

Mechanism of the Formation of Singularities to the Goursat Problem for Diagonal Systems with Linearly Degenerate Characteristic Fields

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Abstract For an inhomogeneous quasilinear hyperbolic system of diagonal form, under the assumptions that the system is linearly degenerate and the C^1 norm of the boundary data is bounded, we show that the mechanism of the formation of singularities of C^1 classical solution to the Goursat problem with C^1 compatibility conditions at the origin must be an ODE type. The similar result is also obtained for the weakly discontinuous solution with C^0 compatibility conditions at the origin.

Keywords formation of singularity; Goursat problem; global C^1 solution; quasilinear hyperbolic system of diagonal form; linearly degenerate characteristic; weakly discontinuous solution.

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1. Introduction

It is well-known that for first order quasilinear hyperbolic systems, generally speaking, the classical solution exists only locally in time and the singularity may appear in a finite time [2–4]. In some special cases, however, the global existence of classical solutions can be obtained. For general quasilinear strictly hyperbolic systems with weakly linearly degenerate characteristic fields, the global existence and uniqueness of C^1 solution to the Cauchy problem can be obtained, provided that the initial data is small in C^1 norm and decays at infinity [4, 12, 13, 17, 18]. This indicates that, for small initial data, the formation of singularities depends strongly on the properties of the eigenvalues. In particular, for homogeneous system of diagonal form, if the system is linearly degenerate and the C^1 norm of the initial data is bounded, then the global existence of classical solutions can also be obtained [4]. And besides, if the system is rich (see [15] for definition), with the aid of Euler-Lagrange change of coordinates for conservation laws, Li, et al. [9] established the global existence and uniqueness of classical solution to the Cauchy problem, provided only that the C^0 norm of the smooth initial data is bounded. While, for quasilinear

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diagonal system with weakly linearly degenerate characteristic fields, under the assumption that the initial data belong to $C_b^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$, when the $L^1(\mathbb{R})$ norm of the initial data or the $C^0(\mathbb{R})$ norm of the first order derivative of initial data is sufficiently small, Li and Peng [7] showed that the Cauchy problem admits a unique global C^1 solution. While, for inhomogeneous system of diagonal form, under the assumption that the system is reducible and linearly degenerate, by means of Lax transformation, when the C^1 norm of the initial data is bounded and the C^0 norm of the C^1 solution $u = u(t, x)$ is uniformly bounded with respect to t , the global classical solution can be obtained [4]. More generally, under the assumption that the system is rich and linearly degenerate, one can similarly get that, if the C^1 solution $u = u(t, x)$ forms singularities in a finite time, then the solution itself (consequently, its first order derivatives) should become unbounded at the point of formation of singularities. This phenomenon is similar to the breakdown of C^1 solution to first order semilinear hyperbolic system with nonlinear right-hand side, especially, to the Riccati's equation. This singularity is referred to as the ODE singularity [1, 8, 14]. Recently, under the assumptions that the system is linearly degenerate but not necessarily rich and the C^1 norm of initial data is bounded, the author [16] proved that the singularity of the C^1 solution to the Cauchy problem must be an ODE type.

There is a big difference between the mixed initial-boundary value problem and the Cauchy problem. For homogeneous strictly hyperbolic system of diagonal form with linearly degenerate characteristic fields, even though the boundary condition is linear, the classical solution to the mixed initial-boundary value problem may blow-up in a finite time [5]. As a result, it is natural to ask whether the C^1 solution $u = u(t, x)$ exists globally with respect to time t , provided that the C^0 norm of the solution can be controlled, that is to say, whether the singularity must be an ODE singularity. For homogeneous reducible hyperbolic system with linearly degenerate characteristic fields, more generally, for homogeneous rich system with linearly degenerate characteristic fields, the answer is positive, since one can use Lax transformation to simplify the system [5]. Recently, Li and Peng [6] showed that, for general homogeneous quasilinear strictly hyperbolic system of diagonal form, the singularity of the mixed initial-boundary value problem is still an ODE singularity. The author [16] established the similar results for the nonhomogeneous diagonal system on the semi-bounded domain.

