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Vertex-Distinguishing E-Total Coloring of the Graphs mC_3 and mC_4

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Abstract Let G be a simple graph. A total coloring f of G is called E-total-coloring if no two adjacent vertices of G receive the same color and no edge of G receives the same color as one of its endpoints. For E-total-coloring f of a graph G and any vertex u of G, let $C_f(u)$ or $C(u)$ denote the set of colors of vertex u and the edges incident to u. We call $C(u)$ the color set of u. If $C(u) \neq C(v)$ for any two different vertices u and v of $V(G)$, then we say that f is a vertex-distinguishing E-total-coloring of G, or a $VDET$ coloring of G for short. The minimum number of colors required for a $VDET$ colorings of G is denoted by $\chi^e_{vt}(G)$, and it is called the VDET chromatic number of G. In this article, we will discuss vertex-distinguishing E-total colorings of the graphs mC_3 and mC_4.

Keywords coloring; E-total coloring; vertex-distinguishing E-total coloring; vertex-distinguishing E-total chromatic number; the vertex-disjoint union of m cycles with length n.

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1. Introduction

The vertex-distinguishing proper edge colorings and vertex-distinguishing general edge colorings had been widely considered in [1–5, 8, 9] and [7, 10–14], respectively. The appropriate chromatic numbers are called the vertex-distinguishing proper edge chromatic number (or strong chromatic number, or observability) and point-distinguishing chromatic index, respectively.

For a total coloring (proper or not) f of G and a vertex v of G, denote by $C_f(v)$, or simply $C(v)$, if no confusion arises, the set of colors used to color the vertex v as well as the edges incident to v. We call $C(v)$ the color set of vertex v. Let $\overline{C}(v)$ be the complementary set of $C(v)$ in the set of all colors we used. Obviously, $|C(v)| \leq d_G(v) + 1$ and the equality holds if the total coloring is proper.

For a proper total coloring, if $C(u) \neq C(v)$, i.e., $\overline{C}(u) \neq \overline{C}(v)$ for any two different vertices u and v, then the coloring is called vertex-distinguishing (proper) total coloring and the minimum number of colors required for a vertex-distinguishing (proper) total coloring is denoted by $\chi_{vt}(G)$.

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This concept had been considered in [6, 15]. The following conjecture had been given in [15].

Conjecture 1 ([15]) Suppose G is a simple graph and n_d is the number of vertices of degree d, $\delta \le d \le \Delta$. Let k be the minimum positive integer such that $\binom{k}{d+1} \ge n_d$ for all d such that $\delta \le d \le \Delta$. Then $\chi_{vt}(G) = k$ or $k+1$.

From [15], we know that the above conjecture is valid for complete graph, complete bipartite graph, path and cycle, etc.

When we define a proper total coloring of a graph G, we need three conditions for a total coloring which are listed as follows:

Condition (v). no two adjacent vertices receive the same color;

Condition (e). no two adjacent edges receive the same color;

Condition (i). no edge receives the same color as one of its endpoints.

If we just consider the total coloring of graph G such that the Conditions (v) and (i) are satisfied, then such a coloring is called E-total coloring of graph G.

Note that E-total coloring is total coloring, but total coloring is not necessarily E-total coloring. The proper total coloring is E-total coloring and E-total coloring is not necessarily proper total coloring.

If f is an E-total coloring of graph G with colors we use being $1, 2, \ldots, k$ and $\forall u, v \in V(G)$, $u \neq v$, we have $C(u) \neq C(v)$, then f is called k-vertex-distinguishing E-total coloring, or k-VDET coloring.

The vertex-distinguishing E-total chromatic number of graph G, denoted by $\chi^e_{vt}(G)$, is the minimum k for which G has a vertex-distinguishing E-total coloring using k colors.

The following proposition is obviously true.

Proposition 1 *For each graph G, we have $\chi^e_{vt}(G) \le \chi_{vt}(G)$.*

For graph G, let n_i denote the number of the vertices of degree i, $\delta \le i \le \Delta$. Suppose

$$\eta(G) = \min\left\{l | \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{i+1} \ge n_\delta + n_{\delta+1} + \cdots + n_i, 1 \le \delta \le i \le \Delta\right\}.$$

Lemma 1 *For each graph G, we have $\chi^e_{vt}(G) \ge \eta(G)$.*

Proof In order to color G by the method of vertex-distinguishing E-total coloring, the number of colors we used is at least l, where l is a positive integer and $\binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{i+1} \ge n_\delta + n_{\delta+1} + \cdots + n_i$ for each $1 \le \delta \le i \le \Delta$, because the vertices of degree $n_\delta, n_{\delta+1}, \ldots, n_i$ should be distinguished by their color sets and each of their color sets contains 2, or 3, or 4, ..., or $i+1$ colors.

Thus $\chi^e_{vt}(G)$ is not less than the minimum value of such l's.

From Lemma 1, we have

Lemma 2 *If G is an r-regular graph, then $\eta(G) = \{l | \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{r+1} \ge |V(G)|\}$.*

Suppose mC_n denotes the vertex-disjoint union of m cycles of lengths n. In this paper we will determine the VDET chromatic numbers of mC_3 and mC_4.

2. Vertex distinguishing E-total chromatic numbers of mC_3

In order to give the vertex-distinguishing E-total coloring of mC_3 and mC_4 conveniently, we define a matrix A_n firstly.

For $n \geq 6$, we construct a matrix A_n such that A_n has $n-1$ rows and $n-1$ columns and the entries of A_n are empty sets or the subsets of $\{1, 2, \dots, n\}$ which contain n and have 2 or 3 elements. Each subset of $\{1, 2, \dots, n\}$ which contains n and has 2 or 3 elements is an element of A_n. The elements of A_n is given as follows.

