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Indecomposable Torsion Modules over Dedekind Domains

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Abstract We give a direct proof to the classification of all indecomposable torsion modules over a Dedekind ring. As an application, we classify all indecomposable locally-nilpotent modules over the polynomial algebra with one variable over a field.

Keywords torison module; Dedekind ring; locally-nilpotent module.

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1. Main result

Let R be a commutative noetherian domain with identity, R-Mod the category of R-modules. Recall that an R-module M is a torsion module if for each element $m \in M$ there exists a nonzero element $a \in R$ such that $a.m = 0$, or equivalently, the localization of M with respect to the zero ideal is zero. Denote by R-Tor the subcategory of R-Mod consisting of torsion R-modules. Note that the subcategory R-Tor is closed under direct summands and arbitrary coproducts. We are interested in classifying indecomposable modules in R-Tor.

Set $\mathrm{Max}(R)$ to be the set of maximal ideals in R. For each $\mathfrak{m} \in \mathrm{Max}(R)$, consider the R-modules $R/\mathfrak{m}^n$ for $n \geq 1$, and $E(R/\mathfrak{m})$ the injective hull of $R/\mathfrak{m}$. Note that these modules are indecomposable and torsion (concerning the injective hull, consult [7, Theorem 18.4 (i) and (v)]). By considering their associated primes and then their lengths, we infer that these modules are pairwise non-isomorphic. We claim that they are all indecomposable torsion R-modules for Dedekind rings R. More precisely, we have

Theorem 1 *Let R be a Dedekind domain. Then the set of R-modules*

$$\{R/\mathfrak{m}^n, E(R/\mathfrak{m}) \mid n \geq 1,\ \mathfrak{m} \in \mathrm{Max}(R)\}$$

is a complete set of pairwise non-isomorphic indecomposable torsion R-modules.

In case $R = \mathbb{Z}$, the ring of integers, the result is known, see for example [5, Theorem 10]. We would like to mention that the method used in [5] works for any principal ideal rings, although no one had ever written down the details explicitly. In this article, we will give a simpler and more direct proof to this result for any discrete valuation rings and apply it to Dedekind domains.

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We apply Theorem 1 to the study of locally-nilpotent modules. Let R be a commutative ring and I an ideal of R. An R-module is said to be I-locally-nilpotent, if for each $m \in M$ there exists $n \geq 0$ such that $I^n.m = 0$. Denote by (R, I)-Lnp the subcategory of R-Mod consisting of I-locally-nilpotent modules. Note that for a domain R and a nonzero ideal I we have (R, I)-Lnp $\subseteq R$-Tor.

Corollary 1 *Let R be a commutative noetherian domain. Assume that $\mathfrak{m}$ is a nonzero maximal ideal which is principle. Then the set of R-modules $\{R/\mathfrak{m}^n, E(R/\mathfrak{m}) \mid n \geq 1)\}$ is a complete set of pairwise non-isomorphic indecomposable $\mathfrak{m}$-locally-nilpotent modules.*

Note that $R = k[x]$ the polynomial algebra in one variable over a field k and $\mathfrak{m} = (x)$ satisfy the conditions above. Thus Corollary 1 classifies all indecomposable locally-nilpotent $k[x]$-modules.

Let us remark that for commutative non-artinian rings R, the structures of (certain classes of) indecomposable R-modules are far from being known, see for example $[1, 4, 6]$. Hopefully our method makes a little contribution in this direction.

2. The Proofs

Let R be a discrete valuation ring (DVR), $\mathfrak{m}$ its maximal ideal. Recall that the ideal $\mathfrak{m}$ is principle, and thus we assume that $\mathfrak{m} = (t)$ is generated by a nonzero element t.

Recall two well-known facts.

Lemma 1 *Let R be a DVR with maximal ideal $\mathfrak{m} = (t)$, M an R-modules. Then $M \simeq E(R/\mathfrak{m})$ if and only if there exists a set $\{e_i \mid i = 0, 1, 2, \ldots\}$ of nonzero elements in M such that $M = \sum_{i \geq 0} Re_i$, $t.e_0 = 0$ and $t.e_i = e_{i-1}$ for $i \geq 1$.*

Proof Denote by $Q(R)$ the fraction field of R. It is well-known that $E(R/\mathfrak{m}) \simeq Q(R)/R$. To see this, recall that as an R-module, $Q(R)$ is the injective hull of R. Note that the DVR R has global dimension one, hence quotient modules of injective modules are injective. Consequently the quotient module $Q(R)/R$ is injective. Note that socle of $Q(R)/R$ is isomorphic to $R/\mathfrak{m}$. This will force that $E(R/\mathfrak{m}) \simeq Q(R)/R$.