This paper is organized as follows. In the next section, under the assumptions that the system is linearly degenerate and the conditions of C^1 compatibility are satisfied at the origin $(t, x) = (0, 0)$, we first show that the singularity of the C^1 solution to the Goursat problem for inhomogeneous quasilinear hyperbolic system of diagonal form must be an ODE singularity. Secondly, we establish the similar result for the Goursat problem with weakly discontinuous boundary conditions. In this case, the conditions of C^0 compatibility are satisfied at the origin $(t, x) = (0, 0)$.

2. Goursat problem with C^1 boundary data

This section is devoted to consider the inhomogeneous quasilinear hyperbolic system of di-

agonal form:

$$\frac{\partial u_i}{\partial t} + \lambda_i(u) \frac{\partial u_i}{\partial x} = F_i(u), \quad i = 1, \dots, n, \quad (2.1)$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector function with respect to (t, x) and $F_i(u)$ ($i = 1, \dots, n$) are given C^1 functions of u .

Suppose that on the domain under consideration, the eigenvalues $\lambda_i(u)$ ($i = 1, \dots, n$) are smooth and satisfy

$$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_n(u). \quad (2.2)$$

Suppose furthermore that $\lambda_i(u)$ ($i = 1, \dots, n$) are linearly degenerate in the sense of P.D. Lax, namely,

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0, \quad i = 1, \dots, n. \quad (2.3)$$

The 1-st characteristic $x = \bar{x}_1(t)$ passing through the origin $O(0, 0)$ satisfies

$$\begin{cases} \frac{d\bar{x}_1}{dt} = \lambda_1(u(t, \bar{x}_1(t))), \\ t = 0 : \bar{x}_1 = 0, \end{cases} \quad (2.4)$$

and the n -th characteristic $x = \tilde{x}_n(t)$ passing through $O(0, 0)$ satisfies

$$\begin{cases} \frac{d\tilde{x}_n}{dt} = \lambda_n(u(t, \tilde{x}_n(t))), \\ t = 0 : \tilde{x}_n = 0. \end{cases} \quad (2.5)$$

At the origin $O(0, 0)$, set

$$u(0, 0) = U_0. \quad (2.6)$$

Suppose that along $x = \bar{x}_1(t)$ and $x = \tilde{x}_n(t)$ we have

$$(u_2(t, \bar{x}_1(t)), \dots, u_n(t, \bar{x}_1(t)))^T = (\bar{U}_2(t), \dots, \bar{U}_n(t))^T \quad (2.7)$$

and

$$(u_1(t, \tilde{x}_n(t)), \dots, u_{n-1}(t, \tilde{x}_n(t)))^T = (\tilde{U}_1(t), \dots, \tilde{U}_{n-1}(t))^T, \quad (2.8)$$

where \bar{U}_i ($i = 2, \dots, n$) and \tilde{U}_j ($j = 1, \dots, n-1$) are both C^1 functions of $t \geq 0$ and its C^1 norms are bounded. Suppose finally that the conditions of C^1 compatibility are satisfied at the origin O .

By the local existence and uniqueness of C^1 solution to the Goursat problem [11], in order to obtain the global existence and uniqueness of C^1 solution $u = u(t, x)$ to the Goursat problem (2.1)–(2.8) for all time $t \geq 0$, it suffices to show that, for any given $T_0 > 0$, if the problem admits a unique C^1 solution $u = u(t, x)$ on the domain $D(T) \stackrel{\text{def.}}{=} \{(t, x) | 0 \leq t \leq T, \bar{x}_1(t) \leq x \leq \tilde{x}_n(t)\}$ ($0 < T < T_0$), then we have the following uniform a priori estimate on the C^1 norm of $u = u(t, x)$:

$$\|u(t, \cdot)\|_{C^1} \stackrel{\text{def.}}{=} \sum_{i=1}^n \left(\|u_i(t, \cdot)\|_{C^0} + \|\partial_x u_i(t, \cdot)\|_{C^0} \right) \leq C_1(T_0), \quad \forall t \in [0, T], \quad (2.9)$$

here and hereafter $C(T_0)$ or $C_i(T_0)$ ($i = 0, 1, \dots$) stand for positive constants independent of T but possibly depending on T_0 , moreover, the meaning of $C(T_0)$ may change from line to line.

