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Tilting Bimodules from Tilting Pairs

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Abstract Tilting pair was introduced by Miyashita in 2001 as a generalization of tilting module. In this paper, we construct a tilting left $\mathrm{End}_\Lambda(C)$-right $\mathrm{End}_\Lambda(T)$-bimodule for a given tilting pairs (C, T) in $\mathrm{mod}\,\Lambda$, where Λ is an Artin algebra.

Keywords selforthogonal module; tilting bimodule; tilting pair.

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0. Introduction

The notion of a tilting module over an Artin algebra Λ was introduced by Brenner and Butler [1]. Tilting modules have been investigated by many authors since then. Tilting theory plays an important role in the representation theory of Artin algebra. Miyashita [2] introduced the notion of tilting pairs in constructing tilting modules with a left tilting series of ideals of an Artin algebra. It is a useful tool in the tilting theory.

In this paper, our aim is to investigate some properties of tilting pairs. For a given n-tilting pair (C, T) in $\mathrm{mod}\,\Lambda$, we obtain that $\mathrm{Hom}_\Lambda(C, T)$ is a tilting left $\mathrm{End}_\Lambda(C)$-right $\mathrm{End}_\Lambda(T)$-bimodule of projective dimension $\leq n$ on both sides.

1. Preliminaries

Throughout this paper, all algebras Λ are Artin algebras and $\mathrm{mod}\,\Lambda$ denotes the category of finitely generated left Λ-modules. We usually view a right Λ-module as a left Λ^{op}-module. By a subcategory of $\mathrm{mod}\,\Lambda$, we always mean a full subcategory closed under isomorphisms. For a Λ-module T, we denote by $\mathrm{add}\,T$ the subcategory of all direct summands of finite sum of copies of T.

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For a subcategory $\mathscr{C}$ of $\operatorname{mod}\Lambda$, we denote by $\hat{\mathscr{C}}$ the subcategory of $\operatorname{mod}\Lambda$ whose objects are the Λ-modules M for which there is a finite exact sequence $0 \to C_n \to \cdots \to C_0 \to M \to 0$ with $C_i \in \mathscr{C}$, and denote by $\dim_{\mathscr{C}}(M)$ the minimal integer n such that there is an exact sequence $0 \to C_n \to \cdots \to C_0 \to M \to 0$ with $C_i \in \mathscr{C}$, and $(\hat{\mathscr{C}})_n$ the category of all $M \in \hat{\mathscr{C}}$ with $\dim_{\mathscr{C}}(M) \le n$. Dually we denote by $\check{\mathscr{C}}$ the subcategory of $\operatorname{mod}\Lambda$ whose objects are the Λ-modules M which admit a finite exact sequence $0 \to M \to C_0 \to \cdots \to C_n \to 0$ with $C_i \in \mathscr{C}$. Similarly, $\operatorname{codim}_{\mathscr{C}}(M)$ and $(\check{\mathscr{C}})_n$ can be defined dually.

Now we recall the notion of tilting and cotilting modules in [3, 4]. A Λ-module T is called n-tilting if (1) $\operatorname{pd}_\Lambda T \le n$, (2) T is selforthogonal, i.e., $\operatorname{Ext}^i_\Lambda(T,T) = 0$ for all $i > 0$, and (3) there is a projective generator P such that $P \in (\check{\operatorname{add}}\, T)_n$. Dually, a Λ-module C is called n-cotilting if (1) $\operatorname{id}_\Lambda C \le n$, (2) C is selforthogonal, and (3) there is an injective cogenerator I such that $I \in (\hat{\operatorname{add}}\, C)_n$.

