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Packings and Coverings of λK_v with 2 Graphs of 6 Vertices and 7 Edges

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Abstract A maximum (v, G, λ)-PD and a minimum (v, G, λ)-CD are studied for 2 graphs of 6 vertices and 7 edges. By means of "difference method" and "holey graph design", we obtain the result: there exists a (v, G_i, λ)-OPD (OCD) for $v \equiv 2, 3, 4, 5, 6 \pmod 7$, $\lambda \geq 1$, $i = 1, 2$.

Keywords G-design; G-packing design; G-covering design.

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1. Introduction

There is a quite long time to the research of the graph packing and covering designs, which involved the simple graphs with less vertices and less edges [1–3], and some special graphs [4, 5]. But there are very few conclusions for the simple graphs with more than five vertices. In this paper, the discussed 2 graphs are listed as follows. For convenience, as a block in a design, the graphs G_1 and G_2 are denoted by (a, b, c, d, e, f) according to the following vertex-label. The related definitions and notations are referred to literature [6].

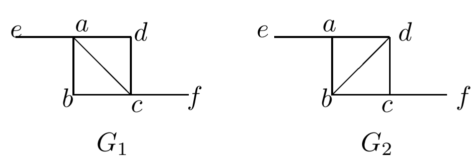

Figure 1 Graphs G_1 and G_2

In what follows, we shall give the constructions of a $\max(v, G_i, \lambda)$-PD and a $\min(v, G_i, \lambda)$-CD for all positive integers v, λ and $i = 1, 2$, all of which are optimal. Our recursive constructions use the following standard "Filling in Holes" method.

Lemma 1.1 ([7]) *For given graph G and positive integers h, w, m, λ, if there exist a G-$HD_\lambda(h^m)$,*

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a G-$ID_\lambda(h+w,w)$ and a (w,G,λ)-OPD (OCD) (or a $(h+w,G,\lambda)$-OPD (OCD)), then a $(mh+w,G,\lambda)$-OPD (OCD) exists, too.

Lemma 1.2 ([7]) *There exist a G_k-$HD(7^{2t+1})$ and a G_k-$HD(14^{t+2})$ for $k=1,2$, $t\geq 1$.*

Lemma 1.3 ([7,8]) *There exists a $(v,G_i,1)$-GD if and only if $v\equiv 0,1 \pmod 7$ for $i=1,2$ and $(v,i)\neq(7,2)$. There exists a $(v,G_i,7)$-GD $(i=1,2)$ for any $v\geq 6$.*

2. Constructions of ID

It is easy to prove that there exists no G_2-$ID(7+w,w)$ for $w=2,5$.

Lemma 2.1 *There exist a G_1-$ID(7+w,w)$ for $2\leq w\leq 6$, and a G_2-$ID(7+w,w)$ for $w=3,4,6$.*

Proof Let G_i-$ID(7+w,w)=(X,W,\mathcal{B})$ for $i=1,2$, where $|\mathcal{B}|=3+w$. Then the family $\mathcal{B}$ consists of the following blocks.

[Graph G_1]

$\underline{w=2:}$ $X=Z_7\bigcup\{x_1,x_2\}$, $W=\{x_1,x_2\}$

$(x_1,3,6,0,1,4),(x_2,4,5,0,6,x_1),(2,1,4,x_1,x_2,0),(1,0,3,x_2,6,4),(2,3,5,6,0,1).$

$\underline{w=3:}$ $X=(Z_3\times Z_3)\bigcup\{x\}$, $W=Z_3\times\{2\}$

$(0_2,x,0_0,1_0,2_0,2_1),(1_1,0_0,0_1,0_2,2_2,x)$ mod $(3,-)$.

$\underline{w=4:}$ $X=Z_7\bigcup\{x_1,\ldots,x_4\}$, $W=\{x_1,\ldots,x_4\}$

$(0,3,1,x_1,x_2,x_4),(1,4,2,x_2,x_3,x_4),(2,5,3,x_1,x_3,x_4),(3,6,4,x_2,x_3,x_4),$
$(4,0,5,x_1,x_3,x_4),(5,1,6,x_2,x_3,x_4),(6,2,0,x_3,x_1,x_4).$

$\underline{w=5:}$ $X=Z_7\bigcup\{x_1,\ldots,x_5\}$, $W=\{x_1,\ldots,x_5\}$

$(x_1,3,6,2,0,x_4),(x_3,6,5,4,2,x_5),(1,2,4,x_1,x_2,x_4),(x_2,4,6,0,5,x_5),$
$(x_4,1,5,0,3,x_1),(x_5,4,0,2,1,x_3),(2,5,3,x_2,x_4,x_5),(1,0,3,x_3,6,4).$

$\underline{w=6:}$ $X=(Z_3\times Z_4)\bigcup\{x\}$, $W=Z_3\times\{2,3\}$

$(0_2,x,0_0,1_0,1_1,2_1),(2_1,0_3,2_0,0_2,2_2,2_3),(0_3,1_1,0_1,x,1_0,2_0)$ mod $(3,-)$.

[Graph G_2] In the G_2-designs, the vertex sets are the same as those of G_1-designs, and the block sets are listed as follows.

$\underline{w=3:}$ $(0_2,x,1_1,1_0,0_1,2_1),(2_1,0_2,2_0,0_0,1_2,0_1)$ mod $(3,-)$.

