A Symbolic Operator Approach to Newton Series

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Abstract In this paper, we present several expansions of the symbolic operator $(1 + E)^x$. Moreover, we derive some series transforms formulas and the Newton generating functions of $\{f(k)\}$.

Keywords symbolic operator; Newton generating function; potential polynomial; hypergeometric function; Stirling number.

1. Introduction

It is known that the symbolic calculus with the shift operator E, the difference operator $\triangle$ and the differential operator D plays an important role in the calculus of finite differences as well as in certain topics of computational methods. Various well-known results can be found in [1–3]. Based on the general theory of the formal power series, all the symbolic expressions used in the calculus could be formally expressed as power series in $\triangle$, E or D over the real or complex number field. For some detailed discussion of power series, see [4] and [5].

As usual, we denote by C^∞ the class of infinitely differentiable real functions defined in $\mathbb{R} = (-\infty, +\infty)$. The operators $\triangle$, E and D may be defined for all $f \in C^\infty$ via the following relations:

$$\triangle f(t) = f(t+1) - f(t), \ Ef(t) = f(t+1), \ Df(t) = \frac{\mathrm{d}}{\mathrm{d}t}f(t).$$

We use the number 1 as an identity operator, viz. $1f(t) = f(t)$. Obviously, these operators satisfy some simple symbolic relations such as

$$E = 1 + \triangle = e^D, \quad \triangle = E - 1 = e^D - 1, \quad D = \log(1 + \triangle),$$

where e^D and $\log(1 + \triangle)$ are defined as the sense of formal power series expansions, i.e.,

$$e^D = \sum_{k \geqslant 0} \frac{1}{k!} D^k,$$

$$\log(1 + \triangle) = \sum_{k \geqslant 1} \frac{(-1)^{k-1}}{k} \triangle^k.$$

Moreover, we may define for any real number a, $E^a f(t) = f(t + a)$. In particular, $E^k f(0) = [E^k f(t)]_{t=0} = f(k)$.

He et al. [6] note that any power series of the form $\sum_{k=0}^{\infty} f(k) x^k$ could be written symbolically as

$$\sum_{k \geqslant 0} f(k) x^k = \sum_{k \geqslant 0} x^k E^k f(0) = \sum_{k \geqslant 0} (xE)^k f(0) = (1 - xE)^{-1} f(0).$$

This shows that the symbolic operator $(1 - xE)^{-1}$ with parameter x can be applied to $f(t)$ (at $t = 0$) to yield a power series or a generating function for $\{f(k)\}$. We observe that the Newton series of the form $\sum_{k=0}^{\infty} f(k) \frac{(x)_k}{k!}$ could be written symbolically as

$$\sum_{k \geqslant 0} f(k) \frac{(x)_k}{k!} = \sum_{k \geqslant 0} \frac{(x)_k}{k!} E^k f(0) = \sum_{k \geqslant 0} \binom{x}{k} E^k f(0) = (1 + E)^x f(0),$$

where the lower factorial polynomial $(x)_k = x(x-1) \cdots (x - k + 1)$. Hence, the symbolic operator $(1 + E)^x$ with parameter x can be applied to $f(t)$ (at $t = 0$) to yield a Newton series or a Newton generating function for $\{f(k)\}$.

In Section 2, we shall give some definitions and lemmas which will be used in the following section. We shall show in Section 3 that $(1 + E)^x$ could be expanded into various series to derive various symbolic operational formulas as well as summation formula for $\sum_{k \geqslant 0} f(k) \frac{(x)_k}{k!}$. Some consequences of the summation formulas and the examples will also be shown in this section. In this paper, we only study the expansions of formal Newton series. For the convergence of Newton series, see [7] and [1].

2. Preliminaries

We shall need the following definitions.

