

Bivariate Splines and Golden Section Based on Theory of Elasticity

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Abstract Splines are important in both mathematics and mechanics. We investigate the relationships between bivariate splines and mechanics in this paper. The mechanical meanings of some univariate splines were viewed based on the analysis of bending beams. For the 2D case, the relationships between a class of quintic bivariate splines with smoothness 3 and bending of thin plates are presented constructively. Furthermore, the variational property of bivariate splines and golden section in splines are also discussed.

Keywords multivariate spline; smoothing cofactor; conformality condition; bending of thin plate; golden section.

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1. Introduction

Splines play important roles in both theories and applications. Multivariate splines are the generalizations of splines from 1D to n D ($n > 1$), which have fruitful achievements widely applied to the fields such as approximation of functions, FEM, CAGD, computer graphics, and so on [1]. It is well known that a cubic spline $s(x)$ corresponds to the deflection curve of an ordinary (infinite) beam under the action of concentrated loads [2]. However, the relationships between multivariate splines and mechanics have not been discussed systematically. We present a new viewpoint of studying multivariate splines and extend their applications to more fields by investigating their mechanical models.

Holladay proved the variational property of the natural cubic spline, which minimizes the bending potential energy of a beam [3]. In the case of two dimensional space, Duchon got the “thin-plate spline”, a radial basis function, which minimizes the bending energy of a thin plate, by replacing a finite domain by the whole plane \mathbb{R}^2 (see [4]). The bivariate splines discussed in

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this paper are derived as the solutions of bending of thin plate with homogeneous boundary conditions, so we obtain the variational property of some bivariate splines based on the mechanical meaning.

The organization of this paper is as follows: Notations and fundamental results of multivariate splines will be given in the next section. Section 3 presents the mechanical meanings of some univariate splines. Section 4 introduces the related theory of thin plate with small deflection. Section 5 is devoted to establishing the relationships between the bending of simply supported polygonal thin plate and a kind of quintic C^3 bivariate splines. The variational property of some bivariate splines and golden section in splines will also be discussed in this section.

2. Notations and results of multivariate splines

In 1975, Wang [5] initiated the “smoothing cofactor conformality method” to study the basic theory of multivariate splines on arbitrary partitions by using the methods of function theory and algebraic geometry. Making use of this method, any problem on multivariate splines can be essentially equivalent to a corresponding algebraic problem. Within the framework of this method, we have the following basic notations and results.

Let Ω be a domain in \mathbb{R}^2 , and \mathbb{P}_k be the collection of all the bivariate polynomials with real coefficients and total degree $\leq k$:

$$\mathbb{P}_k := \left\{ p(x, y) = \sum_{i=0}^k \sum_{j=0}^{k-i} c_{ij} x^i y^j \mid c_{ij} \in \mathbb{R} \right\}. \quad (1)$$

The space of multivariate spline functions is defined by

$$S_k^\mu(\Delta) := \{ s \in C^\mu(\Omega) \mid s|_{\Delta_i} \in \mathbb{P}_k, i = 1, \dots, N \} \quad (2)$$

where Δ is a partition of Ω , $\Delta_i, i = 1, \dots, N$ are the “cells” of Δ , and s is a piecewise polynomial of degree k possessing μ order continuous partial derivatives in Ω . $S_k^{\mu,0}(\Delta)$ denotes the subspace of $S_k^\mu(\Delta)$ with homogeneous boundary condition $s|_{\partial\Omega} = 0$.

Lemma 1 *Let the representations of $z = s(x, y)$ on two arbitrary adjacent cells Δ_i and Δ_j be $z = p_i(x, y)$ and $z = p_j(x, y)$, respectively, where $p_i(x, y), p_j(x, y) \in \mathbb{P}_k$. Then $s(x, y) \in C^\mu(\overline{\Delta_i \cup \Delta_j})$, if and only if there is a polynomial $q_{ij}(x, y) \in \mathbb{P}_{k-(\mu+1)d}$, such that*

$$p_i(x, y) - p_j(x, y) = [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y), \quad (3)$$

where $\overline{\Delta_i}$ and $\overline{\Delta_j}$ have common interior edge $\Gamma_{ij} : l_{ij}(x, y) = 0$, and the irreducible algebraic polynomial $l_{ij}(x, y) \in \mathbb{P}_d$.

