Journal of Mathematical Research & Exposition
Jan., 2011, Vol. 31, No. 1, pp. 100–108
DOI:10.3770/j.issn:1000-341X.2011.01.011
Http://jmre.dlut.edu.cn

A Note on Property (ω)

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Abstract In this note we study the property (ω), a variant of Weyl's theorem introduced by Rakočevič, by means of the new spectrum. We establish for a bounded linear operator defined on a Banach space a necessary and sufficient condition for which both property (ω) and approximate Weyl's theorem hold. As a consequence of the main result, we study the property (ω) and approximate Weyl's theorem for a class of operators which we call the λ-weak-$H(p)$ operators.

Keywords approximate Weyl's theorem; property (ω); Browder operator.

Document code A
MR(2010) Subject Classification 47A53; 47A55; 47A15
Chinese Library Classification O177.2

1. Introduction

Weyl [18] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This "Weyl's theorem" has since been extended to hyponormal and to Toeplitz operators, to seminormal and other operators and to Banach spaces operators. Variants have been discussed by Harte and Lee [6] and Rakočevič [11, 12]. In this note we show how property (ω) and approximate Weyl's theorem (abbrev. a-Weyl's theorem) follow from the relation between Browder spectrum and a variant of the essential approximate point spectrum.

Throughout this paper, X denotes an infinite dimensional complex Banach space, $B(X)$ the algebra of all bounded linear operators on X. For an operator $T \in B(X)$ we shall denote by $n(T)$ the dimension of the kernel $N(T)$, and by $d(T)$ the codimension of the range $R(T)$. We call $T \in B(X)$ an upper semi-Fredholm operator if $n(T) < \infty$ and $R(T)$ is closed; But if $d(T) < \infty$ and $R(T)$ is closed, T is a lower semi-Fredholm operator. An operator $T \in B(X)$ is said to be Fredholm if $R(T)$ is closed and both the deficiency induces $n(T)$ and $d(T)$ are

Received August 10, 2009; Accepted April 27, 2010
Supported by the Fundamental Research Funds for the Central Universities (Grant No. GK200901015), the Support Plan of the New Century Talented Person of Ministry of Education (2006), P.R. China and by Major Subject Foundation of Shanxi.
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finite. If $T \in B(X)$ is an upper (or a lower) semi-Fredholm operator, the index of T, $\mathrm{ind}(T)$, is defined to be $\mathrm{ind}(T) = n(T) - d(T)$. The ascent of T, $\mathrm{asc}(T)$, is the least non-negative integer n such that $N(T^n) = N(T^{n+1})$ and the descent, $\mathrm{des}(T)$, is the least non-negative integer n such that $R(T^n) = R(T^{n+1})$. The operator T is Weyl if it is Fredholm of index zero, and T is said to be Browder if it is Fredholm "of finite ascent and descent". The upper semi-Fredholm spectrum $\sigma_{SF_+}(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined respectively by:

$$\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm}\},$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.$$

Let $\rho(T)$ denote the resolvent set of the operator T and $\sigma(T) = \mathbb{C} \backslash \rho(T)$ denote the usual spectrum of T. We use $\pi_{00}(T)$ to denote the set of isolated eigenvalues λ of T for which $\dim N(T - \lambda I) < \infty$. Also $\pi_{00}^a(T)$ is the set of $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \dim N(T - \lambda I) < \infty$, where $\sigma_a(T)$ denotes the approximate point set of the operator $T \in B(X)$. We say that the Browder's theorem holds for T (see [6]) if

$$\sigma_w(T) = \sigma_b(T),$$

the Weyl's theorem holds for T if

$$\sigma(T) \backslash \sigma_w(T) = \pi_{00}(T),$$

and the a-Weyl's theorem holds for T (see [11]) if

$$\sigma_a(T) \backslash \sigma_{ea}(T) = \pi_{00}^a(T),$$

where $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}$ and $SF_+^-(X) = \{T \in B(X),\ T$ is upper semi-Fredholm of $\mathrm{ind}(T) \leq 0\}$. The concept of a-Weyl's theorem was introduced by Rakočevič: a-Weyl's theorem for $T \Longrightarrow$ Weyl's theorem for T, but the converse is generally false [11].

Sufficient conditions for an operator $T \in B(X)$ to satisfy property (ω) and a-Weyl's theorem have been considered by a number of authors in the recent past [1, 2]. The rest of the paper is organized as follows. In Section 2, we prove our main results and give the necessary and sufficient conditions for T such that both property (ω) and a-Weyl's theorem hold. In Section 3, we show the property (ω) and a-Weyl's theorem for λ-weak-$H(p)$ operators.