Lemma 2.1 Let $\phi = \phi(t, x) \in C^1$ satisfy

$$\frac{\partial \phi}{\partial t} + \frac{\partial(\lambda(t, x)\phi)}{\partial x} = F(t, x), \quad 0 \leq t \leq T, \quad \bar{x}_1(t) \leq x \leq \tilde{x}_n(t), \quad (2.10)$$

where $T > 0$, $\lambda \in C^1$, and $\bar{x}_1(t)$, $\tilde{x}_n(t)$ are defined by (2.4) and (2.5), respectively. Then, for any given point $(t, x) \in D(T)$, we have

$$\begin{aligned} \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} |\phi(t, y)| dy &\leq \int_0^t \int_{\bar{x}_1(s)}^{\tilde{x}_n(s)} |F(s, y)| dy ds + \int_0^t \left(\frac{d\tilde{x}_n(s)}{ds} - \lambda(s, \tilde{x}_n(s)) \right) |\phi(s, \tilde{x}_n(s))| ds + \\ &\int_0^t \left(\lambda(s, \bar{x}_1(s)) - \frac{d\bar{x}_1(s)}{ds} \right) |\phi(s, \bar{x}_1(s))| ds. \end{aligned} \quad (2.11)$$

Proof It follows from (2.10) that

$$|\phi|_t + (\lambda|\phi|)_x = \text{sgn}(\phi)F, \quad (2.12)$$

where $\text{sgn}(f)$ is the signal function of the function f . By means of (2.4), (2.5) and (2.12), we have

$$\begin{aligned} \frac{d}{dt} \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} |\phi(t, y)| dy &= \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} \frac{\partial}{\partial t} |\phi(t, y)| dy + \tilde{x}'_n(t) |\phi(t, \tilde{x}_n(t))| - \bar{x}'_1(t) |\phi(t, \bar{x}_1(t))| \\ &= \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} \text{sgn}(\phi)F dy - \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} (\lambda(t, y) |\phi(t, y)|)_y dy + \\ &\quad \tilde{x}'_n(t) |\phi(t, \tilde{x}_n(t))| - \bar{x}'_1(t) |\phi(t, \bar{x}_1(t))| \\ &\leq \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} |F(t, y)| dy + (\tilde{x}_n(s) - \lambda(s, \tilde{x}_n(s))) |\phi(s, \tilde{x}_n(s))| + \\ &\quad (\lambda(s, \bar{x}_1(s)) - \bar{x}_1(s)) |\phi(s, \bar{x}_1(s))|. \end{aligned}$$

Integrating the above equation and noting $\bar{x}_1(0) = \tilde{x}_n(0) = 0$ immediately yields the desired result. \square

Theorem 2.1 For any given $T_0 > 0$, let $u = u(t, x)$ be a unique C^1 solution to the Goursat problem (2.1)–(2.8) on the domain $D(T)$ ($0 < T < T_0$). Suppose that $u = u(t, x)$ satisfies the following uniform a priori estimate on the domain $D(T)$:

$$\|u(t, \cdot)\|_{C^0} \leq C_0(T_0), \quad \forall t \in [0, T], \quad (2.14)$$

then

$$\|u(t, \cdot)\|_{C^1} \leq C_1(T_0), \quad \forall t \in [0, T], \quad (2.15)$$

and, as a result, the Goursat problem (2.1)–(2.8) admits a unique global C^1 solution $u = u(t, x)$ on the domain $\{(t, x) | t \geq 0, \bar{x}_1(t) \leq x \leq \tilde{x}_n(t)\}$.

Proof Let

$$w_i = \frac{\partial u_i}{\partial x}, \quad i = 1, \dots, n. \quad (2.16)$$

On the one hand, along $\bar{x}_1(t)$, it follows from (2.7) that

$$\frac{\partial u_i}{\partial t}(t, \bar{x}_1(t)) + \lambda_1(u(t, \bar{x}_1(t)))w_i(t, \bar{x}_1(t)) = \bar{U}'_i(t), \quad i = 2, \dots, n. \quad (2.17)$$

On the other hand, by (2.1), we have

$$\frac{\partial u_i}{\partial t}(t, \bar{x}_1(t)) + \lambda_i(u(t, \bar{x}_1(t)))w_i(t, \bar{x}_1(t)) = F_i(u(t, \bar{x}_1(t))), \quad i = 2, \dots, n. \quad (2.18)$$