The "only if" part follows immediately by setting $e_i = 1/t^{i+1} + R$, the residue class of $1/t^{i+1}$ in $Q(R)/R$, for each i. To see the "if" part, note that a nonzero element in $Q(R)/R$ is uniquely written as $a/t^r + R$ for some $r \geq 1$, $a \in R \backslash \mathfrak{m}$. We define a map $\theta : Q(R)/R \longrightarrow M$ by $\theta(0) = 0$ and $\theta(a/t^r + R) = a.e_{r-1}$, for $a \in R \backslash \mathfrak{m}$ and $r \geq 1$. One checks directly that this is a homomorphism of R-modules, and it is bijective. Thus we are done. $\square$

Lemma 2 *Let R be a DVR with maximal ideal $\mathfrak{m} = (t)$. Then for each $n \geq 0$, the quotient ring $R/\mathfrak{m}^n$ is self-injective and each $R/\mathfrak{m}^n$-module M has a decomposition of modules $M \simeq \bigoplus_{i=1}^n (R/\mathfrak{m}^i)^{(\Lambda_i)}$, where each Λ_i is an index set.*

Proof Note that the ring R is Gorenstein of dimension 1 and thus by [7, Theorem 213] the quotient ring $R/\mathfrak{m}^n$ is Gorenstein of dimension 0, that is, self-injective. It is also well known that the set $\{R/\mathfrak{m}^i \mid 1 \leq i \leq n\}$ is a complete set of pairwise non-isomorphic indecomposable finitely generated $R/\mathfrak{m}^n$-modules (view $R/\mathfrak{m}^n$-modules as R-modules, and then use the fundamental theorem of finitely generated modules over principle ideal domains). Then any modules have the decomposition by a general result of Auslander [3, Corollary 4.8] and Ringel-Tachikawa [8, Corollary 4.4]. For completeness we include a direct argument.

Let M be an $R/\mathfrak{m}^n$-module. Consider the submodule $K = \{m \in M \mid t^{n-1}.m = 0\}$ and the quotient module M/K. Since t acts trivially on M/K, the module M/K is a semisimple module. So we have a decomposition $M/K \simeq \bigoplus_{i\in\Lambda} R\bar{e}_i$ where $e_i \in M$ and $\bar{e}_i$ denote the residue classes, each component $R\bar{e}_i$ is isomorphic to $R/\mathfrak{m}$ and Λ is an index set (maybe empty). Then it follows that $M = \sum_{i\in\Lambda} Re_i + K$.

We claim that $\sum_{i\in\Lambda} Re_i = \bigoplus_{i\in\Lambda} Re_i$ and $Re_i \simeq R/\mathfrak{m}^n$. To see this, it suffices to show that any equation $\sum_{i\in\Lambda} a_i.e_i = 0$ (with only finite sum) will imply that each $a_i = 0$. Take $0 \leq l < n$ such that $a_i = b_i t^l$ and at least one of the b_i's is invertible. Hence $\sum_{i\in\Lambda} b_i.e_i \in K$, this means in the quotient module M/K we have $\sum_{i\in\Lambda} b_i.\bar{e}_i = 0$. This will force that each $b_i.\bar{e}_i = 0$ and thus $b_i \in \mathfrak{m}$. A contradiction.

Since $R/\mathfrak{m}^n$ is self-injective, by the claim we infer that the submodule $\sum_{i\in\Lambda} Re_i$ is injective. Hence it is a direct summand of M, say we have $M = (\sum_{i\in\Lambda} Re_i) \oplus M' \simeq (R/\mathfrak{m}^n)^{(\Lambda)} \oplus M'$. Note that the composite $K \longrightarrow M \longrightarrow M'$ is epic, where the left morphism is the inclusion map and the right the canonical projection. Note that t^{n-1} acts trivially on K, and thus also on M'. Therefore, M' can be viewed as an $R/\mathfrak{m}^{n-1}$-module. By the same argument, we can decompose the $R/\mathfrak{m}^{n-1}$-module M', and then we are done by induction. □

For an R-module M, denote by $t_M : M \longrightarrow M$ the homomorphism given by $t_M(m) = t.m$. Note that the module M is torsion if and only if the homomorphism t_M is locally-nilpotent on M, that is, for each $m \in M$ there is $n \geq 1$ such that $t_M^n(m) = 0$.