2. Property (ω) and a-Weyl's theorem

The following variant of Weyl's theorem has been introduced by Rakočevič [12]

Definition 2.1 *$T \in B(X)$ is said to satisfy property (ω) if*

$$\sigma_a(T) \backslash \sigma_{ea}(T) = \pi_{00}(T).$$

Unlike a-Weyl's theorem, the study of property (ω) has been rather neglected, although, exactly like a-Weyl's theorem, property (ω) implies Weyl's theorem, a-Browder's theorem and

Browder's theorem. But what is the relation between a-Weyl's theorem and property (ω)?

Remark 2.2 (1) "a-Weyl's theorem" does not imply "property (ω)".

For example, $A \in B(\ell^2)$ is defined by

$$A(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, x_3, x_4, \ldots),$$

then

(I) $\sigma_a(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$, and $\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$;

(II) $\pi_{00}(A) = \emptyset$, and $\pi_{00}^a(A) = \{0\}$.

So $\sigma_a(A)\backslash\sigma_{ea}(A) = \pi_{00}^a(A)$, but $\sigma_a(A)\backslash\sigma_{ea}(A) \neq \pi_{00}(A)$, which means that a-Weyl's theorem holds for A, but property (ω) fails for A.

(2) "property (ω)" does not imply "a-Weyl's theorem".

For example, $B \in B(\ell^2)$ is defined by

$$B(x_1, x_2, x_3, \ldots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots),$$

and let $T = A \oplus B \in B(\ell^2 \oplus \ell^2)$, where $A \in B(\ell^2)$ is the operator defined in (1). Then

(I) $\sigma_a(T) = \sigma_{ea}(T) = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\}$;

(II) $\pi_{00}(T) = \emptyset$ and $\pi_{00}^a(T) = \{0\}$.

This implies that property (ω) holds for T but a-Weyl's theorem fails for T.

We hope that both property (ω) and a-Weyl's theorem hold for an operator T or property (ω) and a-Weyl's theorem are equivalent. We turn to a variant of the essential approximate point spectrum, involving a condition introduced by saphar [14] and the zero jump condition of Kato [7]. Recall that $T \in B(X)$ is a saphar operator iff $N(T) \subseteq \bigcap_{n=1}^{\infty} R(T^n)$, i.e., the kernel of T is contained in the hyper-range. We might describe the set of λ for which $T - \lambda I$ fails to be a saphar operator as the Saphar spectrum $\sigma_S(T)$ of T. If we also write $\sigma_G(T)$ for the Goldberg spectrum of T, collecting [5, Definition VI.7.1] $\lambda \in \mathbb{C}$ for which $T - \lambda I$ does not have closed range, then neither σ_G nor σ_S behaves well, while their union $\sigma_G \cup \sigma_S$ is a sort of Kate spectrum σ_k, enjoying most of the good spectral properties, such as spectral mapping theorem. The new spectrum set is defined as follows. Let

$$\rho_1(T) = \{\lambda \in \mathbb{C} : \ \dim N(T - \lambda I) < \infty \text{ and there exists } \epsilon > 0 \text{ such that } T - \mu I \in SF_+^-(X)$$
$$\text{and } N(T - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \mu I)^n] \text{ if } 0 < |\mu - \lambda| < \epsilon\}$$

and let $\sigma_1(T) = \mathbb{C}\backslash\rho_1(T)$. Then

$$\sigma_1(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_b(T) \subseteq \sigma(T).$$

We recall that an " a-isoloid " operator is one of the isolated points whose approximate point spectrum are all eigenvalues.

Theorem 2.3 *$\sigma_b(T) = \sigma_1(T) \cup [\sigma(T) \cap \rho_a(T)]$ if and only if T is a-isoloid and both property (ω) and a-weyl's theorem hold for T.*

Proof Suppose that $\sigma_b(T) = \sigma_1(T) \cup [\sigma(T) \cap \rho_a(T)]$.

By definition of $\rho_1(T)$, we know that $\sigma_a(T)\backslash\sigma_{ea}(T) \cup \pi_{00}(T) \subseteq \rho_1(T)\backslash\rho_a(T)$ and $\pi^a_{00}(T) \subseteq \rho_1(T)\backslash\rho_a(T)$, then $\sigma_a(T)\backslash\sigma_{ea}(T) \cup \pi_{00}(T) \subseteq \mathbb{C}\backslash\sigma_b(T)$ and $\pi^a_{00}(T) \subseteq \mathbb{C}\backslash\sigma_b(T)$, and hence both property (ω) and a-Weyl's theorem hold for T. Let $\lambda_0 \in \operatorname{iso}\sigma_a(T)$. If $N(T - \lambda_0 I) = \{0\}$, then $\lambda_0 \notin \sigma_1(T) \cup [\sigma(T) \cap \rho_a(T)]$, which means that $\lambda_0 \notin \sigma_b(T)$. This tells us that $T - \lambda_0 I$ is Browder, then $T - \lambda_0 I$ is invertible. It is in contradiction to the fact that $\lambda_0 \in \operatorname{iso}\sigma_a(T)$. We now have that $N(T - \lambda_0 I) \neq \{0\}$, which means that T is a-isoloid.