$\underline{w=4:}$ $(x_1,0,3,1,6,x_4),(x_2,2,3,5,4,x_3),(x_3,4,0,5,2,x_4),(x_4,6,0,2,5,x_2),$
$(x_1,2,1,4,3,x_4),(x_2,3,4,6,1,x_4),(x_3,6,5,1,0,x_1).$

$\underline{w=6:}$ $(0_0,0_2,2_1,x,1_3,2_0),(0_2,0_1,0_3,1_1,2_0,x),(1_0,0_3,0_0,2_1,0_2,2_0)$ mod $(3,-)$. □

Lemma 2.2 *There exist a G_i-$ID(14+w,w)$ for $i=1,2$, $2\leq w\leq 6$, and a G_2-$ID(14+w,w)$ for $w=9,12$.*

Proof Let G_i-$ID(14+w,w)=(X,\mathcal{B})$ for $i=1,2$, where $|\mathcal{B}|=13+2w$.

[Graph G_1] Let $X=(Z_7\times Z_2)\bigcup\{x_1,\ldots,x_w\}$ for $2\leq w\leq 5$ and $X=((Z_7\bigcup\{A,B\})\times Z_2)\bigcup\{C,D\}$ for $w=6$. The family $\mathcal{B}$ consists of the following blocks.

$\underline{w=2:}$ $(0_0,x_1,1_1,x_2,2_1,4_0),(0_1,3_1,1_1,1_0,4_0,3_0)$ mod $(7,-)$;

$(0_0,6_0,1_0,3_0,4_0,2_0),(2_0,6_0,3_0,5_0,0_0,4_0),(4_0,6_0,5_0,1_0,2_0,0_0).$

<u>$w=3$:</u> $(0_0,x_1,0_1,x_2,1_1,3_0)\mod(7,-)$; $(0_0,x_3,5_1,2_1,3_1,6_0)+i_0\ (2\le i\le 6)$;
$(0_0,6_0,1_0,2_0,5_0,3_0),(0_1,2_1,1_1,6_1,5_1,3_1),(2_0,6_0,3_0,4_0,5_0,0_0),(4_0,6_0,5_0,1_0,0_0,3_0)$,
$(4_1,2_1,3_1,5_1,6_1,0_0),(0_0,x_3,5_1,2_1,6_1,6_0),(1_0,x_3,6_1,3_1,4_1,5_1)$.

<u>$w=4$:</u> $(2_1,x_1,0_0,x_2,1_0,x_4),(0_0,x_3,3_1,3_0,2_0,x_4),(6_1,5_1,0_0,4_1,3_1,1_0)\mod(7,-)$.

<u>$w=5$:</u> $(2_1,x_1,0_0,x_2,4_1,x_4),(0_0,x_3,1_1,0_1,4_1,x_4)$, $\mod(7,-)$;
$(0_0,x_5,3_1,6_1,4_0,5_0)+i_0\ (i=0,1,2,3,5)$;
$(0_0,2_0,1_0,6_0,5_0,3_0),(4_0,2_0,3_0,5_0,1_0,6_0),(4_0,x_5,0_1,3_1,6_0,2_0),(6_0,x_5,2_1,5_1,5_0,4_0)$.

<u>$w=6$:</u> $(0_0,C,0_1,D,4_0,4_1)+i_0\ (i=0,1,2,3,5)$; $(0_0,A_0,2_1,B_0,3_1,1_0)\mod(7,2)$;
$(0_0,2_0,1_0,6_0,5_0,3_0),(4_0,2_0,3_0,5_0,1_0,6_0)$ $\mod(-,2)$;
$(4_0,C,4_1,D,6_0,6_1),(6_0,C,6_1,D,5_0,5_1)$.

[Graph G_2] Let $X=(Z_7\times Z_2)\bigcup\{x_1,x_2\}$ for $w=2$ and $X=Z_{14}\bigcup\{x_1,\dots,x_w\}$ for the other w. The family $\mathcal{B}$ consists of the following blocks.

<u>$w=2$:</u> $(0_0,0_1,3_1,1_1,2_0,x_2)+i_0\ (i=2,3,4,6)$; $(6_1,0_0,5_1,3_0,x_1,1_0)+i_0\ (1\le i\le 5)$;
$(0_0,0_1,3_1,1_1,5_0,x_2),(1_0,1_1,4_1,2_1,0_0,x_2),(5_0,5_1,1_1,6_1,4_0,x_2),(6_1,0_0,5_1,x_1,3_0,1_0)$,
$(5_1,6_0,4_1,2_0,3_0,0_0),(x_1,3_0,x_2,4_0,1_0,2_0),(0_0,3_0,1_0,2_0,6_0,x_2),(x_1,5_0,x_2,6_0,2_0,0_0)$.

<u>$w=3$:</u> $(x_1,3,4,7,0,x_3),(x_1,4,12,10,5,0),(x_1,8,9,12,13,4),(x_2,6,11,4,9,1)$,
$(x_2,1,8,3,12,10),(x_3,1,12,6,13,3),(8,11,10,7,x_2,2),(3,x_3,9,2,0,11)$,
$(13,9,0,10,11,5),(x_1,1,7,9,11,x_3),(x_2,5,3,10,13,11),(13,4,5,8,0,11)$,
$(x_2,0,2,7,11,x_1),(12,13,6,7,5,10),(x_3,0,6,8,10,x_1),(1,13,2,5,10,8)$,
$(0,1,2,4,11,6),(5,6,3,9,7,13),(x_3,11,2,12,5,x_2)$.