The following lemma is of interest.

Lemma 3 *Let R be a DVR with maximal ideal $\mathfrak{m} = (t)$. Let M be an R-module satisfying* $\mathrm{Ker}t_M \subseteq \mathrm{Im}t_M^l$ *for some $l \geq 0$. Then as an R-module, we have an isomorphism* $\mathrm{Ker}t_M^{l+1} \simeq (R/\mathfrak{m}^{l+1})^{(\Lambda)}$ *for some index set Λ.*

Proof Set $X = \mathrm{Ker}t_M^{l+1}$. Since t^{l+1} acts trivially on X, the module X can be viewed an $R/\mathfrak{m}^{l+1}$-module. By Lemma 2 we have a decomposition $X \simeq (R/\mathfrak{m}^{l+1})^{(\Lambda)} \oplus \bigoplus_{i=1}^{l}(R/\mathfrak{m}^i)^{(\Lambda_i)}$. If $l = 0$ we are done. Otherwise we will show that for each $i \leq l$ the set Λ_i is empty. If not, we may find an element $e \in X$ such that $t_M^l(e) = 0$ and $e \notin t_M(X)$.

We claim that for each $0 \leq i \leq l$, there exists an element $e_i \in X$ such that $t_M^i(e_i) = 0$ and $e_i \notin t_M(X)$. The case $i = l$ is known by taking $e_l = e$. Use induction on $l - i$. Assume that e_i is chosen. Then $t_M^{i-1}(e_i) \in \mathrm{Ker}t_M \subseteq \mathrm{Im}t_M^l$. Thus there exits $y \in M$ such that $t_M^{i-1}(e_i) = t_M^l(y)$. Note that $0 = t_M^i(e_i) = t_M^{l+1}(y)$ and thus $y \in \mathrm{Ker}t_M^{l+1}$. Set $e_{i-1} = e_i - t_M(y)$, and observe that $t_M^{i-1}(e_{i-1}) = 0$ and $e_{i-1} \notin t_M(X)$. Thus we are done with the claim. By considering the case $i = 0$ of the claim, we see the contradiction. □

The following lemma seems to be very technical.

Lemma 4 *Let R be a DVR with maximal ideal $\mathfrak{m} = (t)$. Set $U = (R\backslash\mathfrak{m}) \cup \{0\}$. Let M be an R-module satisfying $\mathrm{Ker}t_M \subseteq \mathrm{Im}t_M^l$ for some $l \geq 0$. Assume that V is a subset of M satisfying*

(a) $0 \in V$ and $V \subseteq M$ is closed under actions by elements in U;

(b) $V \cap (\mathrm{Im}t_M + \mathrm{Ker}t_M^{l+1}) = \{0\}$ and $M = R.V + (\mathrm{Im}t_M + \mathrm{Ker}t_M^{l+1})$, where $R.V$ denotes the submodule generated by V.

Set $M' = R.V + \mathrm{Im}t_M^{l+1}$ and $M'' = M' \cap \mathrm{Ker}t_M^{l+1}$. Then we have

(1) $M = M' + \mathrm{Ker}t_M^{l+1}$;

(2) $t_M^i(V) \cap \mathrm{Im}t_M^{i+1} = \{0\}$ for $0 \leq i \leq l$;

(3) $M' \cap \mathrm{Ker}t_M \subseteq \mathrm{Im}t_M^{l+1}$;

(4) The submodule M'' also satisfies $\mathrm{Ker}t_{M''} \subseteq \mathrm{Im}t_{M''}^l$.

Proof (1) It suffices to show that for each $i \geq 1$, $M = R.V + \mathrm{Im}t_M^i + \mathrm{Ker}t_M^{l+1}$. The case $i = 1$ is clear. Use induction on i. Assume that we have $M = R.V + \mathrm{Im}t_M^i + \mathrm{Ker}t_M^{l+1}$. Then we have

$$\begin{aligned} M &= R.V + t_M(M) + \mathrm{Ker}t_M^{l+1} = R.V + t_M(R.V + \mathrm{Im}t_M^i + \mathrm{Ker}t_M^{l+1}) + \mathrm{Ker}t_M^{l+1} \\ &= R.V + t_M(\mathrm{Im}t_M^i) + \mathrm{Ker}t_M^{l+1} = R.V + \mathrm{Im}t_M^{i+1} + \mathrm{Ker}t_M^{l+1}. \end{aligned}$$

Thus we are done.