For the converse, we only need to prove that $\sigma_b(T) \subseteq \sigma_1(T) \cup [\sigma(T) \cap \rho_a(T)]$. Let $\lambda_0 \notin \sigma_1(T) \cup [\sigma(T) \cap \rho_a(T)]$. Then $n(T - \lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $T - \lambda I \in SF^-_+(X)$ and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda - \lambda_0| < \epsilon$. We claim that $T - \lambda I$ is bounded from below if $0 < |\lambda - \lambda_0| < \epsilon$. In fact, if $n(T - \lambda_1 I) > 0$ and $0 < |\lambda_1 - \lambda_0| < \epsilon$, then $\lambda_1 \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Since T satisfies property (ω), it follows that $\lambda_1 \in \pi_{00}(T)$. Then $T - \lambda_1 I$ is Browder, which means that $\operatorname{asc}(T - \lambda_1 I) < \infty$. From Lemma 3.4 in [17], we know that $N(T - \lambda_1 I) = N(T - \lambda_1 I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda_1 I)^n] = \{0\}$, it is a contradiction. Hence $T - \lambda I$ is bounded from below if $0 < |\lambda - \lambda_0| < \epsilon$, that is $\lambda_0 \in \operatorname{iso}\sigma_a(T) \cup \rho_a(T)$. If $\lambda_0 \in \operatorname{iso}\sigma_a(T)$, then $\lambda_0 \in \pi^a_{00}(T)$, since T is a-isoloid. Since T has the property (ω) and a-Weyl's theorem holds for T, it follows that $T - \lambda_0 I$ is Browder. This induces that $\lambda_0 \notin \sigma_b(T)$. If $\lambda_0 \in \rho_a(T)$, from the fact $\lambda_0 \notin [\sigma(T) \cap \rho_a(T)]$, $T - \lambda_0 I$ is invertible. Also, $\lambda_0 \notin \sigma_b(T)$. This proves that $\sigma_b(T) = \sigma_1(T) \cup [\sigma(T) \cap \rho_a(T)]$.

$T \in B(X)$ is called left Drazin invertible if $\operatorname{asc}(T) < \infty$ and $R(T^{\operatorname{asc}(T)+1})$ is closed. The left Drazin spectrum $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}$. If $T - \lambda I$ is left Drazin invertible, λ is called the left pole point of T. T is called a-polaroid if all the isolated points of $\sigma_a(T)$ are left poles. a-polaroid operators are a-isoloid. □

Corollary 2.4 *Suppose that T is a-polaroid, then the following statements are equivalent:*

(1) $\sigma_b(T) = \sigma_1(T) \cup [\sigma(T) \cap \rho_a(T)]$;

(2) Property (ω) holds for T;

(3) $\pi_{00}(T) = \pi^a_{00}(T)$ *and a-Weyl's theorem holds for T.*

Proof (1)⇒(2). See Theorem 2.3.

(2)⇒(1). From Theorem 2.3, we need to prove that a-Weyl's theorem holds for T. Since property (ω) holds for T, it follows that $\sigma_a(T)\backslash\sigma_{ea}(T) \subseteq \pi^a_{00}(T)$. Let $\lambda_0 \in \pi^a_{00}(T)$. Then $T - \lambda_0 I$ is left Drazin invertible since T is a-polaroid. Therefore $R[(T - \lambda_0 I)^n]$ is closed for some integer n. This induces that $(T - \lambda_0 I)^n$ is upper semi-Fredholm, and hence $T - \lambda_0 I$ is upper semi-Fredholm. Since $\lambda_0 \in \pi^a_{00}(T)$, it follows that $\operatorname{ind}(T - \lambda_0 I) \leq 0$, which means that $\lambda_0 \in \sigma_a(T)\backslash\sigma_{ea}(T)$. This proves that generalized a-Weyl's theorem holds for T.

(1)⇔ (3). See Theorem 2.3. □

Remark 2.5 (1) The condition "T is a-isoloid" is essential in Theorem 2.3.

For example, $T \in B(\ell^2)$ is defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots),$$

then

$$\sigma_a(T)=\sigma_{ea}(T)=\sigma_b(T)=\{0\}, \pi_{00}(T)=\pi_{00}^a(T), \text{and } \sigma_1(T)\cup[\sigma(T)\cap\rho_a(T)]=\emptyset,$$

which says that both property (ω) and a-Weyl's theorem hold for T while T is not a-isoloid. But $\sigma_b(T)\neq\sigma_1(T)\cup[\sigma(T)\cap\rho_a(T)]$.