The polynomial factor $q_{ij}(x, y)$ defined by (3) is called the smoothing cofactor of $s(x, y)$ across the interior edge $\Gamma_{ij} : l_{ij}(x, y) = 0$ from Δ_i to Δ_j (see [1, 5]).

Given an interior vertex A , we define the Conformality Condition at A by

$$\sum_A [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y) \equiv 0, \quad (4)$$

where \sum_A presents the summation over all the interior edges around A counterclockwise, and $q_{ij}(x, y)$ is the smoothing cofactor on Γ_{ij} . Let A_1, \dots, A_M be all of the interior vertices in Δ . Then the Global Conformality Condition is

$$\sum_{A_v} [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y) \equiv 0, \quad v = 1, \dots, M, \tag{5}$$

where $q_{ij}(x, y)$ satisfies the Conformality Condition corresponding to A_v .

We have the following existence theorem of bivariate splines.

Theorem 1 *Let Δ be any partition of Ω . The multivariate spline function $s(x, y) \in S_k^\mu(\Delta)$ exists, if and only if for every interior edge, there exists a smoothing cofactor of $s(x, y)$, which satisfies the Global Conformality Condition.*

A function $s(x, y) \in S_k^\mu(\Delta)$ is called the non-degenerate bivariate spline, if $s(x, y)$ has at least one non-zero smoothing cofactor.

According to these definitions, the following results can be obtained.

Corollary 1 *If partition Δ is a cross-cut partition and $k \geq \mu + 1$, then there is always a non-degenerate multivariate spline function $s(x, y) \in S_k^\mu(\Delta)$.*

For simplicity, we use the notation S_k^μ to denote the spline space $S_k^\mu(\Delta)$, if Δ is an equidistant partition or a regular triangulation for the case of 1D or 2D respectively, and $S_k^{u,0}$ the subspace of S_k^u with homogeneous boundary conditions. In this paper, we will only restrict ourselves to discussing the mechanical meaning of bivariate splines $s(x, y) \in S_5^{3,0}$. At first, let us consider the mechanical background of univariate splines.

3. Some univariate splines and bending of beams or bars

The related literature on univariate spline is immense and in which some results are common sense already. Here we only introduce some splines on equidistant knots with obvious mechanical meaning based on the theory of materials and elasticity [6, 7].

Case 1 Some straight bars joined with hinges bring us the simplest spline $s(x) \in S_1^0$ as shown in Figure 1.



Figure 1 Bars joined with hinges

Case 2 The deflection curves of a prismatic bar subject to some suitable couples are splines in S_2^1 , which is shown in Figure 2.

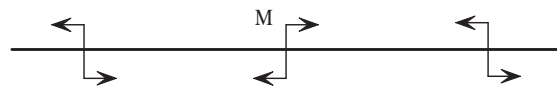


Figure 2 Bar subject to couples

Case 3 The famous cubic spline $s(x) \in S_3^2$ corresponds to the deflection curve of an ordinary (infinite) beam under the action of concentrated loads [2], which is shown in Figure 3.

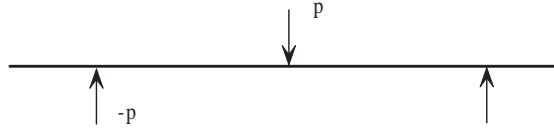


Figure 3 Beam under the action of concentrated loads

Case 4 A simply supported beam bent by piecewise uniformly distributed loads brings us a quartic C^3 spline, i.e., a spline in S_4^3 .

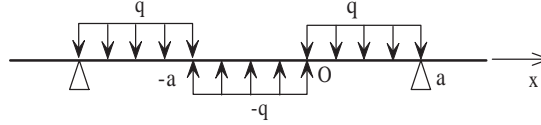


Figure 4 Beam acted by piecewise distributed loads

In Case 4, we establish Cartesian coordinates as shown in Figure 4, where $q > 0$ denotes the density of the distributed loads. Without loss of generality, let us consider the two symmetric intervals aside the origin O . The deflection curves of the beam on these two intervals are:

$$v_1(x) = \frac{q}{24D}(-x^4 + 2ax^3 - a^3x), \quad x \in [0, a],$$

$$v_2(x) = \frac{q}{24D}(x^4 + 2ax^3 - a^3x), \quad x \in [-a, 0],$$

in which D is the flexural rigidity of the beam.