<u>$w=4$:</u> $(x_1,5,9,6,11,x_2),(x_2,5,12,7,6,4),(x_2,10,1,11,0,x_4),(13,11,2,12,3,x_2)$,
$(x_2,3,7,4,13,9),(x_3,12,1,6,10,x_1),(x_2,8,9,12,1,x_1),(13,0,11,5,2,9)$,
$(x_1,0,12,3,4,x_4),(x_1,7,11,8,12,3),(x_3,1,3,8,7,9),(x_3,4,6,11,13,8)$,
$(x_4,3,5,10,13,1),(x_4,0,2,7,5,9),(x_4,2,3,6,9,x_3),(x_1,2,8,10,13,0)$,
$(0,1,2,4,6,x_3),(13,6,10,7,8,12),(x_3,9,10,0,5,13),(1,13,4,9,7,10)$,
$(x_4,4,5,8,11,2)$.

<u>$w=5$:</u> $(x_1,0,4,1,11,x_2),(x_1,7,11,8,13,x_4),(x_1,2,6,3,4,13),(x_5,1,9,7,2,4)$,
$(6,x_2,7,5,x_1,13),(x_5,10,1,11,9,x_3),(x_2,1,8,3,9,13),(x_4,3,7,4,9,0)$,
$(3,x_5,8,0,12,x_2),(x_5,12,1,6,13,x_4),(x_3,9,11,3,7,0),(13,1,2,5,0,7)$,
$(9,x_1,12,5,8,13),(x_4,6,7,10,13,12),(13,4,6,11,9,8),(x_2,0,9,10,2,12)$,
$(5,x_3,6,0,x_5,9),(x_3,8,2,10,13,4),(2,x_4,12,0,9,8),(13,3,5,10,2,11)$,
$(x_4,5,4,8,7,x_5),(x_3,4,10,12,11,x_1),(x_2,11,2,12,13,x_3)$.

<u>$w=6$:</u> $(1,x_2,3,8,x_4,13),(8,x_1,11,7,13,x_3),(x_1,0,4,1,13,2),(x_2,0,9,10,2,4)$,
$(5,x_3,6,0,13,x_6),(x_3,4,10,12,1,x_1),(x_1,2,6,3,4,10),(2,x_3,10,8,7,13)$,
$(x_4,5,4,8,13,x_2),(x_3,9,11,3,13,x_4),(x_4,3,7,4,9,13),(2,x_4,12,0,13,8)$,
$(6,x_4,10,7,8,x_6),(6,x_2,7,5,x_1,x_3),(4,x_5,9,6,13,2),(0,x_5,12,3,13,6)$,
$(x_5,1,9,7,13,x_2),(5,x_1,12,9,11,13),(0,x_6,12,7,8,5),(11,x_6,9,8,0,13)$,
$(x_6,3,10,5,13,2),(x_5,10,1,11,8,12),(1,13,11,6,3,4),(x_6,1,5,2,4,x_5)$,
$(x_2,11,2,12,13,x_5)$.

<u>$w=9$:</u> $(0,x_1,4,1,x_6,11),(10,x_2,12,4,x_1,1),(5,x_1,9,6,x_3,x_2),(7,x_2,1,13,5,x_5)$,

$(8,x_3,12,9,x_2,10),(7,x_1,11,8,x_4,x_6),(5,x_2,11,0,3,x_5),(11,x_3,1,10,3,x_6),$
$(3,x_3,13,4,x_4,x_8),(4,x_4,13,0,x_5,2),(2,x_4,11,1,10,x_7),(3,x_6,5,10,x_7,x_8),$
$(6,x_6,9,7,x_3,11),(7,x_5,12,3,x_7,13),(4,x_6,12,8,x_9,x_7),(6,x_7,8,13,x_8,3),$
$(7,x_9,8,10,12,x_8),(10,x_7,1,9,x_8,x_9),(2,x_9,5,12,8,11),(8,x_4,9,5,0,x_9),$
$(2,x_1,13,3,x_5,x_6),(6,x_9,13,11,1,10),(3,x_8,7,1,x_9,4),(10,x_4,12,6,x_5,x_1),$
$(12,x_8,2,11,0,7),(2,x_3,7,0,9,11),(0,x_8,4,9,x_9,x_7),(5,x_5,9,13,1,3),$
$(x_7,2,4,5,0,6),(8,x_5,0,6,1,10),(2,x_2,3,6,x_6,0).$

