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Fixed Point and Asymptotical Stability in p-Moment of Neutral Stochastic Differential Equations with Mixed Delays

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Abstract In this paper the asymptotical stability in p-moment of neutral stochastic differential equations with discrete and distributed time-varying delays is discussed. The authors apply the fixed-point theory rather than the Lyapunov functions. We give a sufficient condition for asymptotical stability in p-moment when the coefficient functions of equations are not required to be fixed values. Since more general form of system is considered, this paper improves Luo Jiaowan's results.

Keywords fixed point; stochastic; delay; neutral; asymptotically stable in p-moment; Kernel.

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1. Introduction

In the recent years, a great number of people have studied the stability of stochastic differential equations using Lyapunov functions and obtained good results, for examples, Liao [1], Mao [2], Yang [3] and so on. Lots of difficulties, unfortunately, were encountered by these researchers. Luckily, Burton and co-authors used the fixed point theory to investigate the stability for deterministic systems, where some of these problems existing in the Lyapunov functions were resolved by direct method [5–7]. Lately, Luo [8–10] successfully applied such method in the stability of stochastic systems. The Lyapunov function method, which has been studied and perfected in the last over one hundred years, is a mature theory for the stability discussion of differential equations. On the contrary, the fixed point theory is just on the early stage. Thus, we believe that it is possible for investigators to obtain better results using the latter method, which is fresh and full of potential, than the former one.

In [9], Luo studied the mean square asymptotic stability of a linear neutral stochastic differ-

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ential equation. He studied the system of the form

$$\begin{aligned} d[x(t) - q(t)x(t-\tau(t))] =& [a(t)x(t) + b(t)x(t-\tau(t))]\mathrm{d}t + \\ & [c(t)x(t) + e(t)x(t-\delta(t))]\mathrm{d}W(t), \quad t \geq 0 \end{aligned}$$

where $a(t)$, $b(t)$, $c(t)$, $e(t)$, $q(t) \in C(R^+, R))$, $\tau(t)$, $\delta(t) \in C(R^+, R^+)$, satisfy $t - \tau(t) \to \infty$, $t - \delta(t) \to \infty$ as $t \to \infty$. The system considers the discrete time-varying delays only. If we take the distributed time-varying delays into account, the above system becomes

$$\begin{aligned} d[x(t) - q(t)x(t-\tau(t))] =& [a(t)x(t) + b(t)x(t-\tau(t)) + \int_{-\infty}^{t} K(t-u)f(x(u))\mathrm{d}u]\mathrm{d}t + \\ & [c(t)x(t) + e(t)x(t-\delta(t))]\mathrm{d}W(t), \quad t \geq 0 \end{aligned}$$

where the kernel function $K(s)$ is a real-valued nonnegative piecewise continuous function defined on $[0, \infty)$ and satisfies $\int_0^\infty K(s)\mathrm{d}s = 1$. In this paper, we will discuss more general system. Moreover, only the mean square asymptotic stability was considered in [9]. We will consider the asymptotical stability in p-moment ($p \geq 2$).

The rest of this paper is organized as follows. In Section 2, we introduce the basic notations, equation which is studied in this paper, and the definitions. In Section 3, we give our main results. In Sections 4 and 5, we give the remarks and example, respectively.

2. Preliminaries

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$-contains all $\mathbb{P}$-null sets. Let $\mathbb{W}(t)$ be a standard 1-dimensional Brownian motion defined on the probability space. Let $C(A, B)$ denote the family of functions from A to B that are right continuous and have limit on the left. Operator norm is denoted by $\| A \| = \sup(|Ax| : x = 1)$. Let $p > 0$, $t \geq 0$. Denote by $L^p_{\mathcal{F}_t}$ the family of all $\mathcal{F}_t$-measurable, $C(A, B)$-valued random variables $\xi = \xi(\theta), \theta \in A$ satisfying $\sup \mathbb{E}|\xi(\theta)|^p < \infty$. Let H, K be two real separable Hilbert spaces and we denote by $\langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle_K$ their inner products and by $\|\cdot\|_H, \|\cdot\|_K$ their vector norms, respectively. We denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from K into H.

The mappings $a(t)$, $b(t)$, $c(t)$, $e(t)$, $q(t) \in C(R^+, R^+))$, $\tau(t) \in C(R^+, R^+)$, $\delta(t) \in C(R^+, R^+)$ satisfy $t - \tau(t) \to \infty$, $t - \delta(t) \to \infty$ as $t \to \infty$.