(2) The condition "T satisfies property (ω)" is essential in Theorem 2.3.

For example, let T be the operator A defined in Remark 2.2. Then T is a-isoloid and a-Weyl's theorem holds for T while T has not property (ω). Also a straightforward calculation shows that

$$\sigma_1(T)\cup[\sigma(T)\cap\rho(T)]=\{\lambda\in\mathbb{C}:0<|\lambda|\leq 1\},\quad \sigma_b(T)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\},$$

which implies that $\sigma_b(T)\neq\sigma_1(T)\cup[\sigma(T)\cap\rho_a(T)]$.

(3) The condition "a-Weyl's theorem holds for T" is essential in Theorem 2.3.

Let T be defined as operator T in Remark 2.2 (2). Then T is a-isoloid and satisfies property (ω) while a-Weyl's theorem fails for T. But $\sigma_b(T)\neq\sigma_1(T)\cup[\sigma(T)\cap\rho_a(T)]$ since $\sigma_1(T)\cup[\sigma(T)\cap\rho_a(T)]=\{\lambda\in\mathbb{C}:0<|\lambda|\leq 1\}$ and $\sigma_b(T)=\{\lambda\in\mathbb{C}:|\lambda|\leq 1\}$.

In the following, let $H(T)$ be the class of all complex-valued functions which are analytic on a neighborhood of $\sigma(T)$ and are not constant on any component of $\sigma(T)$.

Theorem 2.6 *If $T\in B(X)$, then*

$$\text{ind}(T-\lambda I)\cdot\text{ind}(T-\mu I)\geq 0 \textit{ for each pair } \lambda,\ \mu\in\mathbb{C}\backslash\sigma_{SF_+}(T)$$

if and only if

$$f(\sigma_1(T))\subseteq\sigma_1(f(T)) \textit{ for any } f\in H(T).$$

Proof Suppose that $\text{ind}(T-\lambda I)\cdot\text{ind}(T-\mu I)\geq 0$ for each pair $\lambda,\ \mu\in\mathbb{C}\backslash\sigma_{SF_+}(T)$.

For any $f\in H(T)$, let $\mu_0\notin\sigma_1(f(T))$. Then $\dim N(f(T)-\mu_0 I)<\infty$ and there exists $\epsilon>0$ such that $f(T)-\mu I\in SF_+^-(X)$ and $N(f(T)-\mu I)\subseteq\bigcap_{n=1}^{\infty}R[(f(T)-\mu I)^n]$ if $0<|\mu-\mu_0|<\epsilon$. Therefore μ is not in $\sigma_k(f(T))=f(\sigma_k(T))$ (see [16, Satz 6]) if $0<|\mu-\mu_0|<\epsilon$, where $\sigma_k(T)=\sigma_G(T)\cup\sigma_S(T)$.

Let $f(T)-\mu_0 I=(T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_k I)^{n_k}g(T)$, where $\lambda_i\neq\lambda_j$ and $g(T)$ is invertible. Since $N[f(T)-\mu_0 I]\supseteq N[(T-\lambda_i I)^{n_i}]$ and $n(f(T)-\mu_0 I)<\infty$, it follows that $n(T-\lambda_i I)<\infty$ for every λ_i, $1\leq i\leq k$. In what follows, we will prove that $\lambda_i\notin\sigma_1(T)$ for all $1\leq i\leq k$. By continuity of $f(\lambda)$ and the fact that the solutions of equation $f(\lambda)=f(\lambda_i)=\mu_0$ are finite, for every λ_i, there exists $\delta_i>0$ such that $0<|f(\lambda)-f(\lambda_i)|=|f(\lambda)-\mu_0|<\epsilon$ if $0<|\lambda-\lambda_i|<\delta_i$. Then $f(T)-f(\lambda)I\in SF_+^-(X)$ and $f(\lambda)$ is not in $\sigma_k(f(T))=f(\sigma_k(T))$, which means that $\lambda\notin\sigma_k(T)$. For any λ such that $0<|\lambda-\lambda_i|<\delta_i$, let $f(T)-f(\lambda)I=(T-\lambda I)^{m_\lambda}(T-\lambda_1' I)^{m_1}(T-\lambda_2' I)^{m_2}\cdots(T-\lambda_t' I)^{m_t}h(T)$, where $\lambda_i'\neq\lambda_j'(i\neq j)$, $\lambda_i'\neq\lambda$, and $h(T)$ is invertible. Since $f(T)-f(\lambda)I\in SF_+^-(X)$, it follows that $T-\lambda_i' I$ and $T-\lambda I$ are upper semi-Fredholm operators for all λ_i, $i=1,2,\ldots,t$. Thus $\text{ind}[(T-\lambda I)^{m_\lambda}]+\sum_{i=1}^{t}\text{ind}[(T-\lambda_i' I)^{m_i}]\leq 0$. So by condition, $\text{ind}(T-\lambda I)\leq 0$. We get that $T-\lambda I\in SF_+^-(X)$. Now we have proved that: $\dim N(T-\lambda_i I)<\infty$ and there exists $\delta_i>0$ such that $T-\lambda I\in SF_+^-(X)$ and $N(T-\lambda I)\subseteq\bigcap_{n=1}^{\infty}R[(T-\lambda I)^n]$ if $0<|\lambda-\lambda_i|<\delta_i$. Then $\lambda_i\notin\sigma_1(T)$, and hence $\mu_0\notin f(\sigma_1(T))$.