Consequently, we conclude from (2.17)–(2.18) and (2.2) that

$$w_i(t, \bar{x}_1(t)) = \frac{1}{\lambda_i(u(t, \bar{x}_1(t))) - \lambda_1(u(t, \bar{x}_1(t)))} (F_i(u(t, \bar{x}_1(t))) - \bar{U}'_i(t)), \quad i = 2, \dots, n. \quad (2.19)$$

Likewise, we get

$$w_j(t, \tilde{x}_n(t)) = \frac{1}{\lambda_n(u(t, \tilde{x}_n(t))) - \lambda_j(u(t, \tilde{x}_n(t)))} (\tilde{U}'_j(t) - F_j(u(t, \tilde{x}_n(t))), \quad j = 1, \dots, n-1. \quad (2.20)$$

Differentiating (2.1) with respect to x gives

$$\frac{\partial w_i}{\partial t} + \frac{\partial}{\partial x}(\lambda_i(u)w_i) = \sum_{k=1}^n \frac{\partial F_i(u)}{\partial u_k} w_k, \quad i = 1, \dots, n, \quad (2.21)$$

as a result,

$$\frac{\partial |w_i|}{\partial t} + \frac{\partial}{\partial x}(\lambda_i(u)|w_i|) = \text{sgn}(w_i) \sum_{k=1}^n \frac{\partial F_i(u)}{\partial u_k} w_k, \quad i = 1, \dots, n. \quad (2.22)$$

For any given point $(t, x) \in D(T)$, applying Lemma 2.1 to (2.22) and noting assumptions (2.7), (2.8) and (2.14), for $i = 1, \dots, n$, we have

$$\int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} |w_i(t, y)| dy \leq C(T_0) + C(T_0) \int_0^t \sum_{k=1}^n \int_{\bar{x}_1(s)}^{\tilde{x}_n(s)} |w_k(s, y)| dy ds. \quad (2.23)$$

Then, summing with respect to i and using Gronwall's inequality, we get

$$\sum_{i=1}^n \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} |w_i(t, y)| dy \leq C(T_0), \quad \forall t \in [0, T]. \quad (2.24)$$

Let $A(t, x)$ be any given point in $D(T)$. For $i = 1, \dots, n$, we denote by $C_i : x = x_i(\tau)$ the i -th characteristic passing through the point $A(t, x)$.

Case i The i -th characteristic intersects the 1-st characteristic $\bar{x}_1(t)$ at the point $B(t_i, \bar{x}_1(t_i))$, namely,

$$\begin{cases} \frac{dx_i}{d\tau} = \lambda_i(u(\tau, x_i(\tau))), & t_i \leq \tau \leq t; \\ \tau = t_i : x_i = \bar{x}_1(t_i). \end{cases} \quad (2.25)$$

In this case, there are two possibilities for the j -th ($j \neq i$) characteristic $C_j : x = x_j(\tau)$ passing through $A(t, x)$:

a) It intersects the n -th characteristic $\tilde{x}_n(t)$ at the point $C(t_j, \tilde{x}_n(t_j))$. Note that system (2.22) can be rewritten as

$$d(|w_i(t, x)|(dx - \lambda_i(u)dt)) = \text{sgn}(w_i) \sum_{k=1}^n \frac{\partial F_i(u)}{\partial u_k} w_k dt \wedge dx, \quad i = 1, \dots, n. \quad (2.26)$$

On the domain $ABOC$, using Stokes formula

$$\oint_{ABOC} f ds = \iint_{ABOC} df dx dt \quad (2.27)$$

and noting (2.14), (2.19)–(2.20) and (2.24), we obtain

$$\begin{aligned} \left| \int_{C_j} |w_i(t, x)| (\lambda_j(u) - \lambda_i(u)) dt \right| &\leq \int_0^{t_i} (\lambda_i(u(s, \bar{x}_1(s))) - \lambda_1(u(s, \bar{x}_1(s)))) |w_i(s, \bar{x}_1(s))| ds + \\ &\quad \int_0^{t_j} (\lambda_n(u(s, \tilde{x}_n(s))) - \lambda_i(u(s, \tilde{x}_n(s)))) |w_i(s, \tilde{x}_n(s))| ds + \\ &\quad C(T_0) \iint_{ABOC} \sum_{i=1}^n |w_i(t, x)| dx dt \\ &\leq C(T_0). \end{aligned} \quad (2.28)$$