Consider the following neutral stochastic differential equations with variable delays of the form

$$\begin{aligned} d[x(t) - q(t)x(t-\tau(t))] =& [a(t)x(t) + b(t)x(t-\tau(t)) + \int_{-\infty}^{t} K(t-u)f(x(u))\mathrm{d}u]\mathrm{d}t + \\ & [c(t)x(t) + e(t)x(t-\delta(t))]\mathrm{d}W(t), \quad t \geq 0 \end{aligned} \tag{2.1}$$

with the initial condition

$$x_0 = \phi(t) \in C([m(0), 0], R),$$

where $x(t)$ is the state variable,

$$m(0) = \min\{\inf(s-\tau(s), s \geq 0), \inf(s-\delta(s), s \geq 0)\}$$

and we always assume that $f(0) = 0$ and there exists a positive constant L such that

$$\parallel f(x) - f(y) \parallel_H \leq L \parallel x - y \parallel_H . \tag{2.2}$$

The kernel function $K(s)$ is a real-valued nonnegative piecewise continuous function defined on $[0, \infty)$ and satisfies $\int_0^\infty K(s)\mathrm{d}s = 1$.

Definition 2.1 *Let $p \geq 2$ be an integer. Equation (2.1) is said to be stable in p-moment, if for arbitrarily given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\parallel \xi \parallel_D < \delta$ guarantees*

$$\mathbb{E}\{\sup_{t \geq 0} \parallel x(t) \parallel_H^p\} \leq \varepsilon > 0.$$

In particular, when $p = 2$, we say it is mean square stable.

Definition 2.2 *Let $p \geq 2$ be an integer. Equation (2.1) is said to be asymptotically stable in p-moment, if it is stable in p-moment and for any $\xi \in D^b_{\mathcal{F}_0}([m(0), 0], H)$,*

$$\lim_{T \to \infty} \mathbb{E}\{\sup_{t \geq T} \parallel x(t) \parallel_H^p\} = 0.$$

In particular, when $p = 2$, we say it is mean square asymptotically stable.

Lemma 2.1 (B-D-G inequality, from [12, p.194]) *For any $r \geq 1$ and for arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(\cdot)$,*

$$\sup_{s \in [0,t]} \mathbb{E}\| \int_0^s \Phi(u)\mathrm{d}W(u)\|_H^{2r} \leq (r(2r-1))^r (\int_0^t (\mathbb{E} \parallel \Phi(s) \parallel_{\mathcal{L}_2^0}^{2r})^{\frac{1}{r}} \mathrm{d}s)^r, \quad t \in [0, T]$$

Lemma 2.2 (C_p inequality, from [13, p.657]) *For any stochastic variable ξ, η,*

$$\mathbb{E}(|\xi + \eta|^p) \leq C_p(\mathbb{E}|\xi|^p + \mathbb{E}|\eta|^p)$$

where

$$C_p = \begin{cases} 2^{p-1}, & \text{if } p \geq 1 \\ 1, & \text{if } 0 < p \leq 1 \end{cases}$$

and it is generalizable. For any stochastic variable ξ_k,

$$\mathbb{E}(|\sum_{k=1}^n \xi_k|^p) \leq C_p \sum_{k=1}^n \mathbb{E}(|\xi_k|^p)$$

where

$$C_p = \begin{cases} n^{p-1}, & \text{if } p \geq 1 \\ 1, & \text{if } 0 < p \leq 1 \end{cases} .$$

3. Main results

Theorem 3.1 *Let $\tau(t)$ be derivable, $p \geq 2$ be an integer and L be the same in (2.2). Assume that there exist a constant $\alpha \in (0,1)$ and a continuous function $h(t) : [0,\infty) \to \mathbb{R}$ such that for $t \geq 0$*

(i) $\int_0^t h(u)\mathrm{d}u \to \infty$ *as* $t \to \infty$, $\liminf_{t\to\infty} \int_0^t h(u)\mathrm{d}u > -\infty$.

(ii)
$$|q(t)|^p + \int_{t-\tau(t)}^t |a(s)+h(s)|^p\mathrm{d}s+$$
$$\int_0^t e^{-p\int_s^t h(u)\mathrm{d}u}|(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s))+b(s)-q(s)h(s)|^p\mathrm{d}s+$$
$$\int_0^t e^{-p\int_s^t h(u)\mathrm{d}u}|h(s)|^p(\int_{s-\tau(s)}^s |a(u)+h(u)|^p\mathrm{d}u)\mathrm{d}s+$$
$$L^p\int_0^t e^{-p\int_s^t h(u)\mathrm{d}u}(\int_0^\infty |K(u)|^p\mathrm{d}u)\mathrm{d}s+$$
$$\frac{1}{2}(2p(p-1))^{p/2}(\int_0^t e^{-2\int_s^t h(u)\mathrm{d}u}(|c(s)|^p+|e(s)|^p)^{2/p}\mathrm{d}s)^{p/2}$$
$$\leq \alpha < 1.$$

Then the zero solution of (2.1) is asymptotically stable in p-moment.