If conversely there exist $\lambda_0,\, \mu_0 \in \mathbb{C}\backslash\sigma_{SF_+}(T)$ for which

$$\mathrm{ind}(T-\lambda_0 I) = -m < 0 < k = \mathrm{ind}(T-\mu_0 I),$$

let $f(T) = (T-\lambda_0 I)^k\ (\lambda-\mu_0)^m$ if k is finite or else let $f(T) = (T-\lambda_0)(T-\mu_0 I)$. Then $0 \notin \sigma_1(f(T))$, and hence $0 \notin f(\sigma_1(T))$. This implies that $\lambda_0 \notin \sigma_1(T)$ and $\mu_0 \notin \sigma_1(T)$. But by perturbation theorem of upper semi-Fredholm operator, we know that $\lambda_0 \in \sigma_1(T)$. It is a contradiction. $\square$

We can prove that if $\sigma_b(T) = \sigma_1(T)\cup[\sigma(T)\cap\rho_a(T)]$, then $\mathrm{ind}(T-\lambda I)\mathrm{ind}(T-\mu I) \geq 0$ for each pair $\lambda,\mu \in \mathbb{C}\backslash\sigma_{SF_+}(T)$ if and only if $f(\sigma_1(T)) = \sigma_1(f(T))$ for any $f \in H(T)$. If $\sigma_b(T) = \sigma_1(T)$, then for any $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\mathrm{ind}(T-\lambda I) \geq 0$. In fact, If there exists $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$ such that $\mathrm{ind}(T-\lambda I) < 0$, then $\lambda \in \sigma_1(T)$ and hence $T-\lambda I$ is Browder. This means that $\mathrm{ind}(T-\lambda I) = 0$, leading to a contradiction. Thus if $\sigma_b(T) = \sigma_1(T)$, we have that $f(\sigma_1(T)) = \sigma_1(f(T))$ for any $f \in H(T)$. In this case, we have that $\sigma_b(f(T)) = f(\sigma_b(T)) = f(\sigma_1(T)) = \sigma_1(f(T))$. Using Theorem 2.3, we know that for any $f \in H(T)$, $f(T)$ is a-isoloid and both property (ω) and a-Weyl's theorem hold for $f(T)$. That is:

Theorem 2.7 *Suppose $\sigma_b(T) = \sigma_1(T)$, then*

(1) $f(\sigma_1(T) = \sigma_1(f(T))$ for any $f \in H(T)$;

(2) For any $f \in H(T)$, $f(T)$ is a-isoloid and both property (ω) and a-Weyl's theorem hold for $f(T)$.

Example 2.8 Let $T \in B(\ell^2)$ be defined by

$$T(x_1,x_2,x_3,\ldots) = (x_2,x_3,x_4,\ldots).$$

Then $\sigma_b(T) = \sigma_1(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Using Theorem 2.7, we see that for any $f \in H(T)$, $f(T)$ is a-isoloid and both property (ω) and a-Weyl's theorem hold for $f(T)$.

Theorem 2.9 *Let $\sigma_b(T) = \sigma_1(T)$ and F be a finite rank operator that commutes with T. Then $T+F$ is a-isoloid and satisfies property (ω) and a-Weyl's theorem.*