It is not difficult to verify that they are of C^3 across the knot O . Hence, the whole deflection curve constructs a spline curve of degree 4 and smoothness 3.

Within the theory of beam with small deflection in [6], there are four approximate differential relations among the five physical quantities, namely, density of distributed loads q , shearing force Q , bending moment M , rotating angle θ and deflection v , which are all functions of variable x ,

$$\frac{dQ}{dx} = q, \quad \frac{dM}{dx} = Q, \quad \frac{d\theta}{dx} = \frac{M}{D}, \quad \frac{dv}{dx} = \theta, \quad (6)$$

where D is the flexural rigidity of the beam.

If q is a piecewise constant function on equally spaced partition, then Q, M, θ and v will belong to spline spaces S_1^0, S_2^1, S_3^2 and S_4^3 , so the spline in S_4^3 derived from a piecewise uniformly loaded beam contains the above three kinds of splines, and they also have obvious mechanical meaning. More generally, if the density of distributed loads q acted on an elastic beam is a spline in S_k^u itself, then Q, M, θ and v will belong to $S_{k+1}^{u+1}, S_{k+2}^{u+2}, S_{k+3}^{u+3}$ and S_{k+4}^{u+4} , respectively. These results were essentially determined by their differential relations (6).

We have established the mechanical background of a class of bivariate splines on regular triangulation or general three directional meshes in [8, 9], now we will generalize these results to the case of uniformly loaded thin plates in the following two sections.

4. Bending of simply supported polygonal plates

4.1. Equilibrium equation and boundary conditions

The bending properties of a plate depend greatly on its thickness as compared with its other dimensions. We only discuss the bending of thin plates with small deflection under the following assumptions [10]: 1) There is no deformation in the middle plane of the plate, in other words, the plane remains neutral during bending; 2) Points of the plate lying initially on a normal-to-the-middle plane of the plate remain on the normal-to-the-middle surface of the plate after bending. Then the deflection of the plate w is a function of the two coordinates in the plane of the plate, that is, $w = w(x, y)$; 3) The normal stresses in the direction transverse to the plate can be disregarded; 4) The materials of the plate is homogeneous, continuous, isotropic and linearly elastic.

Under the above assumptions, the problem of bending of plates is equivalent to solving the equilibrium equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}, \quad (7)$$

where D is the flexural rigidity of the plate, and q is the lateral load acting on the plate.

For a simply supported polygonal plate, the analytical representation of the boundary condition is

$$(w)_{l=0} = 0, \quad \left(\frac{\partial^2 w}{\partial n^2}\right)_{l=0} = 0, \quad (8)$$

where $l = 0$ is the equation of any rectilinear edge of the plate, and n denotes the external normal direction of the edge.

4.2. Reduction of bending of a plate

Equation (7) can be represented in the following form

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = -q, \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M}{D},$$

by introducing a new notation

$$M = \frac{M_x + M_y}{1 + \nu} = -D \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right), \quad (9)$$

which was discussed in [10].

For a simply supported polygonal plate bent by uniformly distributed loads, the solution of the plate problem can be reduced to solve two Dirichlet's Problems of Poisson's equation in succession as follows:

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = -q, \quad M|_{l=0} = 0, \quad (10)$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M}{D}, \quad w|_{l=0} = 0, \quad (11)$$

Equations (10) and (11) are of the same kind as that obtained for a uniformly stretched and laterally loaded membrane with homogeneous boundary condition [10].