Proof Denote by S the Banach space of all $\mathcal{F}$-adapted process $\psi(t,\omega) : [m(0),\infty)\times\Omega \to \mathbb{R}$, which is almost surely continuous in t for fixed $\omega \in \Omega$. Moreover, $\psi(s,\omega) = \phi(s)$ for $s \in [m(0),0]$ and $\mathbb{E} \parallel \psi(t,\omega) \parallel_H^p \to 0$ as $t \to \infty$.

It is then routine to check that S is a Banach space when it is equipped with a norm defined by

$$\parallel \phi \parallel_S := \sup_{t\geq 0} \mathbb{E} \parallel \phi(y) \parallel_H^p$$

for each $\phi \in S$. Define an operator $\theta : S \to S$ by $\theta(x)(t) = \phi(t)$ for $t \in [m(0),0]$ and for $t \geq 0$

$$\begin{aligned}\theta(x)(t) =&[\phi(0)-q(0)\phi(-\tau(0)) - \int_{-\tau(0)}^0 (a(s)+h(s))\phi(s)\mathrm{d}s]e^{-\int_0^t h(u)\mathrm{d}u}+\\ &[q(t)x(t-\tau(t)) + \int_{t-\tau(t)}^t ((a(s)+h(s))x(s)\mathrm{d}s)]+\\ &\int_0^t e^{-\int_s^t h(u)\mathrm{d}u}[(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s))+b(s)-q(s)h(s)]\times\\ &x(s-\tau(s))\mathrm{d}s - \int_0^t e^{-\int_s^t h(u)\mathrm{d}u}h(s)(\int_{s-\tau(s)}^s (a(u)+h(u))x(u)\mathrm{d}u)\mathrm{d}s+\\ &\int_0^t e^{-\int_s^t h(u)\mathrm{d}u}[\int_{-\infty}^s K(s-u)f(x(u))\mathrm{d}u]\mathrm{d}s+\\ &\int_0^t e^{-\int_s^t h(u)\mathrm{d}u}[c(s)x(s)+e(s)x(s-\delta(s))]\mathrm{d}w(s)\\ =&\sum_{i=1}^6 I_i(t).\end{aligned} \tag{3.1}$$

We first verify the continuity in p-moment of θ on $[0,\infty)$. Let $x\in S, t_1\geq 0$, and $|r|$ be sufficiently small. By using Lemma 2.2, we get

$$\mathbb{E}\parallel \theta(x)(t_1+r)-\theta(x)(t_1)\parallel_H^p \leq 6^{p-1}\sum_{i=1}^{6}\mathbb{E}\parallel I_i(t_1+r)-I_i(t_1)\parallel_H^p . \tag{3.2}$$

Considering the terms on the right-hand side of (3.2), we have

$$\begin{aligned}
\mathbb{E}\parallel I_1(t_1+r)-I_1(t_1)\parallel_H^p =& \mathbb{E}\|[\phi(0)-q(0)\phi(-\tau(0))-\int_{-\tau(0)}^{0}(a(s)+h(s))\phi(s)\mathrm{d}s] \\
&(e^{-\int_0^{t_1+r}h(u)\mathrm{d}u}-e^{-\int_0^{t_1}h(u)\mathrm{d}u})\|_H^p \\
=&\mathbb{E}\|\phi(0)-q(0)\phi(-\tau(0))-\int_{-\tau(0)}^{0}(a(s)+h(s))\phi(s)\mathrm{d}s\|_H^p \\
&|e^{-\int_0^{t_1+r}h(u)\mathrm{d}u}-e^{-\int_0^{t_1}h(u)\mathrm{d}u}|^p \\
\to& 0, \quad \text{as} \quad r\to 0.
\end{aligned}$$

In the same way, it is easy to know that $\mathbb{E}\parallel I_i(t_1+r)-I_i(t_1)\parallel_H^p\to 0$, $i=2,3,4$, as $r\to 0$. When $i=5$,