By (2.2) and (2.14), there exists a positive constant δ_0 such that

$$|\lambda_j(u) - \lambda_i(u)| \geq \delta_0, \quad \forall j \neq i \text{ and } |u| \leq C_0(T_0). \quad (2.29)$$

Thus we infer from (2.28)–(2.29) that

$$\int_{C_j} |w_i(t, x)| dt \leq C(T_0), \quad \forall j \neq i. \quad (2.30)$$

b) It intersects the 1-st characteristic $\bar{x}_1(t)$ at the point $C^*(t_j^*, \bar{x}_1(t_j^*))$. Similar to (2.28), on the domain ABC^* , we get

$$\begin{aligned} \left| \int_{C_j} |w_i(t, x)| (\lambda_j(u) - \lambda_i(u)) dt \right| &\leq \left| \int_{t_j^*}^{t_i} (\lambda_i(u(s, \bar{x}_1(s))) - \lambda_1(u(s, \bar{x}_1(s)))) |w_i(s, \bar{x}_1(s))| ds \right| + \\ &\quad C(T_0) \iint_{ABC^*} \sum_{i=1}^n |w_i(t, x)| dx dt \\ &\leq C(T_0). \end{aligned} \quad (2.31)$$

It follows from (2.29) and (2.31) that

$$\int_{C_j} |w_i(t, x)| dt \leq C(T_0), \quad \forall j \neq i. \quad (2.32)$$

Case ii The i -th characteristic intersects the n -th characteristic $\tilde{x}_n(t)$ passing through the origin at the point $D(t'_i, \tilde{x}_n(t'_i))$. Similar to case i, it is easy to see that

$$\int_{C_j} |w_i(t, x)| dt \leq C(T_0), \quad \forall j \neq i. \quad (2.33)$$

With the aid of (2.3), system (2.21) can be rewritten as

$$\frac{\partial w_i}{\partial t} + \lambda_i(u) \frac{\partial w_i}{\partial x} = \sum_{\substack{k=1 \\ k \neq i}}^n \frac{\partial F_i(u)}{\partial u_k} w_k + \left(\frac{\partial F_i(u)}{\partial u_i} - \sum_{\substack{k=1 \\ k \neq i}}^n \frac{\partial \lambda_i(u)}{\partial u_k} w_k \right) w_i. \quad (2.34)$$

Given any point $(t, x) \in D(T)$.

If the i -th characteristic C_i passing through the point (t, x) intersects the 1-st characteristic $\bar{x}_1(t)$ passing through the origin O at the point $(t_i, \bar{x}_1(t_i))$, then $i \in \{2, \dots, n\}$. Integrating the above equation along C_i and noting (2.19) yields

$$w_i(t, x) = w_i(t_i, \bar{x}_1(t_i)) \exp \left\{ \int_{t_i}^t \left(\frac{\partial F_i}{\partial u_i} - \sum_{\substack{k=1 \\ k \neq i}}^n \frac{\partial \lambda_i}{\partial u_k} w_k \right) (s, x_i(s)) ds \right\} +$$

$$\int_{t_i}^t \exp\left\{\int_s^t \left(\frac{\partial F_i}{\partial u_i} - \sum_{k \neq i} \frac{\partial \lambda_i}{\partial u_k} w_k\right)(\tau, x_i(\tau)) d\tau\right\} \sum_{k \neq i} \frac{\partial F_i}{\partial u_k} w_k(s, x_i(s)) ds. \quad (2.35)$$

It follows from (2.14), (2.19), (2.29) and (2.33) that

$$\|w_i(t, \cdot)\|_{C^0} \leq C(T_0), \quad \forall t \in [0, T] \quad i \in \{2, \dots, n\}. \quad (2.36)$$