5. Splines in $S_5^{3,0}$ and bending of simply supported polygonal plates

5.1. Bending of simply supported equilateral triangular plate

A uniformly loaded equilateral triangular plate is shown in Figure 5, whose boundary is simply supported. For the sake of convenience, we disregard the units (same in the following text). Let the length of the edge be $2a$, and assume that the density of distributed loads $q = q_0 > 0$ is a constant. Then the solution of equation (10) can be obtained similarly to the result in [10] (see page 94), say, $M = M(x, y)$ is a cubic polynomial

$$M(x, y) = -\frac{q_0}{4\sqrt{3}a} \left[y(\sqrt{3}x - y)(\sqrt{3}x + y - 2\sqrt{3}a) \right].$$

Hence, equation (11) becomes

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{q_0}{4\sqrt{3}aD} \left[y(\sqrt{3}x - y)(\sqrt{3}x + y - 2\sqrt{3}a) \right], \quad w|_{l=0} = 0. \quad (12)$$

Then the equation of deflection surface of the plate can be obtained easily by using the method of indeterminate coefficients

$$w(x, y) = \frac{q_0}{64\sqrt{3}aD} \left[y(\sqrt{3}x - y)(\sqrt{3}x + y - 2\sqrt{3}a) \right] \left[(x - a)^2 + \left(y - \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right]. \quad (13)$$

Evidently,

$$y = 0, \sqrt{3}x - y = 0, \sqrt{3}x + y - 2\sqrt{3}a = 0 \text{ and } (x - a)^2 + \left(y - \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} = 0,$$

are the equations of the three edges and circumcircle of the triangle plate.

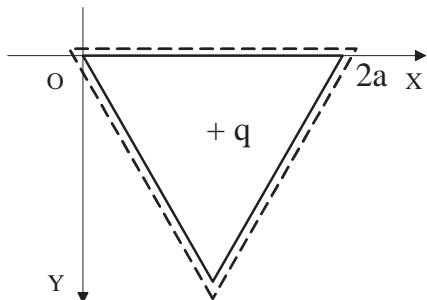


Figure 5 Equilateral triangular plate

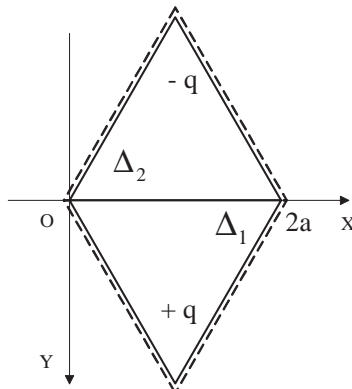


Figure 6 Rhombus plate

5.2. Bending of simply supported rhombus plate

Now let us consider a simply supported rhombus plate which consists of two regular triangles. Take the coordinate axes as shown in Figure 6 and assume that the length of each edge is $2a$. Then the common edge can be regarded as “interior edge” and the two triangles are “cells”

denoted by Δ_1 and Δ_2 respectively. $+q$ and $-q$ ($q = q_0 > 0$) denote the density of distributed loads acted on each cell.

It is easy to see that the deflection vanishes at the interior edge from symmetry and the effect is the same as simply supported along the interior edge. Hence the bending of the plates on cells Δ_1 and Δ_2 is identical to the bending of simply supported equilateral triangular plate which has been discussed previously. Therefore, the deflection equation of the rhombus plate on Δ_1 and Δ_2 can be readily obtained from equation (13)

$$w_1(x, y) = \left[y(\sqrt{3}x - y)(\sqrt{3}x + y - 2\sqrt{3}a) \right] \left[(x - a)^2 + \left(y - \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_1,$$

$$w_2(x, y) = - \left[y(\sqrt{3}x + y)(\sqrt{3}x - y - 2\sqrt{3}a) \right] \left[(x - a)^2 + \left(y + \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_2.$$

Here we have neglected the constant factor $\frac{q_0}{64\sqrt{3}aD}$ which is included in equation (13).

It is easy to verify that they just constitute a quintic C^3 spline according to “smoothing cofactor conformality method” [1, 5].

5.3. Bending of simply supported regular hexagon plate

To verify the Conformality Condition at an interior vertex, we consider the bending of a simply supported regular hexagon plate as shown in Figure 7, without loss of generality. The coordinates of the six vertices are $A(2a, 0)$, $B(a, \sqrt{3}a)$, $C(-a, \sqrt{3}a)$, $D(-2a, 0)$, $E(-a, -\sqrt{3}a)$ and $F(a, -\sqrt{3}a)$ respectively. Connecting three diagonals AD , BE and CF , we get regular triangulation and six triangular cells, which are indicated by Δ_i , $i = 1, 2, \dots, 6$.