$$\begin{aligned}
&\mathbb{E}\parallel I_5(t_1+r)-I_5(t_1)\parallel_H^p \\
&=\mathbb{E}\|\int_0^{t_1+r}e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}[\int_{-\infty}^{s}K(s-u)f(x(u))\mathrm{d}u]\mathrm{d}s- \\
&\quad\int_0^{t_1}e^{-\int_s^{t_1}h(u)\mathrm{d}u}[\int_{-\infty}^{s}K(s-u)f(x(u))\mathrm{d}u]\mathrm{d}s\|_H^p \\
&=\mathbb{E}\|\int_0^{t_1}(e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}-e^{-\int_s^{t_1}h(u)\mathrm{d}u})[\int_{-\infty}^{s}K(s-u)f(x(u))\mathrm{d}u]\mathrm{d}s+ \\
&\quad\int_{t_1}^{t_1+r}e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}[\int_{-\infty}^{s}K(s-u)f(x(u))\mathrm{d}u]\mathrm{d}s\|_H^p \\
&\to 0, \quad \text{as} \quad r\to 0
\end{aligned}$$

and Lemmas 2.1 and 2.2 yield

$$\begin{aligned}
&\mathbb{E}\parallel I_6(t_1+r)-I_6(t_1)\parallel_H^p \\
&=\mathbb{E}\|\int_0^{t_1+r}e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}[c(s)x(s)+e(s)x(s-\delta(s))]\mathrm{d}w(s)- \\
&\quad\int_0^{t_1}e^{-\int_s^{t_1}h(u)\mathrm{d}u}[c(s)x(s)+e(s)x(s-\delta(s))]\mathrm{d}w(s)\|_H^p \\
&=\mathbb{E}\|\int_0^{t_1}(e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}-e^{-\int_s^{t_1}h(u)\mathrm{d}u})[c(s)x(s)+e(s)x(s-\delta(s))]\mathrm{d}w(s)+ \\
&\quad\int_{t_1}^{t_1+r}e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}[c(s)x(s)+e(s)x(s-\delta(s))]\mathrm{d}w(s)\|_H^p \\
&\leq 2^{p-1}\{\mathbb{E}\parallel\int_0^{t_1}(e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}-e^{-\int_s^{t_1}h(u)\mathrm{d}u})[c(s)x(s)+e(s)x(s-\delta(s))]\mathrm{d}w(s)\parallel_H^p + \\
&\quad\mathbb{E}\parallel\int_{t_1}^{t_1+r}e^{-\int_s^{t_1+r}h(u)\mathrm{d}u}[c(s)x(s)+e(s)x(s-\delta(s))]\mathrm{d}w(s)\parallel_H^p\}
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{p-1}(p(p-1)/2)^{p/2}\{(\int_0^{t_1}(\mathbb{E} \parallel (e^{-\int_s^{t_1+r} h(u)\mathrm{d}u} - e^{-\int_s^{t_1} h(u)\mathrm{d}u})[c(s)x(s)+ \\
&\quad e(s)x(s-\delta(s))] \parallel_H^p)^{2/p}\mathrm{d}s)^{p/2}+ \\
&\quad (\int_{t_1}^{t_1+r}(\mathbb{E} \parallel e^{-\int_s^{t_1+r} h(u)\mathrm{d}u}[c(s)x(s)+e(s)x(s-\delta(s))] \parallel_H^p)^{2/p}\mathrm{d}s)^{p/2}\} \\
&\to 0 \quad \text{as} \quad r \to 0.
\end{aligned}
$$

Thus, θ is indeed continuous in p-moment on $[0,\infty)$.

Next, we show that $\theta(S) \subset S$. It follows from (3.1) that

$$
\mathbb{E} \parallel \theta(x)(t) \parallel_H^p \leq 6^{p-1}\sum_{i=1}^{6}\mathbb{E} \parallel I_i(t) \parallel_H^p . \tag{3.3}
$$

Now we estimate the terms on the right side of (3.3). Firstly, by the condition (i) we obtain

$$
\begin{aligned}
6^{p-1}\mathbb{E} \parallel I_1(t) \parallel_H^p &= 6^{p-1}\|[\phi(0)-q(0)\phi(-\tau(0)) - \int_{-\tau(0)}^{0}(a(s)+h(s))\phi(s)\mathrm{d}s]e^{-\int_0^t h(u)\mathrm{d}u}\|_H^p \\
&= 6^{p-1}\mathbb{E}\|\phi(0)-q(0)\phi(-\tau(0)) - \int_{-\tau(0)}^{0}(a(s)+h(s))\phi(s)\mathrm{d}s\|_H^p e^{-p\int_0^t h(u)\mathrm{d}u} \\
&\to 0 \quad \text{as} \quad t\to\infty.
\end{aligned}
$$