(2) Use induction on i to show $t_M^i(V) \cap \mathrm{Im}t_M^{i+1} = \{0\}$ for $0 \leq i \leq l$. The case $i = 0$ is clear. Assume that $t_M^i(V) \cap \mathrm{Im}t_M^{i+1} = \{0\}$ for $i < l$. Consider $m \in t_M^{i+1}(V) \cap \mathrm{Im}t_M^{i+2}$. Hence $m = t_M^{i+1}(v) = t^{i+2}(m')$ with $v \in V$ and $m' \in M$. Then $t_M^i(v) - t_M^{i+1}(m') \in \mathrm{Ker}t_M \subseteq \mathrm{Im}t_M^l \subseteq \mathrm{Im}t_M^{i+1}$ (note that $i + 1 \leq l$). Hence $t_M^i(v) \in \mathrm{Im}t_M^{i+1} \cap t_M^i(V)$. By induction we have $t_M^i(v) = 0$ and thus $m = 0$. Hence we get $t_M^{i+1}(V) \cap \mathrm{Im}t_M^{i+2} = \{0\}$.

(3) First note that each element in R can be written as $\sum_{i=0}^l c_i t^i + t^{l+1}a$ for some $c_i \in U$ and $a \in R$, and note that the subset $U \subseteq R$ is closed under multiplication and V is closed under actions by elements in U. From these facts one infers that any element m' in M' may be written as $m' = \sum_{i=0}^l t_M^i(v_i) + t_M^{l+1}(m)$ for some $v_i \in V$ and $m \in M$. Consider $m' \in M' \cap \mathrm{Ker}t_M$. Since $\mathrm{Ker}t_M \subseteq \mathrm{Im}t_M^l$, we have $m' \in \mathrm{Im}t_M^l$. This forces that $v_0 \in V \cap \mathrm{Im}t_M$ and thus by (2) we have $v_0 = 0$. Hence we have $m' = \sum_{i=1}^l t_M^i(v_i) + t_M^{l+1}(m)$ and use the same argument we get $t_M^1(v_1) = 0$ and then by induction we deduce that $t_M^i(v_i) = 0$ for all $0 \leq i \leq l-1$. Thus $m' = t_M^l(v_l) + t_M^{l+1}(m)$. Since $t_M(m') = 0$, we get $v_l + t_M(m) \in \mathrm{Ker}t_M^{l+1}$, and this forces $v_l \in V \cap (\mathrm{Im}t_M + \mathrm{Ker}t_M^{l+1}) = \{0\}$. Hence $m' = t_M^{l+1}(m)$, and this shows (3).

(4) It suffices to show that $M' \cap \mathrm{Ker}t_M \subseteq t_M^l(M' \cap \mathrm{Ker}t_M^{l+1})$. By (3), we have $M' \cap \mathrm{Ker}t_M \subseteq t_M^{l+1}(M) = t_M^{l+1}(M' + \mathrm{Ker}t_M^{l+1}) = t_M^{l+1}(M')$ while the first equality uses (1). Consider $m' \in M' \cap \mathrm{Ker}t_M$. Thus $m' = t_M^{l+1}(m)$ for some $m \in M'$. Note that $t_M(m') = 0$, and this forces $t_M(m) \in M' \cap \mathrm{Ker}t_M^{l+1}$. Hence $m' \in t_M^l(M' \cap \mathrm{Ker}t_M^{l+1})$. We are done. □

Now we are in the position to show the local version of our main theorem. Recall that a local Dedekind domain is nothing but a DVR.

Proposition 1 *Let R be a DVR with maximal ideal $\mathfrak{m} = (t)$. Then the set of R-modules $\{R/\mathfrak{m}^n, E(R/\mathfrak{m}) \mid n \geq 1)\}$ is a complete set of pairwise non-isomorphic indecomposable torsion R-modules.*

Proof As we pointed out in Section 1 that the listed modules are indecomposable and torsion and they are pairwise non-isomorphic. So it suffices to show any indecomposable torsion R-module is isomorphic to one in the list. Otherwise, choose an R-module M such that it is indecomposable, torsion and isomorphic to none of the modules in the list.