Definition 1 δ is Sheppard central difference operator defined by the relation $\delta f(t) = f(t + \frac{1}{2}) - f(t - \frac{1}{2})$, so that [2]

$$\delta = \triangle E^{-\frac{1}{2}} = \frac{\triangle}{E^{\frac{1}{2}}}, \quad \delta^{2k} = \triangle^{2k} E^{-k}.$$

Definition 2 The partial Bell polynomials are defined as follows:

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1! c_2! \cdots (1!)^{c_1} (2!)^{c_2} \cdots} x_1^{c_1} x_2^{c_2} \cdots, \tag{1}$$

where the summation takes place over all integers $c_1, c_2, c_3, \cdots \geqslant 0$, such that:

$$c_1 + 2c_2 + 3c_3 + \cdots = n, \quad c_1 + c_2 + c_3 + \cdots = k.$$

The potential polynomials $P_n^{(x)}$ are defined by:

$$P_0^{(x)} = 1, \quad P_n^{(x)} = P_n^{(x)}(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant k \leqslant n} (x)_k B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \ n \geqslant 1. \tag{2}$$

Definition 3 *The hypergeometric function $_2F_1(a,b;c;x)$ is defined by the series*

$$\sum_{n=0}^{\infty} \frac{\langle a\rangle_n \langle b\rangle_n}{\langle c\rangle_n n!} x^n$$

for $|x| < 1$, and by continuation elsewhere, where the upper factorial polynomial $\langle x\rangle_n = x(x + 1)\cdots(x + n - 1)$ (see [8]).

In next section, we will use some simple and well-known propositions which may be stated as lemmas as follows.

Lemma 1 *For all positive integer j, we have*

$$\sum_{k=0}^{\infty} \frac{(x)_k k^j}{k!} = 2^x P_j^{(x)}(\frac{1}{2}, \frac{1}{2}, \ldots).$$

Lemma 2 *For $x \geqslant 1$, we have Everett's symbolic expression*

$$E^x = \sum_{k=0}^{\infty} \left(\binom{x+k}{2k+1} \frac{\triangle^{2k}}{E^{k-1}} - \binom{x+k-1}{2k+1} \frac{\triangle^{2k}}{E^k} \right).$$

Lemma 3 *Gauss's symbolic expression for E^x is given by*

$$E^x = \sum_{k=0}^{\infty} \left(\binom{x+k}{2k} \frac{\triangle^{2k}}{E^k} + \binom{x+k}{2k+1} \frac{\triangle^{2k+1}}{E^{k+1}} \right).$$

3. Main results

In this section, we will show various expansions of $(1 + E)^x$.

Proposition 1 *The operator $(1 + E)^x$ has following formal symbolic expansions:*

$$(1 + E)^x = 2^x \sum_{k \geqslant 0} \frac{(x)_k}{k!} \frac{1}{2^k} \triangle^k, \tag{3}$$

$$(1 + E)^x = 2^x \sum_{j=0}^{\infty} P_j^{(x)}(\frac{1}{2}, \frac{1}{2}, \ldots) \frac{D^j}{j!}, \tag{4}$$

$$(1 + E)^x = 1 + \sum_{l=0}^{\infty} \left[\binom{x}{l+1} {}_2F_1\left(\begin{matrix} l+1-x,\ 2l+2 \\ l+2 \end{matrix}; -1 \right) \frac{\triangle^{2l}}{E^{l-1}} - \right.$$
$$\left. \binom{x}{l+2} {}_2F_1\left(\begin{matrix} l+2-x,\ 2l+2 \\ l+3 \end{matrix}; -1 \right) \frac{\triangle^{2l}}{E^l} \right], \tag{5}$$

$$(1 + E)^x = \sum_{l=0}^{\infty} \left[\binom{x}{l} {}_2F_1\left(\begin{matrix} l-x,\ 2l+1 \\ l+1 \end{matrix}; -1 \right) \frac{\triangle^{2l}}{E^l} + \right.$$
$$\left. \binom{x}{l+1} {}_2F_1\left(\begin{matrix} l+1-x,\ 2l+2 \\ l+2 \end{matrix}; -1 \right) \frac{\triangle^{2l+1}}{E^{l+1}} \right]. \tag{6}$$

Proof Obviously, (3) can be derived as follows:

$$(1 + E)^x = (2 + \triangle)^x = 2^x \left(1 + \frac{\triangle}{2} \right)^x = 2^x \sum_{k \geqslant 0} \binom{x}{k} \left(\frac{\triangle}{2} \right)^k = 2^x \sum_{k \geqslant 0} \frac{(x)_k}{k! 2^k} \triangle^k.$$