Proof Using Theorem 2.3, we need to prove that $\sigma_b(T+F) \subseteq \sigma_1(T+F)\cup[\sigma(T+F)\cap\rho_a(T+F)]$. Let $\lambda_0 \notin \sigma_1(T+F)\cup[\sigma(T+F)\cap\rho_a(T+F)]$. Then $n(T+F-\lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $T+F-\lambda I \in SF_+^-(X)$ and $N(T+F-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T+F-\lambda I)^n]$ if $0 < |\lambda-\lambda_0| < \epsilon$. Then $n(T-\lambda_0 I) < \infty$ and $T-\lambda I \in SF_+^-(X)$ if $0 < |\lambda-\lambda_0| < \epsilon$. Since a-Weyl's theorem holds for T, it follows that $\mathrm{asc}(T-\lambda I) < \infty$ if $0 < |\lambda-\lambda_0| < \epsilon$. Then $T+F-\lambda I$ has finite ascent [13], hence $T+F-\lambda I$ is bounded from below, which induces that $\lambda_0 \in \mathrm{iso}\,\sigma_a(T+F)\cup\rho_a(T+F)$. If $\lambda_0 \in \mathrm{iso}\,\sigma_a(T+F)$, then $\lambda_0 \in \mathrm{iso}\,\sigma_a(T)\cup\rho_a(T)$. Then $\lambda_0 \notin \sigma_1(T)$, which means that $T-\lambda_0 I$ is Browder. Therefore $T+F-\lambda_0 I$ is Browder, that is, $\lambda_0 \notin \sigma_b(T+F)$. If $\lambda_0 \in \rho_a(T+F)$, then $T+F-\lambda_0 I$ is invertible since $\lambda_0 \notin [\sigma(T+F)\cap\rho_a(T+F)]$. Thus $\sigma_b(T+F) \subseteq \sigma_1(T+F)\cup[\sigma(T+F)\cap\rho_a(T+F)]$, which means that $T+F$ is a-isoloid and satisfies property (ω) and a-Weyl's theorem. $\square$

3. Property (ω) and a-Weyl's theorem for λ-weak-$H(p)$ operators

In this section, X denotes an infinite dimensional Hilbert space. For operator $A \in B(X)$, the analytic core of A is the subspace

$$K(A) = \{x \in X : Ax_{n+1} = x_n, Ax_1 = x, \|x_n\| \leq c^n\|x\|(n = 1, 2, \ldots) \text{ for some } c > 0, x_n \in X\},$$

and the quasi-nilpotent part of A is the subspace

$$H_0(A) = \{x \in X : \lim_{n\to\infty} \|A^n x\|^{\frac{1}{n}} = 0\}.$$

The spaces $K(A)$ and $H_0(A)$ are hyperinvariant under A and satisfy $N(A^n) \subseteq H_0(A)$, $K(A) \subseteq R(A^n)$ for all $n \in \mathbb{N}$ and $AK(A) = K(A)$. We refer to the recent book of Aiena [3, 8, 9] for more information about these subspaces.

Now let us introduce the class $H(p)$ formed by the operators $T \in B(X)$ such that for every $\lambda \in \mathbb{C}$ there exists an integer d_λ for which $H_0(T-\lambda I) = N[(T-\lambda I)^{d_\lambda}]$. This class is considerably large since it contains every totally paranormal and subscalar operator, and consequently, every M-hyponormal, p-hyponormal and log-hyponormal operator. We know that if $T \in H(p)$, then Weyl's theorem holds for T and T^*, where T^* is the adjoint of T (see [10, Theorem 3.1]).

Definition 3.1 *Let $\lambda \in \mathbb{C}$. $T \in B(X)$ is called a λ-weak-$H(p)$ operator if for any $\mu \in \mathbb{C}$, there exists an integer d_μ for which $K(A-\lambda I) \cap H_0(A-\mu I) = N[(A-\mu I)^{d_\mu}]$. $A \in B(X)$ is called an analytic λ-weak-$H(P)$ operator if there exists some $g \in H(A)$ such that for any $\mu \in \mathbb{C}$, there is an integer $d_\mu \geq 1$ for which $K(A-\lambda I) \cap H_0(g(A)-\mu I) = N[(g(A)-\mu I)^{d_\mu}]$ holds.*

Let $\lambda \in \mathbb{C}\backslash\sigma(T)$. Then $K(T-\lambda I) = X$, and hence $H(P) \subseteq \lambda$-weak-$H(P)$. But if there exists $\lambda \in \mathbb{C}\backslash\sigma(T)$ such that T is a λ-weak-$H(P)$ operator, then $T \in H(P)$.

Example 3.2 $T \in B(\ell^2)$ is defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots),$$

then for any $\lambda \in \mathbb{C}$, $K(T) \cap H_0(T-\lambda I) = N(T-\lambda I) = \{0\}$, so A is a 0-weak-$H(P)$ operator. We can prove that Weyl's theorem holds for T but Weyl's theorem fails for T^* since $\sigma(T^*) = \sigma_w(T^*) = \{0\}$, $\pi_{00}(T^*) = \{0\}$.

From Example 3.2, strict inclusion can occur in $H(P) \subseteq \lambda$-weak-$H(P)$. Also we know that Weyl's theorem does not transfer from T to T^* for operators in λ-weak-$H(p)$.