If the i -th characteristic C_i passing through the point (t, x) intersects the n -th characteristic $\tilde{x}_n(t)$ passing through the origin O at the point $(t_i^*, \tilde{x}_n(t_i^*))$, then $i \in \{1, \dots, n-1\}$. Likewise, we have

$$\begin{aligned} w_i(t, x) = & w_i(t_i^*, \tilde{x}_n(t_i^*)) \exp\left\{\int_{t_i^*}^t \left(\frac{\partial F_i}{\partial u_i} - \sum_{k \neq i} \frac{\partial \lambda_i}{\partial u_k} w_k\right)(s, x_i(s)) ds\right\} + \\ & \int_{t_i^*}^t \exp\left\{\int_s^t \left(\frac{\partial F_i}{\partial u_i} - \sum_{k \neq i} \frac{\partial \lambda_i}{\partial u_k} w_k\right)(\tau, x_i(\tau)) d\tau\right\} \sum_{k \neq i} \frac{\partial F_i}{\partial u_k} w_k(s, x_i(s)) ds, \end{aligned} \quad (2.37)$$

and then

$$\|w_i(t, \cdot)\|_{C^0} \leq C(T_0), \quad \forall t \in [0, T] \quad i \in \{1, \dots, n-1\}. \quad (2.38)$$

Combining (2.36) with (2.38) completes the proof of Theorem 2.1. \square

Remark 2.1 If system (2.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity. Without loss of generality, we assume that

$$\lambda_1(u) < \dots < \lambda_k(u) \equiv \dots \equiv \lambda_{k+p}(u) < \dots < \lambda_n(u), \quad p > 1. \quad (2.39)$$

Then the conclusion of Theorem 2.1 is still valid (see Yang [16]).

Remark 2.2 Suppose that system (2.1) is either a strictly hyperbolic system or a system with characteristics with constant multiplicity. Suppose furthermore that in (2.1)

$$F_i(u) \equiv 0, \quad i = 1, \dots, n. \quad (2.40)$$

Then the conclusion of Theorem 2.1 is still valid even without the assumption (2.14) (see [4, 9]).

3. Goursat problem with weakly discontinuous boundary data

In this section, we consider the Goursat problem for system (2.1) with weakly discontinuous boundary data, namely, unlike the last section, we only assume that the conditions of C^0 compatibility are satisfied at the origin O .

We begin with the definition of the weakly discontinuous solution.

Definition 3.1 ([10]) *A continuous and piecewise C^1 vector function*

$$u = u(t, x) = \begin{cases} u_-(t, x), & x \leq x_k(t), \\ u_+(t, x), & x \geq x_k(t) \end{cases} \quad (3.1)$$

is a weakly discontinuous solution containing the k -th weak discontinuity $x = x_k(t)$ for system (2.1), if $u = u(t, x)$ satisfies system (2.1) in the classical sense on both sides of $x = x_k(t)$,

$$u_-(t, x_k(t)) = u_+(t, x_k(t)) \quad (3.2)$$

and $x = x_k(t)$ is the corresponding k -th characteristic:

$$\frac{dx_k(t)}{dt} = \lambda_k(u_-(t, x_k(t))) = \lambda_k(u_+(t, x_k(t))), \quad (3.3)$$

and moreover, the first order derivatives of $u(t, x)$ have the first kind of discontinuity on $x = x_k(t)$.

By the local existence and uniqueness of weakly discontinuous solution to the Goursat problem [11], there exists a sufficiently small $t_0 > 0$ such that the Goursat problem (2.1)–(2.8) admits a unique weakly discontinuous solution $u = u(t, x)$ at most containing $n - 2$ discontinuities $x = x_k(t)$ ($k = 2, \dots, n - 1$) on the domain $D(t_0) = \{(t, x) | 0 \leq t \leq t_0, \bar{x}_1(t) \leq x \leq \tilde{x}_n(t)\} = \bigcup_{l=1}^{n-1} R_l(t_0)$:

$$u = u(t, x) = u^{(l)}(t, x), \quad (t, x) \in R_l(t_0), \quad l = 1, \dots, n - 1, \quad (3.4)$$

where

$$R_l(t_0) = \{(t, x) | 0 \leq t \leq t_0, x_l(t) \leq x \leq x_{l+1}(t)\}, \quad l = 1, \dots, n - 1. \quad (3.5)$$

To obtain the global weakly discontinuous solution $u = u(t, x)$ to the Goursat problem (2.1)–(2.8) for all time $t \geq 0$, it suffices to prove that, for any given $T_0 > 0$, if the problem admits a unique weakly discontinuous solution $u = u(t, x)$ on the domain $D(T)$ ($0 < T < T_0$), then we have the following uniform a priori estimates on the C^0 norm of $u(t, x)$ and on the piecewise C^0 norm of $\frac{\partial u(t, x)}{\partial x}$:

$$\sum_{i=1}^n \left(\|u_i(t, \cdot)\|_{C^0} + \sum_{l=1}^{n-1} \|\partial_x u_i^{(l)}(t, \cdot)\|_{C^0} \right) \leq C_1(T_0), \quad \forall t \in [0, T]. \quad (3.6)$$