To prove (4), using $E = e^D$ and Lemma 1 we have

$$(1 + E)^x = (1 + e^D)^x = \sum_{k=0}^{\infty} \binom{x}{k} e^{kD} = \sum_{k=0}^{\infty} \frac{(x)_k}{k!} \sum_{j=0}^{\infty} \frac{(kD)^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{D^j}{j!} \sum_{k=0}^{\infty} \frac{(x)_k k^j}{k!} = 2^x \sum_{j=0}^{\infty} P_j^{(x)}(\frac{1}{2}, \frac{1}{2}, \dots) \frac{D^j}{j!}.$$

By using Lemmas 2 and 3, we can derive (5) as follows.

$$(1 + E)^x - 1 = \sum_{k \geqslant 1} \binom{x}{k} E^k = \sum_{k \geqslant 1} \binom{x}{k} \sum_{l=0}^{\infty} \left(\binom{k+l}{2l+1} \frac{\triangle^{2l}}{E^{l-1}} - \binom{k+l-1}{2l+1} \frac{\triangle^{2l}}{E^l} \right)$$

$$= \sum_{l=0}^{\infty} \left[\frac{\triangle^{2l}}{E^{l-1}} \sum_{k \geqslant l+1} \binom{x}{k} \binom{k+l}{2l+1} - \frac{\triangle^{2l}}{E^l} \sum_{k \geqslant l+2} \binom{x}{k} \binom{k+l-1}{2l+1} \right]$$

$$= \sum_{l=0}^{\infty} \left[\frac{\triangle^{2l}}{E^{l-1}} \sum_{j \geqslant 0} \binom{x}{j+l+1} \binom{j+2l+1}{2l+1} - \right.$$

$$\left. \frac{\triangle^{2l}}{E^l} \sum_{j \geqslant 0} \binom{x}{j+l+2} \binom{j+2l+1}{2l+1} \right]$$

$$= \sum_{l=0}^{\infty} \left[\frac{\triangle^{2l}}{E^{l-1}} \binom{x}{l+1} \sum_{j \geqslant 0} \frac{\langle l+1-x \rangle_j \langle 2l+2 \rangle_j}{\langle l+2 \rangle_j j!} (-1)^j - \right.$$

$$\left. \frac{\triangle^{2l}}{E^l} \binom{x}{l+2} \sum_{j \geqslant 0} \frac{\langle l+2-x \rangle_j \langle 2l+2 \rangle_j}{\langle l+3 \rangle_j j!} (-1)^j \right]$$

$$= \sum_{l=0}^{\infty} \left[\binom{x}{l+1} {}_2F_1 \left(\begin{matrix} l+1-x, 2l+2 \\ l+2 \end{matrix} ; -1 \right) \frac{\triangle^{2l}}{E^{l-1}} - \right.$$

$$\left. \binom{x}{l+2} {}_2F_1 \left(\begin{matrix} l+2-x, 2l+2 \\ l+3 \end{matrix} ; -1 \right) \frac{\triangle^{2l}}{E^l} \right].$$

Similarly, we get formula (6).

Applying the symbolic expansions (3–6) to a function $f(t)$ at $t = 0$, we have

Proposition 2 *Let $f(t)$ and $h(t)$ be two functions, and let $h(t)$ be infinitely differentiable at $t = 0$. Then we have formally*

$$\sum_{k=0}^{\infty} f(k) \frac{(x)_k}{k!} = 2^x \sum_{k=0}^{\infty} \frac{(x)_k}{k! 2^k} \triangle^k f(0), \tag{7}$$

$$\sum_{k=0}^{\infty} h(k) \frac{(x)_k}{k!} = 2^x \sum_{j=0}^{\infty} P_j^{(x)} \left(\frac{1}{2}, \frac{1}{2}, \dots \right) \frac{D^j h(0)}{j!}, \tag{8}$$

$$\sum_{k=1}^{\infty} f(k) \frac{(x)_k}{k!} = \sum_{l=0}^{\infty} \left[\binom{x}{l+1} {}_2F_1 \left(\begin{matrix} l+1-x, 2l+2 \\ l+2 \end{matrix} ; -1 \right) \delta^{2l} f(1) - \right.$$