$\underline{w=12}$: $(8,x_3,12,9,x_2,0),(4,x_4,13,0,x_9,10),(10,x_2,12,4,x_8,x_1),(3,x_6,5,10,x_9,1),$
$(7,x_5,12,3,x_7,13),(11,x_3,1,10,x_{10},x_5),(3,x_8,7,1,x_7,11),(4,x_6,12,8,x_7,10),$
$(10,x_4,12,6,x_1,x_{12}),(0,x_1,4,1,x_6,x_5),(2,x_1,13,3,x_5,x_8),(3,x_3,13,4,x_4,x_6),$
$(5,x_1,9,6,x_8,x_2),(7,x_1,11,8,x_{12},x_5),(2,x_4,11,1,x_6,x_7),(12,x_{10},8,1,x_7,0),$
$(2,x_9,5,12,10,x_{12}),(12,x_8,2,11,x_{11},x_{12}),(x_{12},1,x_{11},6,10,0),(7,x_{11},13,2,12,x_{10}),$
$(6,x_9,13,11,x_{10},x_{12}),(6,x_6,9,7,x_3,x_{10}),(7,x_2,1,13,x_4,x_9),(5,x_5,9,13,x_3,x_9),$
$(0,x_8,4,9,x_{12},x_{10}),(5,x_2,11,0,x_{10},x_6),(10,x_7,1,9,x_{11},x_6),(8,x_5,0,6,x_8,x_9),$
$(8,x_4,9,5,2,x_{12}),(7,x_9,8,10,x_{10},x_{12}),(2,x_3,7,0,x_{10},4),(4,x_{12},3,11,6,8),$
$(6,x_7,8,13,x_8,x_{11}),(9,x_{11},5,11,3,7),(2,x_2,3,6,9,5),(3,x_{10},10,0,x_{11},x_5),$
$(x_7,2,4,5,0,x_{11}).$ □

3. Packings and coverings for $\lambda = 1$

In what follows, the symbols C_n, P_n and $St(n)$ denote the graphs respectively: cycle with n vertices, path with n vertices, and star with n terminal vertices.

Lemma 3.1 *There exist a $(7+w,G_1,1)$-OPD (OCD) for $2\le w\le 6$, and a $(7+w,G_2,1)$-OPD (OCD) for $w=3,4,6$.*

Proof Let $(7+w,G_i,1)$-$OPD=(X,\mathcal{A}_{i}(w))$, where X is taken from the definition of vertex sets in G_i-$ID(7+w,w)$ except for specification, and generally $\mathcal{A}_{i}(w)=(\mathcal{B}_{i}(w)-\mathcal{C})\bigcup\mathcal{C}'\bigcup\mathcal{D}$, where $\mathcal{B}_{i}(w)$ is the block set of G_i-$ID(7+w,w)$ constructed in Lemma 2.1, $\mathcal{C}'$ is the modification of $\mathcal{C}$. $B_m(x\to y)$ (or $B_m(x\leftrightarrow y)$) denotes that we replace x with y (or exchange x and y) in the mth block of $\mathcal{B}_{i}(w)$.

For $w=2,3$ and $i=1,2$, a $(7+w,G_i,1)$-OPD is just the G_i-$ID(7+w,w)$, and $L(\mathcal{A}_{i}(2))=P_2$, $L(\mathcal{A}_{i}(3))=C_3$ except that $(i,w)\neq(2,2)$ (ref. Lemma 2.1). As well, the leave-edge graph $L(A_{i}(6))=P_2$, $i=1,2$, will be omitted, since the value of the end point in P_2 does not affect the constructions from OPD to OCD and from $\lambda=1$ to $\lambda>1$.

$\mathcal{A}_{1}(4)$: $\mathcal{C}: B_1,B_2,B_6,B_7$; $\mathcal{C}': B_1(x_2\to x_3)$, $B_2(x_4\to 0)$, $B_6(x_4\to 0)$, $B_7(x_4\leftrightarrow 0)$.
$L(\mathcal{A}_{1}(4))=\{(x_1,x_3),(x_1,x_4),(x_1,x_2),(x_2,x_3),(x_2,x_4),(x_2,0)\}$.

$\mathcal{A}_{1}(5)$: $\mathcal{C}: B_1,B_2$; $\mathcal{C}': B_1(0\to x_3)$, $B_2(2\to x_2)$; $\mathcal{D}:(x_5,x_2,x_1,x_4,x_3,0)$.
$L(\mathcal{A}_{1}(5))=\{(2,x_3),(x_2,x_4),(x_3,x_4)\}$.

$\mathcal{A}_{1}(6)$: $\mathcal{D}:(x_1,x_2,x_3,x_4,x_5,x_6),(x_5,x_4,x_6,x_2,x_3,x_1)$.

$\mathcal{A}_{2}(4)$: $\mathcal{C}: B_1$; $\mathcal{C}': B_1(6\to x_3)$. $L(\mathcal{A}_{2}(4))=\{(6,x_1),(x_1,x_2),(x_1,x_4),(x_2,x_4),(x_2,x_3),(x_3,x_4)\}$.

$\mathcal{A}_{2}(6)$: $X=Z_{11}\bigcup\{x_1,x_2\}$ $\quad(3,0,5,1,x_1,x_2)\quad$ mod 11.

Obviously, each $L(\mathcal{A}_{i}(w))$ is a subgraph of G_i, so each OCD can be obtained by adding a block containing this $L(\mathcal{A}_{i}(w))$. □

Lemma 3.2 *There exists a $(14+w, G_i, 1)$-OPD (OCD) for $2 \le w \le 6,\ i = 1, 2$.*

Proof Let $(14+w, G_i, 1)$-$OPD = (X, \mathcal{A}_{i}(w))$, where X is taken from the definition of vertex set in G_i-$ID(14+w, w)$ except for specification, and generally $\mathcal{A}_{i}(w) = (\mathcal{B}_{i}(w) - \mathcal{C}) \bigcup \mathcal{C}' \bigcup \mathcal{D}$, where $\mathcal{B}_{i}(w)$ is the block set of G_i-$ID(14+w, w)$ constructed in Lemma 2.2, $\mathcal{C}'$ is the modification of $\mathcal{C}$.