Lemma 3.1 Let $\phi = \phi(t, x)$ be piecewise C^1 and satisfy

$$\frac{\partial \phi}{\partial t} + \frac{\partial(\lambda(t, x)\phi)}{\partial x} = F(t, x), \quad 0 \leq t \leq T, \quad \bar{x}_1(t) \leq x \leq \xi(t) \quad \text{or} \quad \xi(t) \leq x \leq \tilde{x}_n(t), \quad (3.7)$$

where $T > 0$, $\lambda \in C^1$, and $\bar{x}_1(t)$, $\tilde{x}_n(t)$ are defined by (2.4) and (2.5), respectively. While $x = \xi(t)$ is a discontinuity passing through the origin $O(0, 0)$ and satisfies

$$\begin{cases} \frac{d\xi(t)}{dt} = \lambda(t, \xi(t)), \\ t = 0 : \xi = 0. \end{cases}$$

Then, for any given point $(t, x) \in D(T)$, we have

$$\begin{aligned} \int_{\bar{x}_1(t)}^{\tilde{x}_n(t)} |\phi(t, y)| dy &\leq \int_0^t \int_{\bar{x}_1(s)}^{\tilde{x}_n(s)} |F(s, y)| dy ds + \int_0^t \left(\frac{d\tilde{x}_n(s)}{ds} - \lambda(s, \tilde{x}_n(s)) \right) |\phi(s, \tilde{x}_n(s))| ds + \\ &\quad \int_0^t \left(\lambda(s, \bar{x}_1(s)) - \frac{d\bar{x}_1(s)}{ds} \right) |\phi(s, \bar{x}_1(s))| ds. \end{aligned} \quad (3.8)$$

Proof As a matter of fact, it follows that

$$\begin{aligned} \frac{d}{dt} \int_{\bar{x}_1(t)}^{\xi(t)} |\phi(t, y)| dy &= \int_{\bar{x}_1(t)}^{\xi(t)} \text{sgn}(\phi) F(t, y) dy - \int_{\bar{x}_1(t)}^{\xi(t)} \frac{\partial}{\partial y} (\lambda(t, y) |\phi(t, y)|) dy + \\ &\quad \lambda(t, \xi(t)) |\phi(t, \xi(t))| - \frac{d\bar{x}_1(t)}{dt} |\phi(t, \bar{x}_1(t))| \\ &= \int_{\bar{x}_1(t)}^{\xi(t)} \text{sgn}(\phi) F(t, y) dy + (\lambda(t, \bar{x}_1(t)) - \frac{d\bar{x}_1(t)}{dt}) |\phi(t, \bar{x}_1(t))| \end{aligned}$$

$$\leq \int_{\bar{x}_1(t)}^{\xi(t)} |F(t, y)| dy + \left(\lambda(t, \bar{x}_1(t)) - \frac{d\bar{x}_1(t)}{dt} \right) |\phi(t, \bar{x}_1(t))|, \quad (3.9)$$

where $\text{sgn}(f)$ is the sign function of the function f . Likewise,

$$\frac{d}{dt} \int_{\xi(t)}^{\bar{x}_n(t)} |\phi(t, y)| dy \leq \int_{\xi(t)}^{\bar{x}_n(t)} |F(t, y)| dy + \left(\frac{d\bar{x}_n(t)}{dt} - \lambda(t, \bar{x}_n(t)) \right) |\phi(t, \bar{x}_n(t))|. \quad (3.10)$$

Integrating the above two equations from 0 to t gives directly (3.8).