Lemma 3.3 *Let $\lambda_0 \in \mathbb{C}$, and A be an analytic λ_0-weak-$H(P)$ operator. Then*

(1) For any $\lambda \neq \lambda_0$, $\operatorname{asc}(A-\lambda I) < \infty$;

(2) Let $\lambda \in \mathbb{C}$ such that $A-\lambda I$ is an upper semi-Fredholm operator. Then $\operatorname{asc}(A-\lambda I) < \infty$;

(3) Let $\lambda \in \operatorname{iso}\sigma(T)$ and $\lambda \neq \lambda_0$. Then there exists integer $d_\lambda \geq 1$ such that $H_0(T-\lambda I) = N[(T-\lambda I)^{d_\lambda}]$.

Proof Let $g \in H(T)$ satisfy that for any $\mu \in \mathbb{C}$, there exists integer $d_\mu \geq 1$ such that $K(A-\lambda_0 I) \cap H_0(g(A)-\mu I) = N[(g(A)-\mu I)^{d_\mu}]$.

(1) For any $\lambda \neq \lambda_0$, let $g(A)-g(\lambda)I = (A-\lambda I)^{n_\lambda}p(A)h(A)$, where p is a complex polynomial,

$p(\lambda) \neq 0$ and $h(A)$ is invertible. First we claim that $\forall m \in \mathbb{N}$, $N[(A-\lambda I)^m] \subseteq K(A-\lambda_0 I)$. In fact, suppose $x \in N[(A-\lambda I)^m]$, that is, $(A-\lambda I)^m x = 0$. There exists polynomial $f(\cdot)$ satisfying $(\lambda-\lambda_0)^m x = (A-\lambda_0 I)f(A)x$, $x = (A-\lambda_0 I)[\frac{f(A)}{(\lambda-\lambda_0)^m}x]$. Let $c = \|\frac{f(A)}{(\lambda-\lambda_0)^m}\| + 1$, $x_1 = \frac{f(A)}{(\lambda-\lambda_0)^m}x$, $x_n = [\frac{f(A)}{(\lambda-\lambda_0)^m}]^n x$, $\forall n \in \mathbb{N}$. Then $(A-\lambda_0 I)x_1 = x$, $(A-\lambda_0 I)x_{n+1} = x_n$, and $\|x_n\| \leq c^n\|x\|$, which means that $x \in K(A-\lambda_0 I)$. Let $K(A-\lambda_0 I) \cap H_0[g(A)-g(\lambda)I] = N[(g(A)-g(\lambda)I)^d]$, where $d \geq 1$ is integer. Since for any $m \in \mathbb{N}$, $N[(A-\lambda I)^m] \subseteq H_0[g(A)-g(\lambda)I]$, it follows that $N[(A-\lambda I)^m] \subseteq K(A-\lambda_0 I) \cap H_0[g(A)-g(\lambda)I] = N[(g(A)-g(\lambda)I)^d]$. Then $N[(A-\lambda I)^m] \subseteq N[(A-\lambda I)^{n_\lambda}] \oplus N[p(A)]$. Using the fact that $N[(A-\lambda I)^m] \cap N[p(A)] = \{0\}$, we know $N[(A-\lambda I)^m] \subseteq N[(A-\lambda I)^{n_\lambda}]$. Hence for any $\lambda \neq \lambda_0$, $\mathrm{asc}(A-\lambda I) < \infty$.

(2) Let $\lambda \in \mathbb{C}$ and $A-\lambda I$ be upper semi-Fredholm. Then there exists $\epsilon > 0$ such that $\mu \neq \lambda_0$, $A-\mu I$ is upper semi-Fredholm and $N(A-\mu I) \subseteq \bigcap_{n=1}^{\infty} R[(A-\mu I)^n]$ if $0 < |\mu-\lambda| < \epsilon$. This tells us that $\mathrm{asc}(A-\mu I) < \infty$, and hence $N(A-\mu I) = N(A-\mu I) \cap \bigcap_{n=1}^{\infty} R[(A-\mu I)^n] = \{0\}$ (see [17, Theorem 3.4]), which means that $T-\lambda I$ is bounded from below if $0 < |\mu-\lambda| < \epsilon$. Then $\lambda \in \mathrm{iso}\,\sigma_a(A)$, and hence $\mathrm{asc}(A-\lambda I) < \infty$ (see [4, Theorem 11]).