Secondly, Holder's inequality and C_p inequality yield,

$$
\begin{aligned}
&6^{p-1}\mathbb{E} \parallel I_2(t) \parallel_H^p \\
&\quad = 6^{p-1}\mathbb{E} \parallel q(t)x(t-\tau(t)) + \int_{t-\tau(t)}^{t}((a(s)+h(s))x(s)\mathrm{d}s) \parallel_H^p \\
&\quad \leq 6^{p-1}2^{p-1}\{\mathbb{E} \parallel q(t)x(t-\tau(t)) \parallel_H^p + \mathbb{E} \parallel \int_{t-\tau(t)}^{t}((a(s)+h(s))x(s)\mathrm{d}s) \parallel_H^p\} \\
&\quad \leq 6^{p-1}2^{p-1}|q(t)|^p\mathbb{E} \parallel x(t-\tau(t)) \parallel_H^p + 6^{p-1}2^{p-1}\int_{t-\tau(t)}^{t}|(a(s)+h(s))|^p\mathbb{E} \parallel x(s) \parallel_H^p \mathrm{d}s.
\end{aligned}
$$

For any $x(t) \in S$, $\mathbb{E} \parallel x(t) \parallel_H^p \to 0$, as $t\to\infty$ and $t-\tau(t)\to\infty$, as $t\to\infty$, so $\mathbb{E} \parallel x(t-\tau(t)) \parallel_H^p \to 0$, as $t\to\infty$, and then by condition (ii), we obtain $6^{p-1}\mathbb{E} \parallel I_2(t) \parallel_H^p \to 0$ as $t\to\infty$.

Thirdly, as $i=3$

$$
\begin{aligned}
6^{p-1}\mathbb{E} \parallel I_3(t) \parallel_H^p = &6^{p-1}\mathbb{E} \parallel \int_0^t e^{-\int_s^t h(u)\mathrm{d}u}[(a(s-\tau(s))+h(s-\tau(s))) \\
&(1-\tau'(s))+b(s)-q(s)h(s)]x(s-\tau(s))\mathrm{d}s \parallel_H^p \\
\leq &6^{p-1}\int_0^t e^{-p\int_s^t h(u)\mathrm{d}u}|(a(s-\tau(s))+h(s-\tau(s))) \\
&(1-\tau'(s))+b(s)-q(s)h(s)|^p\mathbb{E} \parallel x(s-\tau(s)) \parallel_H^p \mathrm{d}s.
\end{aligned}
$$

For any $x(t) \in S$, $\mathbb{E} \parallel x(t) \parallel_H^p \to 0$, as $t\to\infty$ and $t-\tau(t)\to\infty$, as $t\to\infty$, so $\mathbb{E} \parallel x(t-\tau(t)) \parallel_H^p \to 0$, as $t\to\infty$, and then by condition (ii), we obtain $6^{p-1}\mathbb{E} \parallel I_3(t) \parallel_H^p \to 0$ as $t\to\infty$.

It is easy to know that $6^{p-1}\mathbb{E} \parallel I_4(t) \parallel_H^p \to 0$ as $t\to\infty$ using the same methods.

When $i = 5$

$$
\begin{aligned}
6^{p-1} \parallel I_5(t) \parallel_H^p &= 6^{p-1}\mathbb{E} \parallel \int_0^t e^{-\int_s^t h(u)\mathrm{d}u}[\int_{-\infty}^s K(s-u)f(x(u))\mathrm{d}u]\mathrm{d}s \parallel_H^p \\
&\leq 6^{p-1}L^p\mathbb{E} \parallel \int_0^t e^{-\int_s^t h(u)\mathrm{d}u}[\int_{-\infty}^s K(s-u) \parallel x(u) \parallel \mathrm{d}u]\mathrm{d}s \parallel_H^p \\
&\leq 6^{p-1}L^p \int_0^t e^{-p\int_s^t h(u)\mathrm{d}u}[\int_{-\infty}^s |K(s-u)|^p\mathbb{E} \parallel x(u) \parallel_H^p \mathrm{d}u]\mathrm{d}s.
\end{aligned}
$$

For any $x(t) \in S$, $\mathbb{E} \parallel x(t) \parallel_H^p \to 0$ as $t \to \infty$, and $\int_{-\infty}^s K(s-u)\mathrm{d}u = \int_0^\infty K(v)\mathrm{d}v$, then by the conditions (i) and (ii), we obtain $6^{p-1}\mathbb{E} \parallel I_5(t) \parallel_H^p \to 0$ as $t \to \infty$.