The first row of A_n is $(\{n,1\}, \{n,2\}, \{n,3\}, \dots, \{n, n-1\})$;

The second row of A_n is $(\{n,1,2\}, \{n,2,3\}, \{n,3,4\}, \dots, \{n, n-2, n-1\}, \varnothing)$;

The third row of A_n is $(\{n,1,3\}, \{n,2,4\}, \{n,3,5\}, \dots, \{n, n-3, n-1\}, \varnothing, \varnothing)$;

The fourth row of A_n is $(\{n,1,4\}, \{n,2,5\}, \{n,3,6\}, \dots, \{n, n-4, n-1\}, \varnothing, \varnothing, \varnothing)$;

$\dots$

The $(n-2)$-th row of A_n is $(\{n,1,n-2\}, \{n,2,n-1\}, \varnothing, \varnothing, \dots, \varnothing)$;

The $(n-1)$-th row of A_n is $(\{n,1,n-1\}, \varnothing, \varnothing, \dots, \varnothing)$;

For example

$$A_6 = \begin{pmatrix} \{6,1\} & \{6,2\} & \{6,3\} & \{6,4\} & \{6,5\} \\ \{6,1,2\} & \{6,2,3\} & \{6,3,4\} & \{6,4,5\} & \varnothing \\ \{6,1,3\} & \{6,2,4\} & \{6,3,5\} & \varnothing & \varnothing \\ \{6,1,4\} & \{6,2,5\} & \varnothing & \varnothing & \varnothing \\ \{6,1,5\} & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}$$

Suppose $1 \leq i_1 < i_2 < \cdots < i_r \leq n-1$, $1 \leq j_1 < j_2 < \cdots < j_s \leq n-1$. The submatrix $A_n[i_1, i_2, \dots, i_r | j_1, j_2, \dots, j_s]$ is the $r \times s$ matrix obtained from A_n by removing the rows with indices not in $\{i_1, i_2, \dots, i_r\}$ and columns with indices not in $\{j_1, j_2, \dots, j_s\}$. A 3×2 sub-matrix B of A_n is called good if there exists an E-total coloring method of the vertex-disjoint union of two C_3's such that the color sets of all the vertices of $2C_3$ are just all the entries of B.

For example, $A_6[2,3,4|1,2]$ is good, since all the entries of $A_6[2,3,4|1,2]$ are the color sets of all the vertices of $2C_3$ in the following vertex-distinguishing E-total coloring:

Figure 1 VDTC coloring of $2C_3$

In this section, we will use the following three types of VDET coloring of C_3.

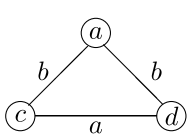

Figure 2 Coloring $f(a, b; c, d)$

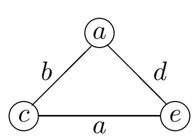

Figure 3 Coloring $g(a; b, c; d, e)$

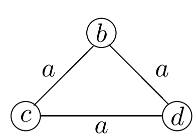

Figure 4 Coloring $h(a; b, c, d)$

The first type coloring is denoted by $f(a,b;c,d)$. Under $f(a,b;c,d)$, the color sets of three vertices are $\{a,b,c\},\{a,b,d\}$ and $\{a,b\}$.

The second type coloring is denoted by $g(a;b,c;d,e)$. Under $g(a;b,c;d,e)$, the color sets of three vertices are $\{a,b,c\},\{a,d,e\}$ and $\{a,b,d\}$.

The third type coloring is denoted by $h(a;b,c,d)$. Under $h(a;b,c,d)$, the color sets of three vertices are $\{a,b\},\{a,c\}$ and $\{a,d\}$.

Lemma 3 *$A_n[i,i+1,i+2 \mid j,j+1]$ is a good 3×2 submatrix when $i\equiv 2(\bmod 3)$, $j\equiv 1(\bmod 2)$ and no entry of $A_n[i,i+1,i+2 \mid j,j+1]$ is $\emptyset$.*

Proof If $i=2$, then

$$A_n[i,i+1,i+2 \mid j,j+1]=\begin{pmatrix}\{n,j,j+1\} & \{n,j+1,j+2\}\\ \{n,j,j+2\} & \{n,j+1,j+3\}\\ \{n,j,j+3\} & \{n,j+1,j+4\}\end{pmatrix}.$$

Obviously, $\{n,j,j+2\},\{n,j+1,j+3\},\{n,j+1,j+2\}$ are the color sets of all vertices of C_3 under some vertex distinguishing E-total coloring. So are $\{n,j,j+3\},\{n,j+1,j+4\},\{n,j,j+1\}$.

If $i\geq 5$, then

$$A_n[i,i+1,i+2 \mid j,j+1]=\begin{pmatrix}\{n,j,j+i-1\} & \{n,j+1,j+i\}\\ \{n,j,j+i\} & \{n,j+1,j+i+1\}\\ \{n,j,j+i+1\} & \{n,j+1,j+i+2\}\end{pmatrix}.$$

Obviously, $\{n,j,j+i-1\},\{n,j+1,j+i\},\{n,j,j+i\}$ are the color sets of all vertices of C_3 under some vertex distinguishing E-total coloring. So are $\{n,j+1,j+i+1\},\{n,j,j+i+1\},\{n,j+1,j+i+2\}$

The proof is completed. □

Lemma 4 *If $n\equiv 1,3,4,0(\bmod 6)$ $(n\geq 7)$, then we can decompose all the entries of A_n (except for $\emptyset$) into $\frac{1}{3}\binom{n}{2}$ groups such that each group contains exactly 3 subsets and three subsets in each group are exactly the color sets of all vertices of C_3 under some vertex-distinguishing E-total coloring.*

Proof By Lemma 3, we only consider the entries of A_n which are not empty set and are not in any good 3×2 submatrices of A_n described in Lemma 3. Such entries are called left subsets in this proof.

Case 1 $n\equiv 1(\bmod 6)$.

For $i\equiv 1(\bmod 6)$, $1\leq i\leq n-1$, we consider such entries (left subsets) which are in the i-th, $(i+1)$-th columns and are not in the first row: $\{n,i,n-1\},\{n,i,n-2\},\{n,i+1,n-1\}$. Obviously, they may become a desired group.

For the entries which are in the $(i+2)$-th, $(i+3)$-th, $(i+4)$-th, $(i+5)$-th columns and are not in the first row, $\{n,i+2,n-2\},\{n,i+3,n-1\},\{n,i+3,n-2\}$ may become a desired group, and $\{n,i+2,n-1\},\{n,i+2,n-3\},\{n,i+4,n-1\}$ may become another desired group.

Of course $\{\{n,i\},\{n,i+1\},\{n,i+2\}\}$ is a desired group, $i=1,4,7,\ldots,n-3$.

Case 2 $n \equiv 3 (\bmod 6)$.