$$\left. \binom{x}{l+2} {}_2F_1 \left(\begin{matrix} l+2-x, 2l+2 \\ l+3 \end{matrix} ; -1 \right) \delta^{2l} f(0) \right], \tag{9}$$

$$\sum_{k=0}^{\infty} f(k)\frac{(x)_k}{k!} = \sum_{l=0}^{\infty}\left[\binom{x}{l} {}_2F_1\left(\begin{array}{c} l-x,\, 2l+1 \\ l+1 \end{array}; -1\right) \delta^{2l} f(0) + \right.$$
$$\left. \binom{x}{l+1} {}_2F_1\left(\begin{array}{c} l+1-x,\, 2l+2 \\ l+2 \end{array}; -1\right) \left(\delta^{2l} f(0) - \delta^{2l} f(-1)\right)\right]. \tag{10}$$

Remark 1 Let $x = -1$ in (7), (9) and (10). Then we get three series transforms

$$\sum_{k=0}^{\infty} (-1)^k f(k) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \triangle^k f(0), \tag{11}$$

$$\sum_{k=1}^{\infty} (-1)^k f(k) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{4^{l+1}} \left(\delta^{2l} f(1) + \delta^{2l} f(0)\right), \tag{12}$$

$$\sum_{k=0}^{\infty} (-1)^k f(k) = \sum_{l=0}^{\infty} \frac{(-1)^l}{4^{l+1}} \left(\delta^{2l} f(0) + \delta^{2l} f(-1)\right), \tag{13}$$

which are He et al.'s (4.3), (4.4) and (4.5) in [6]. In [6], they have also shown these series transform can be used to convert a slowly convergent alternating series into rapidly convergent series.

Corollary 1 *If $f(t)$ is a polynomial in t of degree d, we have*

$$\sum_{k=0}^{\infty} f(k)\frac{(x)_k}{k!} = 2^x \sum_{k=0}^{d} \frac{(x)_k}{k!2^k} \triangle^k f(0), \tag{14}$$

$$\sum_{k=0}^{\infty} f(k)\frac{(x)_k}{k!} = 2^x \sum_{j=0}^{d} P_j^{(x)}\left(\frac{1}{2},\frac{1}{2},\dots\right) \frac{D^j f(0)}{j!}, \tag{15}$$

$$\sum_{k=1}^{\infty} f(k)\frac{(x)_k}{k!} = \sum_{l=0}^{[d/2]}\left[\binom{x}{l+1} {}_2F_1\left(\begin{array}{c} l+1-x,\, 2l+2 \\ l+2 \end{array}; -1\right) \delta^{2l} f(1) - \right.$$
$$\left. \binom{x}{l+2} {}_2F_1\left(\begin{array}{c} l+2-x,\, 2l+2 \\ l+3 \end{array}; -1\right) \delta^{2l} f(0)\right], \tag{16}$$

$$\sum_{k=0}^{\infty} f(k)\frac{(x)_k}{k!} = \sum_{l=0}^{[d/2]}\left[\binom{x}{l} {}_2F_1\left(\begin{array}{c} l-x,\, 2l+1 \\ l+1 \end{array}; -1\right) \delta^{2l} f(0) + \right.$$
$$\left. \binom{x}{l+1} {}_2F_1\left(\begin{array}{c} l+1-x,\, 2l+2 \\ l+2 \end{array}; -1\right) \left(\delta^{2l} f(0) - \delta^{2l} f(-1)\right)\right]. \tag{17}$$

Obviously, (14)–(17) can be used to obtain Newton generating functions of $\{f(k)\}$.