Last when $i = 6$, using Holder's inequality and Lemma 2.1, we can get

$$
\begin{aligned}
&6^{p-1}\mathbb{E} \parallel I_6(t) \parallel_H^p \\
&= 6^{p-1}\mathbb{E} \parallel \int_0^t e^{-\int_s^t h(u)\mathrm{d}u}[c(s)x(s) + e(s)x(s-\delta(s))]\mathrm{d}w(s) \parallel_H^p \\
&\leq 6^{p-1}(p(p-1)/2)^{p/2}\{\int_0^t (\mathbb{E} \parallel e^{-\int_s^t h(u)\mathrm{d}u}[c(s)x(s) + e(s)x(s-\delta(s))] \parallel_H^p)^{2/p}\mathrm{d}s\}^{p/2} \\
&= 6^{p-1}(p(p-1)/2)^{p/2}\{\int_0^t e^{-2\int_s^t h(u)\mathrm{d}u}(\mathbb{E} \parallel c(s)x(s) + e(s)x(s-\delta(s)) \parallel_H^p)^{2/p}\mathrm{d}s\}^{p/2} \\
&\leq 6^{p-1}(p(p-1)/2)^{p/2}\{\int_0^t e^{-2\int_s^t h(u)\mathrm{d}u}[2^{p-1}(\mathbb{E} \parallel c(s)x(s) \parallel_H^p + \\
&\quad \mathbb{E} \parallel e(s)x(s-\delta(s)) \parallel_H^p)]^{2/p}\mathrm{d}s\}^{p/2} \\
&= 6^{p-1}(p(p-1)/2)^{p/2}2^{p-1}\{\int_0^t e^{-2\int_s^t h(u)\mathrm{d}u}[(|c(s)|^p\mathbb{E} \parallel x(s) \parallel_H^p + \\
&\quad |e(s)|^p\mathbb{E} \parallel x(s-\delta(s)) \parallel_H^p)]^{2/p}\mathrm{d}s\}^{p/2}.
\end{aligned}
$$

For any $x(t) \in S$, $\mathbb{E} \parallel x(t) \parallel_H^p \to 0$, as $t \to \infty$ and $t - \delta(t) \to \infty$, as $t \to \infty$, so $\mathbb{E} \parallel x(t-\tau(t)) \parallel_H^p \to 0$, as $t \to \infty$, and then by condition (ii), we obtain $6^{p-1}\mathbb{E} \parallel I_6(t) \parallel_H^p \to 0$ as $t \to \infty$.

We conclude that $\theta(S) \subset S$.

At the end of this proof, we will show that θ is contractive.

For $x, y \in S$, from the condition (ii), (2.2), Lemmas 2.1 and 2.2, we can obtain

$$
\begin{aligned}
&\sup_{s\in[0,t]} \mathbb{E} \parallel \theta(x)(s) - \theta(y)(s) \parallel_H^P \\
&= \sup_{s\in[0,t]} \mathbb{E} \parallel q(s)(x(s-\tau(s)) - y(s-\tau(s))) + \int_{s-\tau(s)}^s ((a(v) + h(v))(x(v) - y(v)\mathrm{d}v) + \\
&\quad \int_0^s e^{-\int_v^s h(u)\mathrm{d}u}[(a(v-\tau(v)) + h(v-\tau(v)))(1-\tau'(v)) + b(v) - q(v)h(v)] \\
&\quad (x(v-\tau(v)) - y(v-\tau(v))\mathrm{d}v - \\
&\quad \int_0^s e^{-\int_v^s h(u)\mathrm{d}u}h(v)(\int_{v-\tau(v)}^v (a(u) + h(u))(x(u) - y(u))\mathrm{d}u)\mathrm{d}v + \\
&\quad \int_0^s e^{-\int_v^s h(u)\mathrm{d}u}h(v)(\int_{-\infty}^v K(v-u)(f(x(u)) - f(y(u)))\mathrm{d}u)\mathrm{d}v +
\end{aligned}
$$

$$\int_0^s e^{-\int_v^s h(u)\mathrm{d}u}[c(v)(x(v)-y(v))+e(v)(x(v-\delta(v))-y(v-\delta(v)))]\mathrm{d}w(v)\parallel_H^P$$

$$\leq 5^{p-1}\sup_{s\in[0,t]}\{|q(s)|^p\mathbb{E}\parallel x(s-\tau(s))-y(s-\tau(s))\parallel_H^P+$$

$$\int_{s-\tau(s)}^s|(a(v)+h(v)|^p\mathbb{E}\parallel x(v)-y(v)\parallel_H^P\mathrm{d}v)+$$

$$\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|(a(v-\tau(v))+h(v-\tau(v)))(1-\tau'(v))+b(v)-q(v)h(v)|^p$$

$$\mathbb{E}\parallel x(v-\tau(v))-y(v-\tau(v))\parallel_H^P)\mathrm{d}v+$$

$$\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|h(v)|^p(\int_{v-\tau(v)}^v|a(u)+h(u)|^p\mathbb{E}\parallel x(u)-y(u)\parallel_H^P\mathrm{d}u)\mathrm{d}v+$$