For $i \equiv 3(\bmod 6)$, $3 \leq i \leq n-12$, the left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+5)$-th columns and not in the first row can become 3 groups (obviously):

$\{\{n,i,n-2\},\{n,i,n-1\},\{n,i+1,n-1\}\},\{\{n,i+2,n-3\},\{n,i+2,n-1\},\{n,i+4,n-1\}\},\{\{n,i+3,n-2\},\{n,i+3,n-1\},\{n,i+2,n-2\}\}$.

For the left subsets which are in the first, second, $(n-6)$-th, $(n-5)$-th, $(n-4)$-th, $(n-3)$-th, $(n-2)$-th columns and are not in the first row as well as the subsets $\{n,n-1\}$, $\{n,n-2\}$, they can become four desired groups: $\{\{n,1,n-1\}, \{n,n-1,n-6\}, \{n,n-1\}\}, \{\{n,n-4,n-2\}, \{n,n-2,n-6\}, \{n,n-2\}\}, \{\{n,n-5,n-1\}, \{n,n-3,n-2\}, \{n,n-3,n-1\}\}, \{\{n,n-4,n-1\}, \{n,n-4,n-3\}, \{n,n-1,n-2\}\}$.

We can easily partition the subsets in the first row (except for $\{n,n-1\}$, $\{n,n-2\}$) into $\frac{n-3}{3}$ desired groups: $\{\{n,i\}, \{n,i+1\}, \{n,i+2\}\}$, $n=1,4,7,\ldots,n-5$.

Case 3 $n \equiv 4 (\bmod 6)$.

Assume that $i \equiv 1(\bmod 6)$, $1 \leq i \leq n-15$ (if $n \geq 16$). The left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+5)$-th columns and not in the first row can become 3 groups (obviously):

$\{\{n,i,n-2\},\{n,i,n-1\},\{n,i+1,n-1\}\},\{\{n,i+2,n-2\},\{n,i+3,n-1\},\{n,i+3,n-2\}\},\{\{n,i+4,n-1\},\{n,i+2,n-1\},\{n,i+2,n-3\}\}$.

The left subsets which are in the $(n-9)$-th, $(n-8)$-th, $\ldots$, $(n-3)$-th, $(n-2)$-th columns and are not in the first row can become four groups: $\{\{n,n-9,n-2\}, \{n,n-1,n-8\}, \{n,n-9,n-1\}\}, \{\{n,n-7,n-3\}, \{n,n-2,n-1\}, \{n,n-7,n-2\}\}, \{\{n,n-7,n-1\}, \{n,n-3,n-2\}, \{n,n-3,n-1\}\}, \{\{n,n-6,n-1\}, \{n,n-1,n-5\}, \{n,n-6,n-2\}\}$.

Of course $\{\{n,i\}, \{n,i+1\}, \{n,i+2\}\}$ are desired groups, $i=1,4,7,\ldots,n-3$.

Case 4 $n \equiv 0 (\bmod 6)$.

Suppose $i \equiv 5(\bmod 6)$, $5 \leq i \leq n-13$ (if $n \geq 18$). We can decompose the left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+5)$-th columns and not in the first row into 3 groups:

$\{\{n,i,n-2\},\{n,i+1,n-1\},\{n,i+1,n-2\}\},\{\{n,i+2,n-1\},\{n,i,n-1\},\{n,i,n-3\}\},\{\{n,i+4,n-1\},\{n,i+4,n-2\},\{n,i+5,n-1\}\}$.

For the left subsets which are in the first, second, third, fourth, $(n-7)$-th, $(n-6)$-th, $\ldots,(n-2)$-th, $(n-1)$-th columns and are not in first row as well as $\{n,n-1\},\{n,n-2\}$, we can decompose them into five groups: $\{\{n,1,n-1\},\{n,n-3,n-7\},\{n,n-3,n-1\}\},\{\{n,4,n-1\},\{n,3,n-1\},\{n,3,n-2\}\},\{\{n,n-7,n-2\},\{n,n-6,n-2\},\{n,n-6,n-1\}\},\{\{n,n-7,n-1\},\{n,n-1\},\{n,n-5,n-1\}\},\{\{n,n-2,n-1\},\{n,n-2\},\{n,n-3,n-2\}\}$.

We can decompose the subsets which are in the first row of A_n except for $\{n,n-1\},\{n,n-2\}$ into $\frac{n-3}{3}$ groups: $\{\{n,i\},\{n,i+1\},\{n,i+2\}\}, i=1,4,7,\ldots,n-5$.

The proof is completed. □

Lemma 5 *If $n \equiv 2$, $5 (\bmod 6)$, $n \geq 11$, then we can decompose all the entries of A_n (except for $\emptyset$ and $\{n,2\}$) into $\frac{1}{3}[\binom{n}{2}-1]$ groups such that each group contains exactly 3 subsets and three subsets in each group are the color sets of all vertices of C_3 under some vertex-distinguishing*

E-total coloring.

Proof by Lemma 3, we only consider the entries of A_n which are not empty sets and are not in any good 3×2 submatrices of A_n described in Lemma 3. Such entries are called left subsets in this proof.

Case 1 $n \equiv 2(\bmod 6)$.

For $i \equiv 1(\bmod 6)$, $1 \leq i \leq n-13$ (if $n \geq 14$), the left entries which are in the i-th, ..., $(i+5)$-th columns and not in the first row can be decomposed into 3 desired groups: $\{\{n,i,n-3\},\{n,i,n-1\},\{n,i+2,n-1\}\},\{\{n,i,n-2\},\{n,i+1,n-2\},\{n,i+1,n-1\}\},\{\{n,i+4,n-2\},\{n,i+4,n-1\},\{n,i+5,n-1\}\}$.

The left subsets which are in the $(n-7)$-th , ..., $(n-2)$-th columns and are not in the first row may become 3 desired groups: $\{\{n,n-7,n-3\},\{n,n-7,n-1\},\{n,n-2,n-1\}\},\{\{n,n-6,n-2\},\{n,n-6,n-1\},\{n,n-7,n-2\}\},\{\{n,n-5,n-1\},\{n,n-3,n-1\},\{n,n-3.n-2\}\}$.

We may partiton the entries which are in the first row (except $\{n,2\}$) into $\frac{n-2}{3}$ desired groups: $\{\{n,1\},\{n,3\},\{n,4\}\},\{\{n,i\},\{n,i+1\},\{n,i+2\}\}$, $i=5,8,\ldots,n-3$.