(3) Suppose $\lambda \in \mathrm{iso}\,\sigma(A)$ and $\lambda \neq \lambda_0$, then $X = H_0(A-\lambda I) \oplus K(A-\lambda I)$. Let $A_1 = A|_{H_0(A-\lambda I)}$, $A_2 = A|_{K(A-\lambda I)}$, where $\sigma(A_1) = \{\lambda\}$ and $\sigma(A_2) = \sigma(A)\backslash\{\lambda\}$. Thus $A_2 - \lambda_0 I$ is invertible, we must have that $(A_1-\lambda_0 I)H_0(A-\lambda I) = H_0(A-\lambda I)$. From Proposition 2 in [15], $H_0(A-\lambda I) \subseteq K(A_1-\lambda_0 I) \subseteq K(A-\lambda_0 I)$. Since $H_0(A-\lambda I) \subseteq H_0[g(A)-g(\lambda)I]$, there exists integer $d \geq 1$ such that $H_0(A-\lambda I) \subseteq N[(g(A)-g(\lambda)I)^d]$. Let $g(A)-g(\lambda)I = (A-\lambda I)^{n_\lambda}p(A)h(A)$, where p is polynomial, $p(\lambda) \neq 0$, and $h(A)$ is invertible. Then $H_0(A-\lambda I) \cap N(p(A)) = \{0\}$ (see [10, Lemma 3.5]). But since $N[(g(A)-g(\lambda)I)^d] = N[(A-\lambda I)^{dn_\lambda}] \oplus N[p(A)^d]$, it follows that $H_0(A-\lambda I) \subseteq N[(A-\lambda I)^{dn_\lambda}] \oplus N[p(A)^d]$. Then $H_0(A-\lambda I) = N[(A-\lambda I)^{dn_\lambda}]$. $\square$

Theorem 3.4 *Let T^* be a λ_0-weak-$H(p)$ operator for some λ_0 and $n(T^*-\lambda_0 I) > 0$. Then for any $f \in H(T)$, $f(T)$ is a-isoloid and both property (ω) and a-Weyl's theorem hold for $f(T)$.*

Proof Using Theorem 2.7, we only need to prove that $\sigma_b(T) = \sigma_1(T)$. The inclusion $\sigma_1(T) \subseteq \sigma_b(T)$ is clear. For the converse, let $\lambda \notin \sigma_1(T)$. Then $n(T-\lambda I) < \infty$ and there exists $\epsilon > 0$ such that $T-\mu I \in SF_+^-(X)$, $N(T-\mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\mu I)^n]$ and $\mu \neq \lambda_0$ if $0 < |\mu-\lambda| < \epsilon$. Then $T^*-\mu I$ is lower semi-Fredholm and $\mathrm{ind}(T^*-\mu I) \geq 0$. Since $\mathrm{asc}(T^*-\mu I) < \infty$ (Lemma 3.3), it follows that $T^*-\mu I$ is a Weyl operator. But since $\mathrm{asc}(T^*-\mu I) < \infty$ (Lemma 3.3), we have that $T^*-\mu I$ is Browder and hence $T-\mu I$ is Browder. Then $N(T-\mu I) = N(T-\mu I) \cap \bigcap_{n=1}^{\infty} R[(T-\mu I)^n] = \{0\}$, which means that $T-\mu I$ is invertible if $0 < |\mu-\lambda| < \epsilon$. This shows that $\lambda \in \mathrm{iso}\,\sigma(T) \cup \rho(T)$. Without loss of generality, we suppose that $\lambda \in \sigma(T)$. We claim that $\lambda \neq \lambda_0$. If not, let $\lambda = \lambda_0$. Then $\lambda \in iso\sigma(T^*)$ and hence T^* has the single valued extension property at λ. This means that $K(T^*-\lambda I) \cap H_0(T^*-\lambda I) = \{0\}$. From the Definition 3.1, $N(T^*-\lambda I) = K(T^*-\lambda I) \cap H_0(T^*-\lambda I) = \{0\}$. It is in contradiction to the fact that $n(T^*-\lambda_0 I) > 0$. Then $H_0(T^*-\lambda I) = N[(T^*-\lambda I)^{d_\lambda}]$ for some integer $d_\lambda \geq 1$ (Lemma 3.3). By the fact that $\lambda \in \mathrm{iso}\,\sigma(T^*)$, there is the decomposition $X = H_0(T^*-\lambda I) \oplus K(T^*-\lambda I) = N[(T^*-\lambda I)^{d_\lambda}] \oplus K(T^*-\lambda I)$ (see [15, Theorem 4]). Then λ is a pole of the resolvent of T^* (see [15, Theorem 5]), which means that λ is also a pole of the resolvent of T. Let

$\mathrm{asc}(T-\lambda I) = \mathrm{des}(T-\lambda I) = p$. Then $X = N[(T-\lambda I)^p] \oplus R[(T-\lambda I)^p]$. Since $n(T-\lambda I) < \infty$, it follows that $T-\lambda I$ is Browder. Then $\lambda_0 \notin \sigma_b(T)$. It is proved that $\sigma_b(T) = \sigma_1(T)$. $\square$

Acknowledgment We are grateful to the referees for their helpful comments on this paper.

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