$$L^p\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}(\int_{-\infty}^v|K(v-u)|^p\mathbb{E}\parallel x(u)-y(u)\mathrm{d}u\parallel_H^P\mathrm{d}v\}+$$

$$5^{p-1}\sup_{s\in[0,t]}\mathbb{E}\parallel\int_0^s e^{-\int_v^s h(u)\mathrm{d}u}[c(v)(x(v)-y(v))+$$

$$e(v)(x(v-\delta(v))-y(v-\delta(v)))]\mathrm{d}w(v)\parallel_H^P$$

$$\leq 5^{p-1}\sup_{s\in[0,t]}\{|q(s)|^p\mathbb{E}\parallel x(s-\tau(s))-y(s-\tau(s))\parallel_H^P+$$

$$\int_{s-\tau(s)}^s|(a(v)+h(v)|^p\mathbb{E}\parallel x(v)-y(v)\parallel_H^P\mathrm{d}v+$$

$$\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|(a(v-\tau(v))+h(v-\tau(v)))(1-\tau^{'}(v))+b(v)-q(v)h(v)|^p$$

$$\mathbb{E}\parallel x(v-\tau(v))-y(v-\tau(v))\parallel_H^P)\mathrm{d}v+$$

$$\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|h(v)|^p(\int_{v-\tau(v)}^v|a(u)+h(u)|^p\mathbb{E}\parallel x(u)-y(u)\parallel_H^P\mathrm{d}u)\mathrm{d}v+$$

$$L^p\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}(\int_{-\infty}^v|K(v-u)|^p\mathbb{E}\parallel x(u)-y(u)\mathrm{d}u\parallel_H^P\mathrm{d}v\}+$$

$$5^{p-1}\sup_{s\in[0,t]}(p(p-1)/2)^{p/2}\{\int_0^s e^{-2\int_v^s h(u)\mathrm{d}u}(\mathbb{E}\parallel c(v)(x(v)-y(v))+$$

$$e(v)(x(v-\delta(v))-y(v-\delta(v)))\parallel_H^P)^{2/p}\mathrm{d}v\}^{p/2}$$

$$\leq 5^{p-1}\sup_{s\in[0,t]}\{|q(s)|^p\mathbb{E}\parallel x(s-\tau(s))-y(s-\tau(s))\parallel_H^P+$$

$$\int_{s-\tau(s)}^s|(a(v)+h(v)|^p\mathbb{E}\parallel x(v)-y(v)\parallel_H^P\mathrm{d}v+$$

$$\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|(a(v-\tau(v))+h(v-\tau(v)))(1-\tau'(v))+b(v)-q(v)h(v)|^p+$$

$$\mathbb{E}\parallel x(v-\tau(v))-y(v-\tau(v))\parallel_H^P)\mathrm{d}v+$$

$$\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|h(v)|^p(\int_{v-\tau(v)}^v|a(u)+h(u)|^p\mathbb{E}\parallel x(u)-y(u)\parallel_H^P\mathrm{d}u)\mathrm{d}v\}+$$

$$L^p\int_0^s e^{-\int_v^s h(u)\mathrm{d}u}\int_{-\infty}^v|K^p(v-u)|\mathbb{E}\parallel x(u)-y(u)\mathrm{d}u\parallel_H^P\mathrm{d}v\}+$$

$$
\begin{aligned}
&5^{p-1}\sup_{s\in[0,t]}(p(p-1)/2)^{p/2}2^{p-1}\{\int_0^s e^{-2\int_v^s h(u)\mathrm{d}u}(|c(v)|^p\mathbb{E}\parallel(x(v)-y(v))\parallel_H^P+\\
&|e(v)|^p\mathbb{E}\parallel x(v-\delta(v))-y(v-\delta(v))\parallel_H^P)^{2/p}\mathrm{d}v\}^{p/2}\\
\le\ &5^{p-1}\sup_{s\in[0,t]}\mathbb{E}\parallel x(s)-y(s)\parallel_H^p\{|q(s)|^p+\int_{s-\tau(s)}^s|a(v)+h(v)|^p\mathrm{d}v+\\
&\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|(a(v-\tau(v))+h(v-\tau(v)))(1-\tau'(v))+b(v)-q(s)h(v)|^p\mathrm{d}v+\\
&\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}|h(v)|^p(\int_{v-\tau(v)}^v|a(u)+h(u)|^p\mathrm{d}u)\mathrm{d}v+\\
&L^p\int_0^s e^{-p\int_v^s h(u)\mathrm{d}u}(\int_0^\infty|K^p(u)|\mathrm{d}u)\mathrm{d}v+\\
&(2p(p-1))^{p/2}/2(\int_0^s e^{-2\int_v^s h(u)\mathrm{d}u}(|c(v)|^p+|e(v)|^p)^{2/p}\mathrm{d}v)^{p/2}\}\\
\le\ &\alpha\sup_{s\in[0,t]}\mathbb{E}\parallel x(s)-y(s)\parallel_H^p.
\end{aligned}
$$

Thus we know that θ is a contraction mapping.