Case 2 $n \equiv 5(\bmod 6)$.

Suppose $i \equiv 1(\bmod 6)$, $1 \leq i \leq n-10$. The left subsets which are in the i-th, ..., $(i+5)$-th columns and are not in the first row can become three groups: $\{\{n,i,n-2\},\{n,i+1,n-2\},\{n,i+1,n-1\}\},\{\{n,i,n-1\},\{n,i,n-3\},\{n,i+2,n-1\}\},\{\{n,i+4,n-2\},\{n,i+4,n-1\},\{n,i+5,n-1\}\}$.

For the left subsets which are in the $(n-4)$-th , $(n-3)$-th , $(n-2)$-th columns and are not in the first row, they can become two desired groups: $\{\{n,n-4,n-2\},\{n,n-4,n-3\},\{n,n-3,n-1\}\},\{\{n,n-4,n-1\},\{n,n-3,n-2\},\{n,n-2,n-1\}\}$.

At last, we consider the entries which are in the first row and which are not in any group constructed. We only give $\frac{n-2}{3}$ groups, and $\{\{n,1\},\{n,3\},\{n,4\}\},\{\{n,i\},\{n,i+1\},\{n,i+2\}\}$ is a desired group, $i=5,8,\ldots,n-3$.

The proof is completed. □

It is easy to see that $\chi^e_{vt}(C_3)=4$.

Theorem 1 *If* $\binom{k-1}{2}+\binom{k-1}{3}<3m \leq \binom{k}{2}+\binom{k}{3}$, $m \geq 2$, $k \geq 4$, $m \neq 3$, *then* $\chi^e_{vt}(mC_3)=k$; $\chi^e_{vt}(3C_3)=5$.

Proof Obviously, we have $\chi^e_{vt}(mC_3) \geq \eta(mC_3)=k$. So we need only to give k-VDET coloring of mC_3 in the following.

If $m=2$, then two C_3's can be colored by $f(1,2;3,4)$ and $f(3,4;1,2)$. So $\chi^e_{vt}(2C_3)=4$.

When $m=3$, $\chi^e_{vt}(3C_3) \geq \eta(3C_3)=4$. Suppose $3C_3$ has a 4-VDET coloring. Consider a 4-VDET coloring g of a $3C_3$.

When the colors of 3 edges of C_3 are the same, then the color set of each vertex of C_3 is 2-set (i.e., it has 2 elements or 2 colors). When the number of the different colors of 3 edges of C_3 is 2, then the color set of one vertex of C_3 is 2-set, and the color sets of other two vertices are 3-sets.

When the colors of 3 edges of C_3 are different, then the color set of each vertex of C_3 is 3-set.

$\{1,2,3,4\}$ has six 2-subsets and four 3-subsets. And the color set of each vertex of $3C_3$ under g is 2-subsets or 3-subsets of $\{1,2,3,4\}$. Note that in $3C_3$, there are at least 3 color sets which have 3 colors. We consider two cases as follows:

Case 1 If there exists one C_3, say the first C_3, such that the sets of all vertices of the C_3 are all 3-set, then the color sets of all vertices of other two C_3's must be 2-subsets of $\{1,2,3,4\}$. So the colors of edges of the second C_3 are the same, say 1; The colors of edges of the third C_3 are the same, say 2. This illustrates that there are three 2-subsets which contain 1 and other three 2-subsets contain 2 among six 2-subsets of $\{1,2,3,4\}$. This is a contradiction.

Case 2 The cardinals of the color sets of the three vertices of the first and the second C_3 under g are $2,3,3$; The cardinals, for the third C_3, are $2,2,2$.

Denote the i-th C_3 by C_3^i, then $V(C_3^i)=\{v_1^i,v_2^i,v_3^i\}$, $1\le i\le m$. We may suppose that $C(v_2^1)=\{1,2,3\}$, $C(v_3^1)=\{1,2,4\}$, $C(v_2^2)=\{1,3,4\}$, $C(v_3^2)=\{2,3,4\}$. Without loss of generality we assume $g(v_2^1v_3^1)=1$, $g(v_2^2v_3^2)=3$. The number of the different colors of 3 edges of C_3^i is 2 and $g(v_1^iv_2^i)\ne g(v_2^iv_3^i)\ne g(v_1^iv_3^i)$, $i=1,2$, so $g(v_1^1v_2^1)=g(v_1^1v_3^1)=2$, $g(v_1^2v_2^2)=g(v_1^2v_3^2)=4$. By the characteristic of E-total coloring we know that $g(v_1^1)=1$, $g(v_1^2)=3$. So $C(v_1^1)=\{1,2\}$, $C(v_1^2)=\{3,4\}$. Thus $\{C(v_1^3),\ C(v_2^3),\ C(v_3^3)\}\subseteq\{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$. This is a contradiction for the 3 edges of the third C_3 accept the same color. So $3C_3$ has no 4-VDET coloring. But we can color $3C_3$'s by $f(1,2;3,4), f(3,4;1,2)$ and $h(1;3,4,5)$, respectively. Thus $\chi_{vt}^e(3C_3)=5$.

If $4\le m\le 6$, $\chi_{vt}^e(mC_3)\ge 5$. $6C_3$ can be colored by $f(1,2;3,4)$, $f(3,4;1,2)$, $h(1;3,4,5)$, $h(2;3,4,5)$, $g(5;3,1;2,4)$ and $g(5;1,2;4,3)$. Of course for $4C_3$ and $5C_3$ we can easily obtain their 5-VDET coloring. So $\chi_{vt}^e(mC_3)=5$, if $4\le m\le 6$. Note that $\{5,3\}$, $\{5,4\}$ are not color sets of any vertex under the above 5-VEDT coloring of 6 C_3.

If $7\le m\le 11$, then $\chi_{vt}^e(mC_3)\ge 6$. we give the 6-VDET coloring of $11C_3$ as follows:

Based on the 5-VDET coloring of $6C_3$ given in the preceding paragraph, we color the $7,8,9,10,11$-th C_3 by $f(6,5;1,3)$, $g(6;1,4;2,5)$, $g(6;3,1;2,4)$, $f(6,4;3,5)$, $h(6;1,2,3)$. The resulting coloring is a 6-VDET coloring of $11C_3$.

The restriction of 6-VDET coloring of $11C_3$ on mC_3 $(7\le m\le 10)$ is a 6-VDET coloring of mC_3. So $\chi_{vt}^e(mC_3)=6$, if $7\le m\le 11$.

