + b + 1}\right)^{m_n}.$$  

Moreover, for each positive constant $A_0$ and $\varepsilon_0 > 0$,

$$|G_b(z)| \geq \exp\{x\lambda(r) - A|x| - A\}, \quad z \in C(A_0, \varepsilon_0),$$

(13)

where $C(A_0, \varepsilon_0) = \{z \in \mathcal{C}_{-b} : |z - \lambda_n| \geq \delta_n, n = 1, 2, \ldots\}$, $\delta_n = \varepsilon_0|\lambda_n|^{-A_0}$, $n = 1, 2, \ldots$.

Proof If the system $E(\Lambda, M)$ is incomplete in $L^p_{\alpha}$, then by Hahn-Banach Theorem, there exists a bounded linear function $T$ on $L^p_{\alpha}$ such that

$$\|T\| = 1 \quad \text{and} \quad T(t^{k-1}e^{\lambda_n t}) = 0, \quad k = 1, 2, \ldots, m_n; \quad \lambda_n \in \Lambda.$$  

So by Riesz representation theorem, there exists a $g \in L^q_{\alpha}$, such that $\|T\| = \|g\|_{q, -\alpha}$ and $T(f) = \int_{-\infty}^{+\infty} f(t)g(t)dt$ ($f \in L^p_{\alpha}$), where $\frac{1}{p} + \frac{1}{q} = 1$,

$$L^q_{\alpha} = \{g : \|g\|_{q, -\alpha} = \left(\int_{-\infty}^{+\infty} |g(t)e^{\alpha(t)}|^q dt\right)^{1/q} < +\infty\};$$

$$L^\infty_{\alpha} = \{g : \|g\|_{\infty, -\alpha} = \text{ess sup}\{|g(t)|e^{\alpha(t)} : t \in \mathbb{R}\} < +\infty\}.$$  

For each $b > a_0 + 1$, the function

$$f(z) = \frac{1}{G_b(z)} \int_{-\infty}^{+\infty} e^{t(z+b)} g(t)dt, \quad x > a_0$$

is analytic in $\mathcal{C}_{-1} = \{z = x + iy : x > -1\}$, where $G_b(z)$ is defined by (11) with zeros $\lambda_n - b : n = 1, 2, \ldots$. By the Lemma 2, we have

$$|f(z)| \leq A \exp\{\tilde{\beta}(x) - x\lambda(|z|) + Ax\}, \quad x > 0,$$

where $\tilde{\beta}(x) = \sup\{|x| - \frac{a(t)}{2} : t \in \mathbb{R}\}$. Then (3) and (4) imply that for any $D_1 > D(q)$ and $A_1 > a_2$, there exists $r_0 > b + 1$ such that