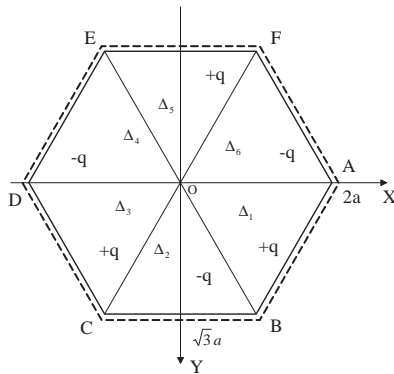


Figure 7 Regular hexagon plate

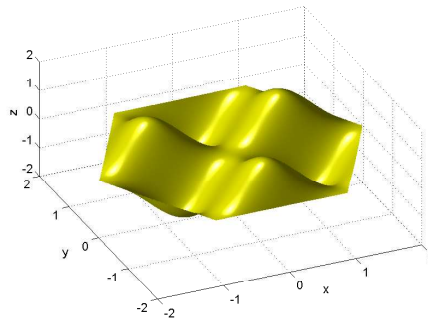


Figure 8 Deflection surface ($a = 1$)

The origin $O(0,0)$ becomes interior vertex and three diagonals can be regarded as interior edges with equations

$$AD : y = 0, BE : \sqrt{3}x - y = 0, DF : \sqrt{3}x + y = 0.$$

Acting uniformly loads with opposite direction on each pair of adjacent cells respectively as shown in the figure, where $q = q_0 > 0$, we obtain the deflection equations of the plate on Δ_i ,

$i = 1, 2, \dots, 6$

$$w_1(x, y) = \left[y(\sqrt{3}x - y)(\sqrt{3}x + y - 2\sqrt{3}a) \right] \left[(x - a)^2 + \left(y - \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_1,$$

$$w_2(x, y) = - \left[(\sqrt{3}x - y)(\sqrt{3}a - y)(\sqrt{3}x + y) \right] \left[x^2 + \left(y - \frac{2a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_2,$$

$$w_3(x, y) = \left[y(\sqrt{3}x + y)(\sqrt{3}x - y + 2\sqrt{3}a) \right] \left[(x + a)^2 + \left(y - \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_3,$$

$$w_4(x, y) = - \left[y(\sqrt{3}x - y)(\sqrt{3}x + y + 2\sqrt{3}a) \right] \left[(x + a)^2 - \left(y + \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_4,$$

$$w_5(x, y) = \left[(\sqrt{3}x - y)(\sqrt{3}x + y)(y + \sqrt{3}a) \right] \left[x^2 + \left(y + \frac{2a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_5,$$

and

$$w_6(x, y) = - \left[y(\sqrt{3}x + y)(\sqrt{3}x - y - 2\sqrt{3}a) \right] \left[(x - a)^2 + \left(y + \frac{a}{\sqrt{3}} \right)^2 - \frac{4a^2}{3} \right], \quad (x, y) \in \Delta_6,$$

where we neglect the constant factor $\frac{q_0}{64\sqrt{3}aD}$.

According to ‘‘smoothing cofactor conformality method’’, we can easily verify that w_i , $i = 1, 2, \dots, 6$, constitute a bivariate quintic spline with smoothness 3. The deformation surface of the plate when $a = 1$ is shown in Figure 8. We note that the regular triangulation can be refined or extended arbitrarily in the middle plane of the plate to the case of multi-‘‘interior vertices’’ as long as it can be realized.

Thus we construct the relationship between a kind of bivariate spline $s(x, y) \in S_5^{3,0}$ and the deformation surfaces of simply supported thin plate subject to piecewise uniformly distributed loads.

Here we have to point out that the simple and elegant expressions of $s(x, y) \in S_5^{3,0}$ merely can be obtained for regular triangulation due to the perfect symmetry. For higher dimensional spaces, we cannot get analogous results because regular polyhedrons cannot always be piled up compactly, i.e., a regular polyhedron partition cannot always be implemented in nD ($n > 2$) space [11].