If $12\le m\le 18$, then $\chi_{vt}^e(mC_3)\ge 7$. Based on the 6-VDET coloring of $11C_3$, we only need color 12-th, ..., 18-th C_3 by the method mentioned in Lemma 4 (the case $n=7$). We can obtain 7-VDET coloring of $18C_3$ and then the restriction of 7-VDET coloring of $18C_3$ on mC_3 $(12\le m\le 17)$ is a 7-VDET coloring of mC_3. The coloring of $18C_3$ has used up all 2-subsets and 3-subsets of $\{1,2,3,4,5,6,7\}$ except $\{5,3\}$, $\{5,4\}$. So we have $\chi_{vt}^e(mC_3)=7$.

If $19\le m\le 28$, then $\chi_{vt}^e(mC_3)\ge 8$. Now in order to prove $\chi_{vt}^e(mC_3)=8$, if $19\le m\le 28$, we give the 8-VDET coloring of $28C_3$. The first $18C_3$ are colored by the 7-VDET coloring of $18C_3$. And the last $10C_3$'s are colored by $h(5;3,4,8)$, $f(8,1;5,7)$, $h(8;2,3,4)$, $g(8;6,1;2,7)$, $f(8,7;3,5)$, $f(8,6;5,7)$, $g(8;2,4;3,1)$, $g(8;1,4;2,5)$, $g(8;4,6;5,3)$, $g(8;3,6;4,7)$. It is easy to see that the above coloring is a 8-VDET coloring of $28C_3$. So far the 8-VDET coloring of $28C_3$ has used up all 2-combinations and 3-combinations of $\{1, 2, \ldots, 8\}$.

Suppose $m \geq 29$ and the result is valid for the vertex-disjoint union of less than $29C_3$'s. We consider mC_3, where $\binom{k-1}{2} + \binom{k-1}{3} + 1 \leq 3m \leq \binom{k}{2} + \binom{k}{3}$, $k \geq 9$.

If $k \equiv 8, 0, 1 (\mathrm{mod}\, 9)$, then by Lemmas 4 and 5 we can obtain VDET coloring of $\frac{1}{3}[\binom{k}{2} + \binom{k}{3}]C_3$ using colors $\{1, 2, \ldots, k\}$. Note that in the case when $n \equiv 8 (\mathrm{mod}\, 9)$, $k \geq 17$, $\{k, 2\}$, $\{k-3, 2\}$, $\{k-6, 2\}$ are the color sets of all vertices of some C_3. We can easily give k-VDET coloring of mC_3 when $\binom{k-1}{2} + \binom{k-1}{3} < 3m < \binom{k}{2} + \binom{k}{3}$.

If $k \equiv 2, 3, 4 (\mathrm{mod}\, 9)$, then by Lemmas 4 and 5 we can obtain VDET coloring of $\frac{1}{3}[\binom{k}{2} + \binom{k}{3} - 1]C_3$ using colors $\{1, 2, \ldots, k\}$ and this coloring has used up all 2-subsets and 3-subsets of $\{1, 2, \ldots, k\}$ but $\{k, 2\}$ (if $k \equiv 2 (\mathrm{mod}\, 9)$) or $\{k-1, 2\}$ (if $k \equiv 3 (\mathrm{mod}\, 9)$) or $\{k-2, 2\}$ (if $k \equiv 4 (\mathrm{mod}\, 9)$). We can easily give k-VDET coloring of mC_3 when $\binom{k-1}{2} + \binom{k-1}{3} < 3m < \binom{k}{2} + \binom{k}{3}$.

If $k \equiv 5, 6, 7 (\mathrm{mod}\, 9)$, then by Lemmas 4 and 5 we can obtain VDET coloring of $\frac{1}{3}[\binom{k}{2} + \binom{k}{3} - 2]C_3$ using colors $\{1, 2, \ldots, k\}$ and this coloring has used up all 2-subsets and 3-subsets of $\{1, 2, \ldots, k\}$ but two 2-combinations $\{k-3, 2\}, \{k, 2\}$ or $\{k-4, 2\}, \{k-1, 2\}$ or $\{k-5, 2\}, \{k-2, 2\}$. We can easily give k-VDET coloring of mK_3 when $\binom{k-1}{2} + \binom{k-1}{3} < 3m < \binom{k}{2} + \binom{k}{3}$.

The proof is completed. $\square$

3. Vertex distinguishing E-total chromatic numbers of mC_4

A 2×2 submatrix B of A_n is called good if there exists an E-total coloring method for C_4 such that the color sets of the all vertices of C_4 are just all the entries of B.

Lemma 6 *$A_n[i, i+1 \mid j, j+1]$ is a good 2×2 sub-matrix when $i \equiv 0 (\mathrm{mod}\, 2)$, $j \equiv 1 (\mathrm{mod}\, 2)$ and no entry of $A_n[i, i+1 \mid j, j+1]$ is $\emptyset$.*

Proof Note that $A_n[i, i+1 \mid j, j+1] = \begin{pmatrix} \{n, j, j+i-1\} & \{n, j+1, j+i\} \\ \{n, j, j+i\} & \{n, j+1, j+i+1\} \end{pmatrix}$.

Obviously, all the entries of $A_n[i, i+1 \mid j, j+1]$ are the color sets of all the vertices of C_4 under the following E-total coloring:

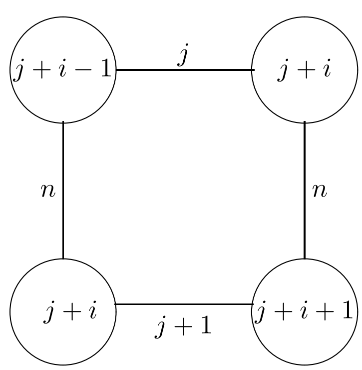

Figure 5 VDET coloring of C_4

The proof is completed. $\square$

Lemma 7 *If $n \equiv 0, 1 (\mathrm{mod}\, 8), n \geq 8$, then we can decompose all the entries of A_n (except for $\emptyset$) into $\frac{1}{4}\binom{n}{2}$ groups such that each group has 4 subsets and four subsets in each group are the color sets of all vertices of C_4 under some vertex-distinguishing E-total coloring of C_4.*

Proof By Lemma 6, we only consider the entries of A_n which are not empty sets and are not in any good 2×2 submatrices of A_n described in Lemma 6. Such entries are called left subsets.