For example, let $f(t) = t^m$ ($m \geqslant 1$). For $k \leqslant m$, we have

$$\triangle^k f(0) = \left[\triangle^k t^m\right]_{t=0} = k! S(m,k),$$

where $S(m,k)$ is the Stirling number of the second kind, i.e., the number of distributions of n distinct balls into k indistinguishable boxes (the order of the boxes does not count) such that no box is empty. From (14), we have

$$\sum_{k=0}^{\infty} k^m \frac{(x)_k}{k!} = 2^x \sum_{k=0}^{m} \frac{(x)_k}{k!2^k} k! S(m,k) = 2^x \sum_{k=0}^{m} \frac{(x)_k}{2^k} S(m,k), \tag{18}$$

i.e., the Newton generating function of $\{k^m\}_{k\in\mathbb{N}}$. For $f(t) = \binom{t}{m}$, we have $\triangle^k f(0) = \delta_{m,k}$, where $\delta_{m,k}$ is the Kronecker symbol with $\delta_{m,k} = 1$ for $m = k$ and zero for $m \neq k$. Following (14), we get

$$\sum_{k=0}^{\infty} \binom{k}{m} \frac{(x)_k}{k!} = 2^x \sum_{k=0}^{m} \frac{(x)_k}{k!2^k} \delta_{m,k} = 2^{x-m} \binom{x}{m}, \tag{19}$$

i.e., the Newton generating function of $\{\binom{k}{m}\}_{k\in\mathbb{N}}$.

Formulas (7)–(10) can be used to derive series transforms formulas. For instance, let $f(t) = \frac{1}{t+1}$. We have $\triangle^k f(0) = \frac{(-1)^k}{k+1}$. Following (7), we have

$$\sum_{k=0}^{\infty} \frac{(x)_k}{(k+1)!} = 2^x \sum_{k=0}^{\infty} \frac{(-1)^k (x)_k}{2^k (k+1)!}, \tag{20}$$

i.e.,

$$_2F_1 \left(\begin{matrix} 1, -x \\ 2 \end{matrix}; -1 \right) = 2^x \, _2F_1 \left(\begin{matrix} 1, -x \\ 2 \end{matrix}; \frac{1}{2} \right). \tag{21}$$

(21) can also be derived by Pfaff's transformation.

Now we consider another example generated by function $f(t) = (g(z))^t$, where $g : \mathbb{R} \to \mathbb{R}$ and f is defined on $\mathbb{N}$. Clearly, $\triangle^k f(0) = (g(z) - 1)^k$. Hence, by (7), we have

$$\sum_{k=0}^{\infty} (g(z))^k \frac{(x)_k}{k!} = 2^x \sum_{k=0}^{\infty} \frac{(x)_k}{2^k k!} (g(z) - 1)^k$$

$$= 2^x \sum_{k=0}^{\infty} \binom{x}{k} \left(\frac{g(z) - 1}{2} \right)^k = 2^x \left(1 + \frac{g(z) - 1}{2} \right)^x = (1 + g(z))^x.$$

Similarly, for $g(z) > 0$, we have $D^k f(0) = (\ln g(z))^k$. Following (8), we obtain

$$\sum_{k=0}^{\infty} (g(z))^k \frac{(x)_k}{k!} = 2^x \sum_{k=0}^{\infty} P_k^{(x)} \left(\frac{1}{2}, \frac{1}{2}, \dots \right) \frac{1}{k!} (\ln g(z))^k.$$

For example, taking $g(z) = e^z$, we get

$$\sum_{k=0}^{\infty} P_k^{(x)} \left(\frac{1}{2}, \frac{1}{2}, \dots \right) \frac{z^k}{k!} = \left(\frac{1 + e^z}{2} \right)^x. \tag{22}$$

References

[1] GELFAND A O. *Finite Difference Computation* [M]. Higher Educational Publishing House, 1955. (in Chinese)
[2] JORDAN C. *Calculus of Finite Differences* [M]. Chelsea, New York, 1965.
[3] MILNE-THOMSON L. M. *The Calculus of Finite Differences* [M]. Macmillan and Co. Ltd., London 1933.
[4] COMTET L. *Advanced Combinatorics* [M]. Reidel, Dordrecht, 1974.
[5] WILF H S. *Generatingfunctionology* [M]. Academic Press, New York, 1990.
[6] HE Tianxiao, HSU L C, SHIUE P J S. et al. *A symbolic operator approach to several summation formulas for power series* [J]. J. Comput. Appl. Math., 2005, **177**(1): 17–33.
[7] WALSH J L. *Interpolation and Approximation by Rational Functions in the Complex Domain* [M]. American Mathematical Society, Providence, R.I. 1965.
[8] ANDREW G E, ASKEY R, ROY R. *Special Functions* [M]. Cambridge University Press, 2000.