Hence by the Contraction Mapping Principle, θ has a unique fixed point $x(t)$ in S, which is a solution of (2.1). With $x(s)=\phi(s)$ on $[m(0),0]$ and $\mathbb{E}\parallel x(t)\parallel_H^P\to 0$ as $t\to\infty$. The proof is completed. □

4. Remarks

Remark 4.1 The system studied in [9] is linear system, while the system we study here is nonlinear. If $f\equiv 0$ in (2.1), the system becomes the linear system which was studied in [9]. Therefore, we considered the more general system than in [9].

Remark 4.2 Only the mean square asymptotic stability was considered in [9], but we considered the asymptotical stability in p-moment ($p\ge 2$).

5. Example

Consider the following neutral stochastic delay differential equation,

$$
\begin{aligned}
&d[x(t)-\frac{\sin t}{12}x(t-t/2)]\\
&=[-4x(t)+l\int_{-\infty}^t e^{-(t-u)}x(u)\mathrm{d}u]\mathrm{d}t+[c_1x(t)+c_2x(t-\frac{t}{3})]\mathrm{d}w(t),\quad t\ge 0,
\end{aligned}\tag{5.1}
$$

where c_1, c_2, l are constants. If we choose $h(t)=4$ in Theorem 3.1, then by Theorem 3.1, the zero solution of (5.1) is asymptotically stable in p-moment, provided that

$$(\frac{1}{12})^p+\frac{1}{4p}(\frac{1}{3})^p+\frac{l^p}{4p^2}+\frac{(p(p-1))^{p/2}}{2}(|c_1|^p+|c_2|^p)<1.$$

The conclusion can be verified by the numerical simulation (see Figures 1 and 2).

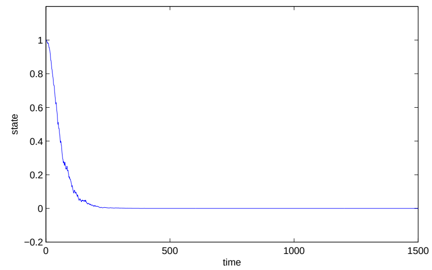

Figure 1 $l=1$, $c_1=2$, $c_2=3$ and $p=2$

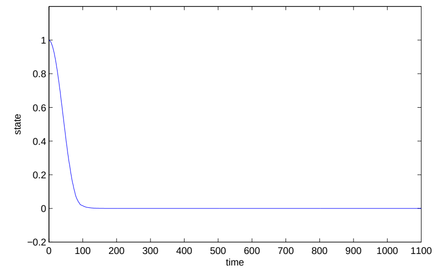

Figure 2 $l=1$, $c_1=4$, $c_2=6$ and $p=4$

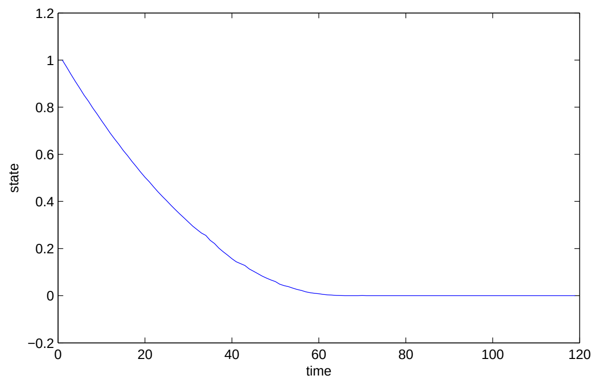

Figure 3 $l=1$, $c_1=4$, $c_2=6$ and $p=4$

And when we consider the case without distributed time-varying delays, and the equation is described by

$$d[x(t)-\frac{\sin t}{12}x(t-t/2)]=[-4x(t)]dt+[c_1x(t)+c_2x(t-\frac{t}{3})]\mathrm{d}w(t),\quad t\geq 0, \tag{5.2}$$

where the parameters are the same as the ones in (5.1), the zero solution of (5.2) is asymptotically stable in p-moment, provided that

$$(\frac{1}{12})^p+\frac{1}{4p}(\frac{1}{3})^p+\frac{(p(p-1))^{p/2}}{2}(|c_1|^p+|c_2|^p)<1.$$

The conclusion can be verified by the numerical simulation (see Figure 3).

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