Case 1 $n \equiv 0 (\mathrm{mod}\, 8)$.

When $n \geq 16$, suppose $i \equiv 1 (\mathrm{mod}\, 8)$, $1 \leq i \leq n-15$. We can decompose the left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+7)$-th columns and not in the first row into 3 groups:

$\{\{n, i, n-2\}, \{n, i, n-1\}, \{n, i+2, n-2\}, \{n, i+2, n-1\}\}, \{\{n, i+4, n-2\}, \{n, i+4, n-1\}, \{n, i+6, n-2\}, \{n, i+6, n-1\}\}, \{\{n, i+1, n-1\}, \{n, i+3, n-1\}, \{n, i+5, n-1\}, \{n, i+7, n-1\}\}$.

For $\{n, n-1\}$, $\{n, n-2\}$, $\{n, n-3\}$ as well as the left subsets which are in the $(n-7)$-th, $(n-6)$-th, $\ldots$, $(n-2)$-th, $(n-1)$-th columns and are not in first row, we can decompose them into three groups: $\{\{n, n-2, n-1\}, \{n, n-3, n-2\}, \{n, n-3, n-1\}, \{n, n-1\}\}, \{\{n, n-4, n-1\}, \{n, n-6, n-1\}, \{n, n-3\}, \{n, n-2\}\}, \{\{n, n-7, n-2\}, \{n, n-7, n-1\}, \{n, n-5, n-2\}, \{n, n-5, n-1\}\}$.

We can decompose the subsets which are in the first row of A_n except for $\{n, n-1\}, \{n, n-2\}, \{n, n-3\}$ into $\frac{n-4}{4}$ groups: $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}$, $i = 1, 5, 9, \ldots, n-7$.

Case 2 $n \equiv 1 (\mathrm{mod}\, 8)$.

Suppose $i \equiv 1 (\mathrm{mod}\, 8)$, $1 \leq i \leq n-8$. The left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+7)$-th columns and not in the first row are the sets: $\{n, i, n-1\}$, $\{n, i+2, n-1\}$, $\{n, i+4, n-1\}$, $\{n, i+6, n-1\}$. These sets may become a desired group.

Of course, the subsets in the first row may become $\frac{n-1}{4}$ desired groups: $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}\}$, $i = 1, 5, 9, \ldots, n-4$.

The proof is completed. $\square$

Lemma 8 *If $n \equiv 2 (\mathrm{mod}\, 8)$, then we can decompose all the entries of A_n (except for $\emptyset$ and $\{n, n-1\}$) into $\frac{1}{4}[\binom{n}{2} - 1]$ groups such that each group contains exactly 4 subsets and four subsets in each group are the color sets of all vertices of C_4 under some vertex-distinguishing E-total coloring of C_4.*

Proof As before we only consider the left subsets.

For $i \equiv 1 (\mathrm{mod}\, 8)$, $1 \leq i \leq n-9$. We can decompose the left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+7)$-th columns and not in the first row into 3 groups: $\{\{n, i, n-2\}, \{n, i, n-1\}, \{n, i+2, n-2\}, \{n, i+2, n-1\}\}, \{\{n, i+4, n-2\}, \{n, i+4, n-1\}, \{n, i+6, n-2\}, \{n, i+6, n-1\}\}, \{\{n, i+1, n-1\}, \{n, i+3, n-1\}, \{n, i+5, n-1\}, \{n, i+7, n-1\}\}$.

For the entries which are in the first row (except for $\{n, n-1\}$), they may become $\frac{n-2}{4}$ desired groups: $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}$, $i = 1, 5, 9, \ldots, n-5$.

The proof is completed. $\square$

Lemma 9 *If $n \equiv 3 (\mathrm{mod}\, 8)$, then we can decompose all the entries of A_n (except for $\emptyset$, $\{n, n-$*

$2, n-1\}$, $\{n, n-4, n-2\}$ and $\{n, n-3, n-1\}$) into $\frac{1}{4}[\binom{n}{2}-3]$ groups such that each group has 4 subsets and four subsets in each group are the color sets of all vertices of C_4 under some vertex-distinguishing E-total coloring of C_4.

Proof We only consider the left subsets as well as four sets $\{n, n-4, n-2\}$, $\{n, n-4, n-3\}$, $\{n, n-3, n-2\}$, $\{n, n-3, n-1\}$.

Suppose $i \equiv 1 (\bmod 8), 1 \le i \le n-18$. The left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+7)$-th columns and not in the first row are the sets: $\{n, i, n-1\}$, $\{n, i+2, n-1\}$, $\{n, i+4, n-1\}$, $\{n, i+6, n-1\}$. These subsets may become a desired group.

We now consider the left subsets which are in the $(n-10)$-th, $(n-9)$-th, $\ldots$, $(n-6)$-th, $(n-5)$-th columns and are not in the first row and the subsets in the $(n-4)$-th, $(n-3)$-th, $(n-2)$-th, $(n-1)$-th columns (except for $\{n, n-3\}$, $\{n, n-4\}$). Obviously, these subsets (except for $\{n, n-2, n-1\}$, $\{n, n-4, n-2\}$ and $\{n, n-3, n-1\}$) are the color sets of all vertices of $2C_4$ under the following E-total coloring:

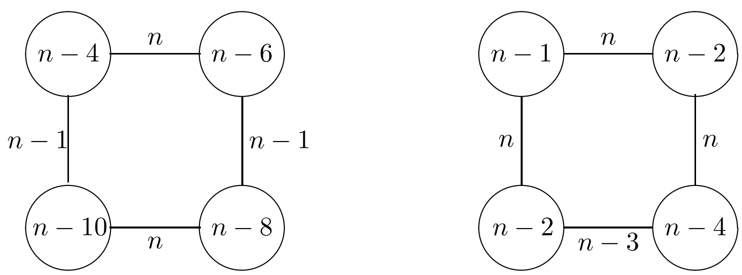

Figure 6 VDET coloring of $2C_4$

We may partition the entries which are in the first row (except for $\{n, n-1\}, \{n, n-2\}$) into $\frac{n-3}{4}$ desired groups: $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}\}, i = 1, 5, 9, \ldots, n-6$.