5.4. The variational property of bivariate splines

It can be seen from the process of calculation that the bivariate splines $s(x, y) \in S_3^{1,0}$ in [9] is the solution of Dirichlet’s Problem of Poisson’s equation with homogeneous boundary condition, i.e.,

$$-\Delta u = f, \quad (x, y) \in \Omega, \quad u|_{\Gamma} = 0, \quad (14)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\Omega \subset \mathbb{R}^2$ is a domain with appropriately smooth boundary Γ . And there is the variational principle between the solution of Dirichlet’s Problem and minimal solution of a functional

$$J(u) = \frac{1}{2}(-\Delta u, u) - (f, u) = \frac{1}{2} \iint_{\Omega} (-\Delta u) u \, dx \, dy - \iint_{\Omega} f \cdot u \, dx \, dy,$$

which can be turned into the following form by using Green's Formula

$$J(u) = \frac{1}{2} \iint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 - f \cdot u \right] dx dy.$$

Theorem 2 (Principle of Minimum Potential Energy) *Assume that $u_* \in C^2(\bar{\Omega})$ be the solution of the boundary value problem (14). Then $J(u_*) \leq J(u)$ for all $u \in H_0^1(\Omega)$; conversely, if $u_* \in C^2(\bar{\Omega}) \cap H_0^1(\Omega)$ is the unique solution of the minimization problem*

$$\min_{u \in H_0^1(\Omega)} \left\{ \frac{1}{2} \iint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 - f \cdot u \right] dx dy \right\}, \quad (15)$$

then u_* is the solution (classical solution) of (14).

We remark that the condition $u_* \in C^2(\bar{\Omega})$ should be satisfied for boundary value problem (14). On the other hand, to look for $u_* \in H_0^1(\Omega)$ such that $J(u_*) = \min\{J(u)\}$, $u \in H_0^1(\Omega)$, it is permitted that $u_* \notin C^2(\bar{\Omega})$, which is called the generalized solution of boundary value problem.

According to Theorem 2, we know that $s(x, y) \in S_3^{1,0}$ (see [9]) is a generalized solution of (14) on the regular triangulation over special domains when f is piecewise constant with opposite signs on the adjacent cells. Meanwhile, it is the solution of the minimization problem (15). The physical meaning is as follows: If (14) is considered to be a model of an elastic membrane fixed on its boundary, $J(u)$ is the *potential energy* associated with the deflection u ; the first two terms in $J(u)$ correspond to the *internal elastic energy* and the third term is the load potential [12]. So $s(x, y) \in S_3^{1,0}$, corresponding to the deformation surface of a simply supported polygonal plate, is the solution which minimizes the potential energy $J(u)$.

Obviously, $s(x, y) \in S_5^{3,0}$ is the solution of the minimization problem (15) when $f \in S_3^{1,0}$.

5.5. The golden section in splines

It is well known that golden section/ratio appears in many basic geometric constructions and other fields. Golden section/ratio also appears in splines.

In Case 4 of Section 3, let us consider the expression on one interval, such as v_1 which can be factorized to the following form

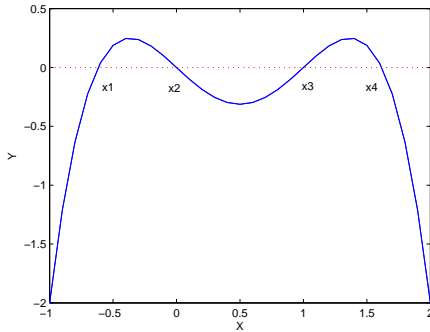
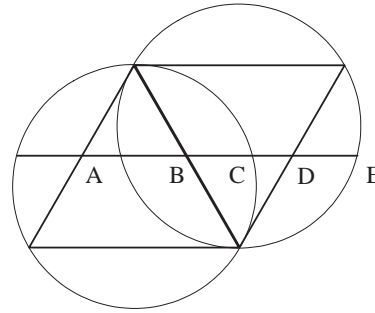
$$v_1(x) = -(x^4 - 2ax^3 + a^3x) = -x(x-a)[x - (1 + \sqrt{5})a/2][x - (1 - \sqrt{5})a/2], \quad (16)$$

where the coefficient $\frac{a}{24D}$ is neglected.