First we claim that $\mathrm{Ker} t_M \nsubseteq \bigcap_{i\geq 0} \mathrm{Im} t_M^i$. Otherwise, we show by induction that for each $n \geq 1$, $\mathrm{Ker} t_M^n \subseteq \mathrm{Im} t_M$. For this, the case $n=1$ is clear. Assume that $\mathrm{Ker} t_M^n \subseteq \mathrm{Im} t_M$. Consider $m \in \mathrm{Ker} t_M^{n+1}$. Thus $t_M^n(m) \in \mathrm{Ker} t_M \subseteq \mathrm{Im} t_M^{n+1}$. That is, there exists $y \in M$ such that $t_M^n(m) = t_M^{n+1}(y)$ and thus $m - t(y) \in \mathrm{Ker} t_M^n \subseteq \mathrm{Im} t_M$ by induction. This forces that $m \in \mathrm{Im} t_M$. This proves that $\mathrm{Ker} t_M^{n+1} \subseteq \mathrm{Im} t_M$. Now since M is a torsion module and equivalently t_M is locally-nilpotent, we have $M = \bigcup_{n\geq 1} \mathrm{Ker} t_M^n \subseteq \mathrm{Im} t_M$, that is, t_M is surjective. However this is impossible. In fact, this allows us to find a set of nonzero elements $\{e_i \mid i \geq 0\}$ in M such that $t_M(e_0) = 0$ and $t_M(e_i) = e_{i-1}$ for $i \geq 1$. However by Lemma 1 the submodule E generated by $\{e_i \mid i \geq 0\}$ is isomorphic to the injective module $E(R/\mathfrak{m})$. Hence the submodule E is a direct summand of M and this forces that $M \simeq E(R/\mathfrak{m})$. A contradiction to the choice of M.

By the claim above, we may choose $l \geq 0$ such that $\mathrm{Ker} t_M \subseteq \mathrm{Im} t_M^l$ and $\mathrm{Ker} t_M \nsubseteq \mathrm{Im} t_M^{l+1}$. Note that we may choose a subset $V \subseteq M$ satisfying the conditions in Lemma 4 : in fact, consider the quotient module $M/(\mathrm{Im} t_M + \mathrm{Ker} t_M^{l+1})$ and note that it is a semisimple module; take a decomposition $M/(\mathrm{Im} t_M + \mathrm{Ker} t_M^{l+1}) = \bigoplus_{i\in\Lambda} R\bar{v}_i$ where Λ is a set, $v_i \in M$ and $\bar{v}_i = v_i + (\mathrm{Im} t_M + \mathrm{Ker} t_M^{l+1})$ the residue class, and each component $R\bar{v}_i$ is a simple module; take

$$V := \{\sum_{i\in\Lambda} u_i.v_i \mid u_i \in U, u_i\text{'s are zero but finitely many } i\text{'s}\}.$$

It is direct to see that this subset V satisfies the conditions (a) and (b).

We are now able to apply Lemma 4. By Lemma 4(3) we infer from $\mathrm{Ker} t_M \nsubseteq \mathrm{Im} t_M^{l+1}$ that M' is a proper submodule of M. Note that $M'' = \mathrm{Ker} t_{M''}^{l+1}$. We may apply Lemma 3 to M'', and we deduce that M'' is direct sum of copies of $R/\mathfrak{m}^{l+1}$ and thus by Lemma 2 the module M'' is an injective $R/\mathfrak{m}^{l+1}$-module. Consider the inclusion $M'' \longrightarrow \mathrm{Ker} t_M^{l+1}$, both of which are viewed as $R/\mathfrak{m}^{l+1}$-modules, and therefore the inclusion map splits. Hence we have a decomposition $\mathrm{Ker} t_M^{l+1} = M'' \oplus H$ of modules. By Lemma 4(1) we get that $M = M' \oplus H$. Note that M is indecomposable and $M' \subseteq M$ is proper, we have $M' = 0$. By Lemma 4(1) again we infer that $M = \mathrm{Ker} t_M^{l+1}$ and by applying Lemma 3 to M we get that the indecomposable module M is isomorphic to $R/\mathfrak{m}^{l+1}$. This contradicts the choice of M and completes the proof. □

Theorem 1 follows immediately from Proposition 1 and the following well-known result.