The proof is completed. □

Lemma 10 *If $n \equiv 4 (\bmod 8)$, then we can decompose all the entries of A_n (except for $\emptyset$, $\{n, n-1\}$, and $\{n, n-3\}$) into $\frac{1}{4}[\binom{n}{2}-2]$ groups such that each group has 4 subsets and four subsets in each group are the color sets of all vertices of C_4 under some vertex-distinguishing E-total coloring.*

Proof We only consider the left subsets.

For $i \equiv 1 (\bmod 8)$, $1 \le i \le n-11$, the left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+7)$-th columns and not in the first row can become 3 desired groups: $\{\{n, i, n-2\}, \{n, i, n-1\}, \{n, i+2, n-2\}, \{n, i+2, n-1\}\}, \{\{n, i+4, n-2\}, \{n, i+4, n-1\}, \{n, i+6, n-2\}, \{n, i+6, n-1\}\}, \{\{n, i+1, n-1\}, \{n, i+3, n-1\}, \{n, i+5, n-1\}, \{n, i+7, n-1\}\}$.

The left subsets which are in the $(n-2)$-th, $(n-3)$-th columns and not in the first row: $\{n, n-2, n-1\}$, $\{n, n-3, n-1\}$, $\{n, n-3, n-2\}$ together with $\{n, n-2\}$ may become a desired group.

We may partion the entries which are in the first row (except for $\{n, n-1\}$, $\{n, n-2\}$, $\{n, n-3\}$) into $\frac{n-4}{4}$ desired groups: $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}\}, i = 1, 5, 9, \ldots, n-7$.

The proof is completed. □

Lemma 11 *If $n \equiv 5 (\bmod 8)$, $n \geq 5$, then we can decompose all the entries of A_n (except for $\emptyset$ and $\{n, n-2, n-1\}$, $\{n, n-4, n-1\}$) into $\frac{1}{4}[\binom{n}{2} - 2]$ groups such that each group contains exactly 4 subsets and four subsets in each group are the color sets of all vertices of C_4 under some vertex-distinguishing E-total coloring.*

Proof As before we consider the left subsets.

When $n \geq 13$, for $i \equiv 1 (\bmod 8)$, $1 \leq i \leq n-12$, we have four left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+7)$-th columns and not in the first row: $\{n, i, n-2\}$, $\{n, i+2, n-1\}$, $\{n, i+4, n-1\}$, $\{n, i+6, n-1\}\}$. These subsets may become a desired group.

For the subsets which are in the first row, we may give $\frac{n-1}{4}$ desired groups, $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}$, $i = 1, 5, 9, \ldots, n-4$.

The proof is completed. □

Lemma 12 *If $n \equiv 6 (\bmod 8)$, $n \geq 6$, then we can decompose all the entries of A_n (except for $\emptyset$, $\{n, n-2, n-1\}$, $\{n, n-3, n-1\}$ and $\{n, n-3, n-2\}$) into $\frac{1}{4}[\binom{n}{2} - 3]$ groups such that each group has 4 subsets and four subsets in each group are the color sets of all vertices of C_4 under some vertex-distinguishing E-total coloring.*

Proof As before, we only consider the left subsets.

When $n \geq 14$, for $i \equiv 1 (\bmod 8)$, $1 \leq i \leq n-13$, we can decompose the left subsets which are in the i-th, $(i+1)$-th, $\ldots$, $(i+7)$-th columns and not in the first row into 3 groups: $\{\{n, i, n-2\}, \{n, i, n-1\}, \{n, i+2, n-2\}, \{n, i+2, n-1\}\}$, $\{\{n, i+4, n-2\}, \{n, i+4, n-1\}, \{n, i+6, n-2\}, \{n, i+6, n-1\}\}$, $\{\{n, i+1, n-1\}, \{n, i+3, n-1\}, \{n, i+5, n-1\}, \{n, i+7, n-1\}\}$.

We have six left subsets which are in the $(n-2)$-th, $(n-3)$-th, $(n-4)$-th, $(n-5)$-th columns and are not in first row. Among these subsets and $\{n, n-1\}$, it is obvious that $\{n, n-1\}$, $\{n, n-4, n-1\}$, $\{n, n-5, n-1\}$, $\{n, n-5, n-2\}$ are the color sets of all vertices of C_4 under the following E-total coloring:

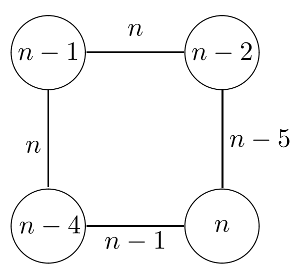

Figure 7 VDET coloring of C_4

At last, we consider the entries which are in the first row (except for $\{n, n-1\}$). They may become $\frac{n-2}{4}$ desired groups, $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}\}$, $i = 1, 5, 9, \ldots, n-5$.

The proof is completed. □

Lemma 13 *If $n \equiv 7 (\bmod 8)$, $n \geq 7$, then we can decompose all the entries of A_n (except for $\emptyset$*

and $\{n, n-2, n-1\}$) into $\frac{1}{4}[\binom{n}{2}-1]$ groups such that each group contains exactly 4 subsets and four subsets in each group are the color sets of all vertices of C_4 under some vertex-distinguishing E-total coloring of C_4.

Proof As before, we only consider the left subsets.

When $n \geq 15$, for $i \equiv 1 (\bmod 8)$, $1 \leq i \leq n-14$, the left subsets which are in the i-th, $(i+1)$-th, ..., $(i+7)$-th columns and not in the first row are the sets: $\{n, i, n-1\}, \{n, i+2, n-1\}, \{n, i+4, n-1\}, \{n, i+6, n-1\}\}$. These subsets may become a desired group.

We have three left subsets which are in the $(n-6)$-th, $(n-5)$-th, ..., $(n-2)$-th columns and are not in first row. For these subsets and the subsets $\{n, n-1\}$, $\{n, n-2\}$, it is obvious that $\{n, n-1\}$, $\{n, n-2\}$, $\{n, n-4, n-1\}$, $\{n, n-6, n-1\}$ are the color sets of all vertices of C_4 under the following E-total coloring:

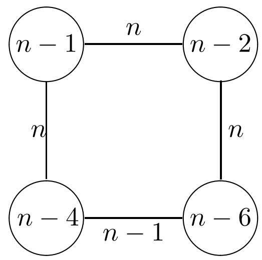

Figure 8 VDET coloring of C_4

For the subsets which are in the first row (except for $\{n, n-1\}, \{n, n-2\}$), we may give $\frac{n-3}{4}$ groups: $\{\{n, i\}, \{n, i+1\}, \{n, i+2\}, \{n, i+3\}, i = 1, 5, 9, \ldots, n-6$.