It can be seen that $x_1 = (1 - \sqrt{5})a/2$, $x_2 = 0$, $x_3 = a$, $x_4 = (1 + \sqrt{5})a/2$ are the four zeros of v_1 , and they satisfy the following equation

$$\frac{|x_2 - x_1|}{|x_3 - x_2|} = \frac{|x_4 - x_3|}{|x_3 - x_2|} = \frac{\sqrt{5} - 1}{2} \approx 0.618, \quad (17)$$

that is to say, the ratio is just the golden section/ratio. It is incredible, but it is the truth. The curve of $v_1(x)$ when $a = 1$ is shown in Figure 9.

Figure 9 Zeros of $v_1(x)(a = 1)$ Figure 10 The Case of $2D$

For the case of bivariate splines, Figure 10 shows parts of the piecewise algebraic curve [1] determined by $\{(x, y) | s(x, y) = 0, s(x, y) \in S_5^{3,0}\}$ on two adjacent triangles, which we have discussed above. Connecting the midpoints of the interior edges A , B and D , and extending the line to the circumcircles, we get the intersections C and E , then

$$\frac{AB}{AC} = \frac{BC}{BD} = \frac{\sqrt{5}-1}{2} \approx 0.618. \quad (18)$$

Golden section appears in these curves again, and they just degenerate to the case of univariate spline along the line AE .

6. Conclusion and future work

There are already some different methods to study multivariate splines, but the relationships between mechanics and multivariate splines have not been established systematically. In this paper, we discuss the mechanical model of some univariate splines at first, and establish the corresponding relationship between the bivariate spline $s(x, y) \in S_5^{3,0}$ on regular triangulation and the deflection surface of a simply supported hexagon plate subject to piecewise distributed loads. Meanwhile, we try to discuss the variational property of bivariate splines. In this paper and [8, 9], we investigate the relations between some bivariate splines and the deflection surfaces of thin plate bent by suitable couples and distributed loads on proper partitions. Interestingly, we point out that the golden section appears in both univariate and bivariate splines. Whereas, it still needs further research whether there exist mechanical backgrounds of more complex multivariate splines. These questions are challenging and remain to be our future work.

References

- [1] WANG Renhong. *Multivariate Spline Functions and Their Applications* [M]. Beijing: Science Press, 2001.
- [2] SCHOENBERG I J. *Contributions to the problem of approximation of equidistant data by analytic functions* [J]. *Quart. Appl. Math.*, 1946, **4**: 45–99; 112–141.
- [3] HOLLADAY J C. *A smoothest curve approximation* [J]. *Math. Tables Aids Comput.*, 1957, **11**: 233–243.
- [4] DUCHON J. *Splines Minimizing Rotation-Invariant Semi-Norms in Sobolev Spaces* [M]. Springer, Berlin, 1977.

- [5] WANG Renhong. *Structure of multivariate splines, and interpolation* [J]. Acta Math. Sinica, 1975, **18**(2): 91–106. (in Chinese)
- [6] GERE J M, TIMOSHENKO S P. *Mechanics of Material* [M]. Third Edition, PWS, Boston, 1990.
- [7] TIMOSHENKO S P, GOODIER J N. *Theory of Elasticity* [M]. Third Edition, McGRAW-HILL, New york, 1970.
- [8] WANG Renhong, CHANG Jincai. *A kind of bivariate spline space over rectangular partition and pure bending of thin plate* [J]. Appl. Math. Mech. (English Ed.), 2007, **28**(7): 963–971.
- [9] WANG Renhong, CHANG Jincai. *The mechanical background of multivariate spline space $S_3^{1,0}(\Delta_c^{(1)})$* [J]. J. Inform. Comput. Sci., 2007, **1**: 299–307.
- [10] TIMOSHENKO S P, WOINOWSKY-KRIEGER S. *Theory of Plates and Shells* [M]. Second Edition, McGRAW-HILL, New york, 1959.
- [11] HILBERT D, COHN-VOSSSEN S. *Geometry and the Imagination* [M]. Beijing: Higher Education Press, 1959. (in Chinese)
- [12] LI Ronghua, FENG Guochen. *Numerical Methods of Differential Equations* [M]. Third Edition, Beijing: Higher Education Press, 1995. (in Chinese)