Lemma 3.2 ([10]) *On the k -th weak discontinuity $x = x_k(t)$, there holds*

$$\partial_x u_i^{(k-1)} = \partial_x u_i^{(k)}, \quad \forall i \neq k. \quad (3.11)$$

Theorem 3.1 *For any given $T_0 > 0$, let $u = u(t, x)$ be a unique weakly discontinuous solution to the Goursat problem (2.1)–(2.8) on the domain $D(T)$ ($0 < T < T_0$). Suppose that $u = u(t, x)$ satisfies the following uniform a priori estimate:*

$$\|u(t, \cdot)\|_{C^0} = \sum_{i=1}^n \|u_i(t, \cdot)\|_{C^0} \leq C_0(T_0), \quad \forall t \in [0, T], \quad (3.12)$$

then

$$\sum_{i=1}^n \left(\|u_i(t, \cdot)\|_{C^0} + \sum_{l=1}^{n-1} \|\partial_x u_i^{(l)}(t, \cdot)\|_{C^0(R_l)} \right) \leq C_1(T_0), \quad \forall t \in [0, T], \quad (3.13)$$

and, as a result, the Goursat problem (2.1)–(2.8) admits a unique global weakly discontinuous solution $u = u(t, x)$ on the domain $\{(t, x) | t \geq 0, \bar{x}_1(t) \leq x \leq \bar{x}_n(t)\}$.

Proof Since the proof is similar to that of Theorem 2.1, here we only give the outline of the proof.

Let

$$w_i = \frac{\partial u_i}{\partial x}, \quad i = 1, \dots, n. \quad (3.14)$$

Differentiating (2.1) with respect to x gives

$$\frac{\partial w_i}{\partial t} + \frac{\partial}{\partial x} (\lambda_i(u) w_i) = \sum_{k=1}^n \frac{\partial F_i(u)}{\partial u_k} w_k, \quad i = 1, \dots, n, \quad (3.15)$$

consequently,

$$\frac{\partial |w_i|}{\partial t} + \frac{\partial}{\partial x} (\lambda_i(u) |w_i|) = \text{sgn}(w_i) \sum_{k=1}^n \frac{\partial F_i(u)}{\partial u_k} w_k, \quad i = 1, \dots, n. \quad (3.16)$$

Similar to the procedure of the last section, by means of Lemma 3.1 and Gronwall inequality, we have

$$\sum_{i=1}^n \int_{\bar{x}_1(t)}^{\bar{x}_n(t)} |w_i(t, y)| dy \leq C(T_0), \quad \forall t \in [0, T]. \quad (3.17)$$

Let $A(t, x)$ be any given point on the domain $R_l(T)$ ($l = 1, \dots, n-1$). For $i = 1, \dots, n$, we denote by $C_i : x = \xi_i(\tau)$ the i -th characteristic passing through the point $A(t, x)$. If it intersects the 1-st characteristic $\bar{x}_1(t)$ passing through the origin O at the point $B(t_i, \bar{x}_1(t_i))$, namely, there

holds

$$\begin{cases} \frac{d\xi_i}{d\tau} = \lambda_i(u(\tau, \xi_i(\tau))), & t_i \leq \tau \leq t; \\ \tau = t_i : \xi_i = \bar{x}_i(t_i). \end{cases}$$

The j -th characteristic $C_j : x = \xi_j(\tau)$ passing through the point $A(t, x)$ intersects the n -th characteristic $\tilde{x}_n(t)$ passing through the origin O at the point $C(t_j, \tilde{x}_n(t_j))$. By Lemma 3.2, for any w_i , there is at most one discontinuity $x = x_i(t)$ passing through the origin O on the domain $ABOC$. Similarly to the treatment in the last section, we infer from Stokes formula that

$$\begin{aligned} \left| \int_{C_j} |w_i(t, x)| (\lambda_j(u) - \lambda_i(u)) dt \right| &\leq \left| \int_B |\varphi'_i(x)| dx \right| + C(T_0) \iint_{ABOC} \sum_{i=1}^n |w_i(t, x)| dx dt \\ &\leq C(T_0). \end{aligned} \quad (3.19)$$

By (2.2) and (3.12), we have

$$\int_{C_j} |w_i(t, x)| dt \leq C(T_0), \quad \forall j \neq i. \quad (3.20)$$

By similar analysis, (3.20) is still valid for other cases.

In what follows, we can follow the procedure of the last section to see that, for any given point $(t, x) \in R_l$ ($l = 1, \dots, n-1$), there holds

$$\|w_i^{(l)}(t, \cdot)\|_{C^0} \leq C(T_0), \quad \forall t \in [0, T], \quad i = 1, \dots, n.$$

This completes the proof of Theorem 3.1. \square

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