Lemma 5 *Let R be a commutative noetherian domain of Krull dimension one, $M \in R$-*Tor *a torsion module. Then we have a decomposition $M = \bigoplus_{\mathfrak{m}\in \mathrm{Max}(R)} \{x \in M \mid \mathfrak{m}^n.x = 0 \text{ for some } n \geq 0\}$ of R-modules. Assume further that R is a Dedekind domain. Then each component in the direct sum is a torsion $R_\mathfrak{m}$-module.*

Proof Note that every nonzero component $\{x \in M \mid \mathfrak{m}^n.x = 0 \text{ for some } n \geq 0\}$ has associated prime $\{\mathfrak{m}\}$. Thus the sum on the right hand side is a direct sum by an argument on their associated primes (say, by [7, Theorem 6.3]). Now it suffices to show that $M = \sum_{\mathfrak{m}\in \mathrm{Max}(R)} \{x \in$

$M \mid \mathfrak{m}^n.x = 0$ for some $n \geq 0\}$. Take $m \in M$ and consider the submodule $N = R.m$ generated by m and denote by $\mathrm{Ann}(N)$ the annihilator ideal of N. Note that the quotient ring $R/\mathrm{Ann}(N)$ is artinian, since R is a domain of Krull dimension 1. By the structure theorem of artinian rings ([2, Theorem 8.7]) we have an isomorphism of rings $R/\mathrm{Ann}(N) \simeq R_1 \times \cdots \times R_s$ where each R_i is a local artinian ring with maximal ideal $\bar{\mathfrak{m}}_i = \mathfrak{m}_i/\mathrm{Ann}(N)$ for some $\mathfrak{m}_i \in \mathrm{Max}(R)$. We may view R_i as the subring $\{(a_1, \ldots, a_s) \in R_1 \times \cdots \times R_s \mid a_j = 0\ \forall j \neq i\}$ of R. Therefore we have a decomposition of modules $N = N_1 \oplus \cdots \oplus N_s$, where each $N_i = R_i N$ is an R_i-module. Since $\bar{\mathfrak{m}}_i$ is nilpotent, we infer that $N_i \subseteq \{x \in M \mid \mathfrak{m}_i^n.x = 0$ for some $n \geq 0\}$ and this forces that $N \subseteq \sum_{\mathfrak{m} \in \mathrm{Max}(R)} \{x \in M \mid \mathfrak{m}^n.x = 0$ for some $n \geq 0\}$. Thus we are done with the decomposition. Note that if R is a Dedekind domain, every localization $R_\mathfrak{m}$ is a DVR. Then the last statement follows from the following easy observation. □

Corollary 1 follows from Proposition 1 and the following easy observation. Just note that in Corollary 1, the localization ring $R_\mathfrak{m}$ is a DVR.

Lemma 6 *Let R be a commutative ring with a maximal ideal $\mathfrak{m}$. Then we have a natural identification of categories $(R, \mathfrak{m})$-Lnp $= (R_\mathfrak{m}, \mathfrak{m}R_\mathfrak{m})$-Lnp. Assume further that $R_\mathfrak{m}$ is a DVR. Then we have $(R, \mathfrak{m})$-Lnp $= R_\mathfrak{m}$-Tor.*

Proof Note that the category $R_\mathfrak{m}$-Mod of $R_\mathfrak{m}$-modules is identified as the subcategory of R-Mod consisting of modules on which the elements outside $\mathfrak{m}$ act invertibly. So to show the lemma, it suffices to show that every $\mathfrak{m}$-locally-nilpotent module lies in $R_\mathfrak{m}$-Mod. For this, let $a \notin \mathfrak{m}$ and $M \in (R, \mathfrak{m})$-Lnp. Since $\mathfrak{m}$ is a maximal ideal, we have $R = Ra + \mathfrak{m}$. In particular, $1 = a'a + t$ for some $a' \in R$ and $t \in \mathfrak{m}$. Since t acts on M locally-nilpotently, it is classical that $1 - t$ is invertible on M (its inverse is given by $\sum_{i \geq 0} t_M^i$). Therefore the action of a on M is invertible. For the last statement, just note that for the DVR $R_\mathfrak{m}$, locally-nilpotent modules coincide with torsion modules. □

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