For $w = 2, 3$ and $i = 1, 2$, a $(14+w, G_i, 1)$-OPD is just the G_i-$ID(14+w, w)$ and $L(\mathcal{A}_{i}(2)) = P_2$, $L(\mathcal{A}_{i}(3)) = C_3$. By the same reason stated in Lemma 3.1, $L(\mathcal{A}_{i}(6)) = P_2$ $(i = 1, 2)$ can be omitted.

$\mathcal{A}_1(4)$: $\mathcal{C}: (2_1, x_1, 0_0, x_2, 1_0, x_4), (3_1, x_1, 1_0, x_2, 2_0, x_4), (0_1, 6_1, 1_0, 5_1, 4_1, 2_0)$.
$(0_0, x_3, 3_1, 3_0, 2_0, x_4), (5_1, x_1, 3_0, x_2, 4_0, x_4)$;
$\mathcal{C}': (2_1, x_1, 0_0, x_2, 1_0, 3_1), (3_1, 2_0, 1_0, x_2, x_3, x_4), (0_1, 6_1, 1_0, 5_1, 4_1, x_1)$,
$(0_0, x_3, x_4, 3_0, 2_0, 3_1), (5_1, x_1, 3_0, x_2, 4_0, 3_1)$.
$L(\mathcal{A}_1(4)) = \{(3_1, x_1), (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4)\}$.

$\mathcal{A}_1(5)$: $\mathcal{C}: (2_1, x_1, 0_0, x_2, 4_1, x_4)$; $\mathcal{C}': (2_1, 0_0, x_1, x_2, 4_1, x_4)$; $\mathcal{D}: (x_4, x_3, x_5, x_2, 0_0, x_1)$.
$L(\mathcal{A}_1(5)) = \{(x_1, x_3), (x_3, x_2), (x_2, 0_0)\}$.

$\mathcal{A}_1(6)$: $\mathcal{D}: (A_0, A_1, B_0, B_1, C, D), (C, B_1, D, A_1, B_0, A_0)$.

$\mathcal{A}_2(4)$: $\mathcal{C}: (x_1, 5, 9, 6, 11, x_2)$; $\mathcal{C}': (x_1, 5, 9, 6, x_3, x_2)$.
$L(\mathcal{A}_2(4)) = \{(11, x_1), (x_1, x_2), (x_1, x_4), (x_2, x_4), (x_2, x_3), (x_3, x_4)\}$.

$\mathcal{A}_2(5)$: $\mathcal{C}: (x_1, 7, 11, 8, 13, x_4), (x_2, 11, 2, 12, 13, x_3), (x_3, 8, 2, 10, 13, 4)$;
$\mathcal{C}': (x_1, 7, 11, 8, x_5, x_4), (x_2, 11, 2, 12, x_4, x_3), (x_3, 8, 2, 10, x_1, 4)$;
$\mathcal{D}: (x_1, 13, x_3, x_2, x_4, x_5)$. $L(\mathcal{A}_2(5)) = \{(x_3, x_4), (x_4, x_5), (x_5, x_2)\}$.

$\mathcal{A}_2(6)$: $X = (Z_9 \times Z_2) \bigcup \{x_1, x_2\}$
$(2_0, 0_1, 3_0, 1_0, x_1, x_2), (0_1, 0_0, 1_1, 3_1, x_1, x_2), (3_0, 0_0, 4_1, 5_1, 7_0, 0_1) \bmod (9, -)$.

Obviously, each $L(\mathcal{A}_{i}(w))$ is a subgraph of G_i, so each OCD can be obtained from the OPD by adding a block containing this $L(\mathcal{A}_{i}(w))$. □

Lemma 3.3 *There exists a G_i-$HD(7^4)$ for $i = 1, 2$.*

Proof Let G_i-$HD(7^4) = (X, \mathcal{B}_i)$ and $X = Z_7 \times Z_4$. Then the family $\mathcal{B}_i$ is listed in the following.

$\mathcal{B}_1$: $(0_0, 5_3, 6_2, 1_1, 0_1, 1_0) \bmod(7, 4)$; $(0_0, 3_3, 0_2, 3_1, 3_2, 3_0) + i_j$ $(0 \le i \le 6,\ j = 0, 1)$.