The proof is completed. □

It is easy to see that $\chi^e_{vt}(C_4) = 4$.

Theorem 2 *If $\binom{k-1}{2} + \binom{k-1}{3} + 1 \leq 4m \leq \binom{k}{2} + \binom{k}{3}$, $m \geq 2$, $k \geq 4$, then $\chi^e_{vt}(mC_4) = k$.*

Proof Obviously we have $\chi^e_{vt}(mC_4) \geq \eta(mC_4) = k$. So we need only to give k-VDET coloring of mC_4 in the following. Denote the i-th C_4 by C_4^i, and $V(C_4^i) = \{v_1^i, v_2^i, v_3^i, v_4^i\}; E(C_4^i) = \{v_1^iv_2^i, v_2^iv_3^i, v_3^iv_4^i, v_4^iv_1^i\}$, $1 \leq i \leq m$.

If $m = 2$, then two C_4 can be colored by 4 colors $1, 2, 3, 4$ as follows:

Let $v_1^1, v_2^1, v_3^1, v_4^1$ receive $2, 3, 2, 4$ respectively; Let $v_1^1v_2^1, v_2^1v_3^1, v_3^1v_4^1, v_4^1v_1^1$ receive $1, 1, 3, 1$, respectively; Let $v_1^2, v_2^2, v_3^2, v_4^2$ receive $3, 4, 3, 1$ respectively and let $v_1^2v_2^2, v_2^2v_3^2, v_3^2v_4^2, v_4^2v_1^2$ receive $2, 2, 4, 2$, respectively. The resulting coloring is obviously 4-VDET coloring of a C_4 and it has used up all 2, 3-combinations of $\{1, 2, 3, 4\}$ but $\{1, 4\}$ and $\{3, 4\}$

So $\chi^e_{vt}(2C_3) = 4$.

If $3 \leq m \leq 5, \chi^e_{vt}(mC_4) \geq 5$. Based on the 4-VDET coloring of $2C_4$ given above, the third, fourth C_4 of $5C_4$ can be colored by the method given in Lemma 11. The fifth C_5 of $5C_4$ can be colored as follows: we assign 1, 3, 1, 3 to $v_1^5, v_2^5, v_3^5, v_4^5$ respectively and assign 4, 4, 5, 4 to $v_1^5v_2^5, v_2^5v_3^5, v_3^5v_4^5, v_4^5v_1^5$, respectively. The resulting coloring is 5-VDET coloring of $5C_5$. Of course for $3C_4$ and $4C_4$, we can easily obtain their 5-VDET colorings.

So $\chi^e_{vt}(mC_4) = 5$, if $3 \le m \le 5$.

Suppose $m \ge 6$, we consider mC_4, where $\binom{k-1}{2} + \binom{k-1}{3} + 1 \le 4m \le \binom{k}{2} + \binom{k}{3}, k \ge 9$.

If $k \equiv 0, 1, 3, 5, 7 (\bmod 8)$, then by Lemmas 7, 9, 11 and 13 we can obtain k-VDET coloring of $(\frac{1}{4}[\binom{k}{2} + \binom{k}{3}])C_4$ using colors $\{1, 2, \ldots, k\}$. Note that in the case when $k \equiv 3 (\bmod 8)$, $k \ge 11$, $\{k-1, k-2\}$, $\{k, k-2, k-1\}$, $\{k, k-3, k-1\}$ and $\{k, k-4, k-2\}$ are the color sets of all vertices of some C_4 under some vertex-distinguishing E-total coloring; when $k \equiv 5 (\bmod 8)$, $k \ge 13$, $\{k, k-2, k-1\}$, $\{k, k-4, k-1\}$, $\{k-1, k-2\}$ and $\{k-1, k-4\}$ are the color sets of all vertices of some C_4 under some vertex-distinguishing E-total coloring; when $k \equiv 7 (\bmod 8)$, $k \ge 7$, $\{k, k-2, k-1\}$, $\{k-1, k-3, k-2\}$, $\{k-1, k-4, k-3\}$ and $\{k-1, k-4, k-2\}$ are the color sets of all vertices of some C_4 under some vertex-distinguishing E-total coloring. We can easily obtain k-VDET coloring of mC_4 when $\binom{k-1}{2} + \binom{k-1}{3} < 4m < \binom{k}{2} + \binom{k}{3}$.

If $k \equiv 2 (\bmod 8)$, then by Lemma 8 we can obtain VDET coloring of $(\frac{1}{4}[\binom{k}{2} + \binom{k}{3} - 1])C_4$ and this coloring has used up all 2, 3-combinations of $\{1, 2, \ldots, k\}$ but $\{k, k-1\}$. We can easily obtain k-VDET coloring of mC_4 when $\binom{k-1}{2} + \binom{k-1}{3} < 4m < \binom{k}{2} + \binom{k}{3}$.

If $k \equiv 4 (\bmod 8)$, $k \ge 12$, then by Lemma 10 we can obtain VDET coloring of $(\frac{1}{4}[\binom{k}{2} + \binom{k}{3} - 2])C_4$ with all colors $1, 2, \ldots, k$. This coloring has used up all 2, 3-combinations but two 2-combinations $\{k, k-1\}$, $\{k, k-3\}$. We can easily obtain k-VDET coloring of mC_4 when $\binom{k-1}{2} + \binom{k-1}{3} < 4m < \binom{k}{2} + \binom{k}{3}$.

If $k \equiv 6 (\bmod 8)$, $k \ge 6$, then by Lemma 12 we can obtain VDET coloring of $(\frac{1}{4}[\binom{k}{2} + \binom{k}{3} - 3])C_4$ with all colors $1, 2, \ldots, k$. This coloring has used up all 2, 3-combinations but three subsets $\{k, k-2, k-1\}$, $\{k, k-3, k-2\}$ and $\{k, k-3, k-1\}$. We can easily obtain k-VDET coloring of mC_4 when $\binom{k-1}{2} + \binom{k-1}{3} < 4m < \binom{k}{2} + \binom{k}{3}$.

The proof is completed. $\Box$

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