$$n(r + q(r)) - n(r) \leq D_1 g(r), \quad r \geq r_0;$$

$$q(r) \leq A_1 r (\log r)^{-1} \leq 2^{-1} r, \quad r \geq r_0.$$
These imply that
\[ n(t) - n(r) \leq D_1(t + q(t) - r), \quad t > r \geq r_0; \]
\[ \lambda(R) - \lambda(r) \leq 2D_1(1 + A_1)(\log R - \log r + 1), \quad R > r \geq r_0. \]
In fact, if \( r \geq r_0 \), let \( p_0(r) = r \), \( p_{k+1}(r) = p_k(r) + q(p_k(r)) \) \((k = 0, 1, 2, \ldots)\). Then \( p_{k+1}(r) \geq p_k(r) + q(r) \) and \( p_k(r) \geq r + kq(r) \) \((k = 0, 1, 2, \ldots)\). So if \( l \geq 0 \) is an integer such that \( p_l(r) \leq t < p_{l+1}(r) \), then
\[ n(t) - n(r) \leq \sum_{k=0}^{l} (n(p_{k+1}(r)) - n(p_k(r))) \leq D_1 \sum_{k=0}^{l} (p_{k+1}(r) - p_k(r)) \]
\[ = D_1(p_l(r) + q(p_l(r)) - r) \leq D_1(t + q(t) - r). \]
Since
\[ \lambda(R) - \lambda(r) \leq \int_r^R \frac{dn(t)}{t}, \quad R > r \geq r_0, \]
integrating by parts gives
\[ \lambda(R) - \lambda(r) \leq 2D_1(1 + A_1)(\log R - \log r + 1), \quad R > r \geq r_0. \]
By Lemma 1, there exists \( a \in \mathbb{R} \) such that (5) holds.
Suppose that there exists a real number \( a \) such that (5) holds. If \( \lambda(r) \) is bounded, then (5) holds for any real number \( a \). So we may think that \( \lambda(r) \) is unbounded on \( r \geq 0 \). Let \( \varphi(t) \) be an even function such that \( \varphi(t) = a(\lambda(t) + a) \) for \( t \geq 0 \) and let \( u(z) \) be the Poisson integral of \( 2\varphi(t) \), i.e.,
\[ u(z) = \frac{x}{\pi} \int_{-\infty}^{+\infty} \frac{2\varphi(t)}{x^2 + (y - t)^2} dt. \quad (14) \]
Then \( u(x + iy) \) is harmonic in the half-plane \( \mathbb{C}_0 = \{ z = x + iy : x > 0 \} \) and there exists a positive constant \( A > 0 \) such that
\[ u(z) \geq \frac{x}{\pi} \int_{-\infty}^{+\infty} \frac{2(\varphi(|z|) - A)}{x^2 + (y - t)^2} dt = \varphi(|z|) - A, \quad x > 0. \]
Therefore, there exists an analytic function \( g(z) \) on \( \mathbb{C}_0 \) such that \( \text{Re} g(z) = u(z) \geq \varphi(|z|) - A \) \((x > 0)\). For \( b > a_0 + 2 \), let
\[ \varphi_b(z) = \frac{G_b(z)}{(1 + z + b)^4} \exp\{-g(z + b)\}, \quad (15) \]
where \( G_b(z) \) is defined by (11). Then \( \varphi_b(z) \) is analytic in \( \mathbb{C}_{-b} = \{ z = x + iy : x > -b \} \). By (12) there exists a positive constant \( A_2 \) such that
\[ |\varphi_b(z)| \leq \frac{A}{1 + y^2} \exp\{\alpha^*(x - 1) + A_2x\}, \quad x > -b. \quad (16) \]
Let
\[ h_b(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi_b(iy)e^{-iy(t + A_2)} dy \]
be the Fourier transform of \( \varphi_b(iy)e^{-iyA_2} \). Then \( h_b(t) \) is bounded and continuous on \( \mathbb{R} \). By Cauchy’s formula,
\[ h_b(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi_b(x + iy)e^{-(x+iy)(t + A_2)} dy, \quad x > -b. \quad (18) \]
By (16), (18) and the formula of the Legendre transform for $\alpha(t)$, we see that $|h_b(t)| \leq A \exp(-\alpha(t) - t) \ (t \geq t_0)$, and that (by taking $x = -b + 1$ in (18)) $|h_b(t)| \leq A \exp((b-1)t) \ (t \leq t_0)$. Therefore, by (1), if $b > a_0 + 3$, $|h_b(t)| \leq A \exp\{-\alpha(t) - |t|\} \ (t \in \mathbb{R})$. By (18) and the inverse Fourier transform formula,

$$\varphi_0(z)e^{-A^2z} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t)e^{zt}dt, \ x > a_0.$$ 

Therefore the bounded linear functional

$$T(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_b(t)h(t)dt, \ h \in L^p_{\alpha}$$

satisfies $T(t^{k-1}e^{\lambda_n t}) = 0 \ (k = 1, 2, \ldots, m_n; \lambda_n \in \Lambda)$, and

$$\|T\| = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{+\infty} |h_b(t)e^{\alpha(t)}|^q dt \right)^{\frac{1}{q}} > 0.$$ 

By the Riesz representation theorem, the space $E(\Lambda, M)$ is incomplete in $L^p_{\alpha}$. This completes the proof of Theorem 1. \qed

References