$\mathcal{B}_2$: $(5_3, 6_2, 1_1, 0_0, 0_1, 1_0) \bmod(7, 4)$; $(3_3, 0_2, 3_1, 0_0, 0_1, 0_3) + i_j$ $(0 \le i \le 6,\ j = 0, 1)$. □

Lemma 3.4 *There exist a $(28+w, G_2, 1)$-OPD (OCD) for $w = 2, 5, 9, 12$ and a $(14+w, G_2, 1)$-OPD (OCD) for $w = 9, 12$.*

Proof $\underline{(30, G_2, 1)\text{-}OPD}$ $X = (Z_7 \times Z_4) \bigcup \{x_1, x_2\}$
$(0_0, x_1, 5_2, 6_1, x_2, 5_0), (4_3, x_2, 0_1, 5_2, x_1, 3_3), (4_2, 0_0, 3_2, 1_1, 6_0, 2_0), (3_1, 0_1, 6_3, 1_1, 5_0, 3_0)$,
$(3_2, 0_2, 5_3, 1_2, 6_1, 4_2), (3_3, 0_3, 2_0, 1_3, 3_2, 2_3), (5_1, 6_2, 0_0, 2_3, 6_3, 1_3), (2_2, 4_3, 4_1, 0_0, 2_1, 6_3) \bmod(7, -)$;
$(0_0, 0_1, 4_0, 5_0, 6_0, 6_1), (6_0, 1_1, 1_0, 5_0, 6_1, 3_0), (0_0, 2_1, 6_0, 2_0, 4_0, 3_0), (1_0, 3_1, 3_0, 0_0, 6_0, 4_0)$,
$(2_0, 4_1, 4_0, 1_0, 3_0, 6_0), (3_0, 5_1, 2_0, 5_0, 6_1, 4_0)$.

$\underline{(33, G_2, 1)\text{-}OPD}$ $X = (Z_{15} \times Z_2) \bigcup \{x_1, x_2, x_3\}$
$(5_0, 0_0, 3_0, 6_1, x_1, x_2), (0_1, 7_1, 2_1, 0_0, x_2, x_3), (4_0, 0_0, 5_1, 8_1, x_3, x_1), (10_1, 0_0, 9_1, 11_1, 6_1, 3_1),$
$(2_0, 0_1, 3_0, 1_0, 8_0, 10_0) \bmod (15, -)$.
$\underline{(23, G_2, 1)\text{-}OPD}$ $X = (Z_3 \times Z_7) \bigcup \{x_1, x_2\}$
$(0_0, x_1, 0_2, 2_1, x_2, 1_2), (0_3, x_1, 1_5, 0_4, 1_3, 1_1), (1_6, x_2, 0_4, 2_5, x_1, 1_4), (0_1, x_2, 1_3, 0_2, 1_4, 0_4),$
$(2_2, 0_0, 2_6, 1_4, 0_1, 1_6), (0_3, 0_0, 1_1, 0_5, 2_1, 0_1), (0_3, 0_1, 0_4, 0_6, 1_2, 1_1), (1_5, 0_2, 1_6, 0_3, 2_5, 1_2),$
$(2_0, 0_2, 0_4, 0_5, 1_0, 2_2), (1_3, 0_0, 1_6, 2_4, 0_6, 0_1), (0_0, 0_1, 1_5, 2_3, 0_4, 1_6), (0_2, 0_0, 0_6, 2_5, 2_6, 1_1) \bmod (3, -)$.
$\underline{(37, G_2, 1)\text{-}OPD}$ $X = (Z_5 \times Z_7) \bigcup \{x_1, x_2\}$
$(0_0, x_1, 2_2, 1_1, x_2, 3_1), (0_3, x_1, 2_5, 2_4, 1_4, 3_5), (4_6, x_2, 0_4, 1_5, x_1, 0_1), (4_3, x_2, 0_1, 3_2, 1_5, 3_0),$
$(0_2, 0_0, 0_3, 4_0, 3_0, 2_3), (3_4, 0_0, 1_5, 2_0, 1_4, 4_0), (4_3, 0_1, 4_4, 1_1, 2_1, 4_2), (4_5, 0_1, 4_6, 2_1, 1_1, 2_6),$
$(4_4, 0_2, 4_5, 1_2, 2_0, 4_0), (4_6, 0_2, 4_3, 2_2, 1_2, 0_0), (4_5, 0_3, 4_6, 1_3, 4_1, 2_0), (4_6, 0_4, 1_3, 1_4, 3_2, 0_1),$
$(3_6, 0_0, 1_6, 0_1, 3_0, 2_6), (2_6, 0_5, 3_4, 0_3, 3_5, 4_6), (1_4, 1_6, 2_0, 0_5, 3_6, 4_3), (2_2, 0_1, 2_0, 0_3, 4_5, 1_4),$
$(2_4, 0_1, 1_5, 0_2, 0_5, 3_5), (4_2, 0_0, 4_1, 0_4, 1_0, 4_6), (4_3, 0_6, 0_2, 0_5, 4_6, 0_3) \bmod (5, -)$.
$\underline{(26, G_2, 1)\text{-}OPD}$ $X = Z_{23} \bigcup \{x_1, x_2, x_3\}$
$(0, 5, 1, 7, x_1, x_2), (0, 10, 2, 11, x_3, 5) \bmod 23$.
$\underline{(40, G_2, 1)\text{-}OPD}$ $X = Z_{37} \bigcup \{x_1, x_2, x_3\}$
$(0, 7, 1, 10, x_1, x_2), (0, 12, 1, 14, x_3, 9), (0, 17, 2, 18, 4, 7) \bmod 37$.

It is easy to see that $L(\mathcal{B}) = P_2$ for $v = 30, 23, 37$ and $L(\mathcal{B}) = C_3$ for $v = 33, 26, 40$. Obviously, each $L(\mathcal{B})$ is a subgraph of G_2, so each OCD can be obtained from the OPD by adding a block containing this $L(\mathcal{B})$. □

By Lemmas 1.1, 2.1, 3.1, 3.3 and 3.4, we get the following lemma.

Lemma 3.5 *There exists a* $(28 + w, G_i, 1)$-OPD (OCD) *for* $2 \leq w \leq 6,\ i = 1, 2$.

Theorem 3.1 *There exists a* $(v, G_i, 1)$-OPD (OCD) *for* $i = 1, 2,\ v \equiv 2, 3, 4, 5, 6 \pmod 7$.

Proof For clearance, we list Tables 1 and 2 to prove the theorem.

v(mod14)	$w = 2, 3, 4, 5, 6$	$7 + w = 9, 10, 11, 12, 13$
HD	14^{t+2}	7^{2t+1}
$ID(v, w)$	$(14 + w, w)$	$(7 + w, w)$
$OPD(OCD)(v)$	$14 + w,\ 28 + w$	$7 + w$

Table 1 Construction of a $(v, G_1, 1)$-OPD (OCD) $(t \geq 1)$

v(mod 14)	$w = 2, 3, 4, 5, 6, 9, 12$	$w = 10, 11, 13$
HD	14^{t+2}	7^{2t+1}
$ID(v, w)$	$(14 + w, w)$	$(7 + w, w)$
$OPD(OCD)(v)$	$14 + w, 28 + w$	$7 + w$

Table 2 Construction of a $(v, G_2, 1)$-OPD (OCD) $(t \geq 1)$

The desired designs in the tables refer to Lemmas 1.2, 2.1, 2.2, 3.1, 3.2, 3.4, 3.5. □

4. Packings and coverings for $\lambda > 1$

Lemma 4.1([5]) *Given positive integers v, λ and μ. Let X be a v set.*

(1) Suppose there exists a (v, G, λ)-$OPD = (X, \mathcal{A})$ with leave-edge graph $L_\lambda(\mathcal{A})$ and $L_\lambda(\mathcal{A}) \subset G$. Then there exists a (v, G, λ)-OCD with the repeat-edge graph $G \backslash L_\lambda(\mathcal{A})$.

(2) Suppose there exist both a (v, G, λ)-$OPD = (X, \mathcal{A})$ (with leave-edge graph $L_\lambda(\mathcal{A})$) and a (v, G, μ)-$OPD = (X, \mathcal{B})$ (with leave-edge graph $L_\mu(\mathcal{B})$). If $|L_\lambda(\mathcal{A})| + |L_\mu(\mathcal{B})| = l_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-$OPD = (X, \mathcal{A} \bigcup \mathcal{B})$ and its leave-edge graph is just $L_\lambda(\mathcal{A}) \bigcup L_\mu(\mathcal{B})$.

(3) Suppose there exist both a (v, G, λ)-$OCD = (X, \mathcal{A})$ (with repeat-edge graph $R_\lambda(\mathcal{A})$) and a (v, G, μ)-$OCD = (X, \mathcal{B})$ (with repeat-edge graph $R_\mu(\mathcal{B})$). If $|R_\lambda(\mathcal{A})| + |R_\mu(\mathcal{B})| = r_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-$OCD = (X, \mathcal{A} \bigcup \mathcal{B})$ and its repeat-edge graph is just $R_\lambda(\mathcal{A}) \bigcup R_\mu(\mathcal{B})$.

(4) Suppose there exist both a (v, G, λ)-$OPD = (X, \mathcal{A})$ (with leave-edge graph $L_\lambda(\mathcal{A})$) and a (v, G, μ)-$OCD = (X, \mathcal{B})$ (with repeat-edge graph $R_\mu(\mathcal{B})$). If $L_\lambda(\mathcal{A}) \supset R_\mu(\mathcal{B})$ and $|L_\lambda(\mathcal{A})| - |R_\mu(\mathcal{B})| = l_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-$OPD = (X, \mathcal{A} \bigcup \mathcal{B})$ with the leave-edge graph $L_\lambda(\mathcal{A}) \backslash R_\mu(\mathcal{B})$.
(5) *Suppose there exist both a (v, G, λ)-$OCD = (X, \mathcal{A})$ (with repeat-edge graph $R_\lambda(\mathcal{A})$), and a (v, G, μ)-$OPD = (X, \mathcal{B})$ (with leave-edge graph $L_\mu(\mathcal{B})$). If $R_\lambda(\mathcal{A}) \supset L_\mu(\mathcal{B})$ and $|R_\lambda(\mathcal{A})| - |L_\mu(\mathcal{B})| = r_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-$OCD = (X, \mathcal{A} \bigcup \mathcal{B})$ with the repeat-edge graph $R_\lambda(\mathcal{A}) \backslash L_\mu(\mathcal{B})$.*

In this section, we only need to consider $1 < \lambda < \lambda_{\min}$, where λ_{min} denotes the minimal λ such that there exists a (v, G_i, λ)-GD for $v \geq |E(G)|$, $i = 1, 2$. Here $\lambda_{\min} = 7$.

Lemma 4.2 *There exist a $(7+w, G_1, \lambda)$-OPD (OCD) for $\lambda > 1$, $w = 2, 6$ and a $(7+w, G_2, \lambda)$-OPD (OCD) for $\lambda > 1$, $w = 6$.*

Proof By Lemmas 3.1 and 4.1, for $1 < \lambda \leq 6$, $L_\lambda = L_1 \bigcup L_{\lambda-1}$, $R_\lambda = G_i \setminus L_\lambda$. □

Lemma 4.3 *There exists a $(7+4, G_i, \lambda)$-OPD (OCD) for $\lambda > 1$ and $i = 1, 2$.*

Proof By Lemmas 3.1 and 4.1, for $1 < \lambda \leq 6$, $L_\lambda = L_{\lambda-1} \setminus R_1$, $R_\lambda = G_i \setminus L_\lambda$. □

Lemma 4.4 *There exist a $(7+3, G_i, \lambda)$-OPD (OCD) for $\lambda > 1$, $i = 1, 2$ and a $(7+5, G_1, \lambda)$-OPD (OCD) for $\lambda > 1$.*

Proof By Lemma 3.1, $L(\mathcal{A}_{1(5)}) = P_4$. Further, for $i = 1, 2$, in G_i-$ID(10, 3) = (Z_7 \bigcup W, W, \mathcal{B})$ constructed in Lemma 2.1, there exists an $x \in W$ such that x adjoins with a pendant vertex, so it is easy to obtain the desired OPD with leave-edge P_4. We list Table 3 for clearance.

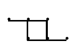
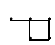
λ	1	2	3	4	5	6
l_λ	3	$6 = 2l_1$	$2 = l_1 - r_2$	$5 = l_1 + l_3$	$1 = l_3 - r_2$	$4 = l_1 + l_5$
L_λ	P_4		P_3		P_2	C_4
r_λ	4	1	5	2	6	3
R_λ	$G_i \setminus P_4$	P_2	$G_i \setminus P_3$	$G_i \setminus L_4$	$G_i \setminus P_2$	$G_i \setminus C_4$

Table 3 Leave (repeat)-edge graphs of the $OPDs$ $(OCDs)$

Lemma 4.5 *There exists a $(14+w, G_i, \lambda)$-OPD (OCD) for $\lambda > 1$, $2 \leq w \leq 6$, and $i = 1, 2$.*

Proof For $w=2,4,6$, the conclusion holds by the proofs of Lemmas 4.2 and 4.3; For $w=5$, by Lemma 3.2, $L(\mathcal{A}_i(5))=P_4$ for $i=1,2$; For $w=3$, similarly to Lemmas 4.4 and 4.5, we can obtain a $(14+w,G_i,\lambda)$-OPD with the leave-edge graph P_4.

$\mathcal{A}_1(3)$: $\mathcal{C}:(0_0,x_1,0_1,x_2,1_1,3_0)$; $\mathcal{C}':(0_0,0_1,x_1,x_2,1_1,x_3)$.

$\mathcal{A}_2(3)$: $\mathcal{C}:(x_1,3,4,7,0,x_3)$; $\mathcal{C}':(x_1,3,4,7,x_2,x_3)$. □

Lemma 4.6 *There exist a $(28+w,G_2,\lambda)$-OPD (OCD) for $\lambda>1$, $w=2,5,9,12$, and a $(14+w,G_2,\lambda)$-OPD (OCD) for $\lambda>1$, $w=9,12$.*

Proof By Lemma 3.4, for $v=30,23,37$, the leave-edge graph of the $(v,G_2,1)$-OPD is P_2. In the following, we will obtain $(v,G_2,1)$-OPD with leave-edge graph P_4 for $v=33,26,40$.

$(33,G_2,1)$-OPD $\mathcal{C}:(5_0,0_0,3_0,6_1,x_1,x_2),(4_1,11_1,6_1,4_0,x_2,x_3),(0_0,11_0,1_1,4_1,x_3,x_1)$.
$\mathcal{C}':(x_3,0_0,3_0,6_1,x_1,x_2),(4_1,11_1,6_1,4_0,x_2,5_0),(0_0,11_0,1_1,4_1,5_0,x_1)$.

$(26,G_2,1)$-OPD $\mathcal{C}:(0,5,1,7,x_1,x_2),(5,15,7,16,x_3,10),(7,17,9,18,x_3,12)$.
$\mathcal{C}':(x_3,5,1,7,x_1,x_2),(5,15,7,16,0,10),(7,17,9,18,0,12)$.

$(40,G_2,1)$-OPD $\mathcal{C}:(0,7,1,10,x_1,x_2),(7,19,8,21,x_3,16),(10,22,11,24,x_3,19)$.
$\mathcal{C}':(x_3,7,1,10,x_1,x_2),(7,19,8,21,0,16),(10,22,11,24,0,19)$.

So the lemma holds by Lemmas 4.2 and 4.4. □

By Lemmas 3.3, 4.2–4.5, we derive the following lemma.

Lemma 4.7 *There exist a $(28+w,G_1,\lambda)$-OPD (OCD) for $2\le w\le 6$, $\lambda>1$, and a $(28+w,G_2,\lambda)$-OPD (OCD) for $w=3,4,6$, $\lambda>1$.*

Similarly to the proof of Theorem 3.1, by Lemmas 1.1, 1.2, 2.1, 2,2, 4.2–4.7, we obtain the following result.

Theorem 4.1 *There exists a (v,G_i,λ)-OPD (OCD) for $\lambda>1$, $i=1,2$ and $v\equiv 2,3,4,5,6 \pmod 7$.*

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