For a subcategory $\mathscr{C}$ (a module T), we define

$$^{\perp}\mathscr{C} = \bigcap_{i\ge1} \operatorname{KerExt}^i_\Lambda(-,\mathscr{C}) = \{M \in \operatorname{mod}\Lambda \,|\, \operatorname{Ext}^i_\Lambda(M,C) = 0 \text{ for all } C \in \mathscr{C} \text{ and } i \ge 1\};$$

$$\mathscr{C}^{\perp} = \bigcap_{i\ge1} \operatorname{KerExt}^i_\Lambda(\mathscr{C},-); \quad {}^{\perp}T = \bigcap_{i\ge1} \operatorname{KerExt}^i_\Lambda(-,T); \quad T^{\perp} = \bigcap_{i\ge1} \operatorname{KerExt}^i_\Lambda(T,-).$$

For a selforthogonal Λ-module T, we denote by ${}_T\mathcal{X}$ the subcategory of $T^{\perp}$ whose objects are the Λ-modules X such that there is an exact sequence $\cdots \to T_m \stackrel{f_m}{\to} T_{m-1} \to \cdots \to T_0 \stackrel{f_0}{\to} X \to 0$ with $T_i \in \operatorname{add} T$ and $\operatorname{Im} f_i \in T^{\perp}$ for all $i \ge 0$.

For convenience, we often denote $\operatorname{Hom}(A,B)$ by (A,B), specially in some commutative diagrams.

2. Orthogonality of modules

The orthogonality of modules is needed for our discussion.

Lemma 2.1 *Assume that T is a selforthogonal module. Then for any $X \in \operatorname{mod}\Lambda$, $Y \in \check{\operatorname{add}}\, T$, we have*

$$\operatorname{Hom}_{\operatorname{End}_\Lambda(T)^{\mathrm{op}}}(\operatorname{Hom}_\Lambda(Y,T), \operatorname{Hom}_\Lambda(X,T)) \cong \operatorname{Hom}_\Lambda(X,Y).$$

In particular, (a) if $X \in \operatorname{add} T$, then

$$\operatorname{End}_{\operatorname{End}_\Lambda(T)^{\mathrm{op}}}(\operatorname{Hom}_\Lambda(X,T)) \cong \operatorname{End}_\Lambda(X);$$

(b) For $Y \in \check{\operatorname{add}}\, T$, we have

$$\operatorname{Hom}_{\operatorname{End}_\Lambda(T)^{\mathrm{op}}}(\operatorname{Hom}_\Lambda(Y,T), T) \cong \operatorname{Hom}_\Lambda(\Lambda, Y) \cong Y.$$

Proof (1) Suppose that $Y \in \operatorname{add} T$. By the additivity of $e_T = \operatorname{Hom}_\Lambda(-,T)$, we can easily see that

$$e_T : \operatorname{Hom}_\Lambda(X,Y) \to \operatorname{Hom}_{\operatorname{End}_\Lambda(T)^{\mathrm{op}}}(\operatorname{Hom}_\Lambda(Y,T), \operatorname{Hom}_\Lambda(X,T))$$

is an isomorphism for $X \in \operatorname{mod}\Lambda$ (see [5, Lemma 3.3]).

(2) Suppose that $Y \in \operatorname{add}\check{} T$. Then there exists an exact sequence

$$0 \to Y \to T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \to T_{n-1} \xrightarrow{f_n} T_n \to 0$$

with $T_i \in \operatorname{add} T$. Since T is selforthogonal, by dimension shifting we have $\operatorname{Ker} f_i \in {}^{\perp}T$ for $i = 1, 2, \ldots, n$. So we have an exact sequence

$$0 \to Y \to T_0 \xrightarrow{f_1} T_1$$

with $\operatorname{Im} f_1, \operatorname{Coker} f_1 \in {}^{\perp}T$. Then we have an exact sequence

$$\operatorname{Hom}_{\Lambda}(T_1, T) \to \operatorname{Hom}_{\Lambda}(T_0, T) \to \operatorname{Hom}_{\Lambda}(Y, T) \to 0.$$

Now applying the left exact functor $\operatorname{Hom}_{\operatorname{End}({}_{\Lambda}T)^{\mathrm{op}}}(-, \operatorname{Hom}_{\Lambda}(X, T))$ to this exact sequence, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \operatorname{Hom}_{\Lambda}(X, Y) & \longrightarrow & \operatorname{Hom}_{\Lambda}(X, T_0) & \longrightarrow & \operatorname{Hom}_{\Lambda}(X, T_1) \\ & \downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 \\ 0 \longrightarrow & ((Y, T), (X, T)) & \longrightarrow & ((T_0, T), (X, T)) & \longrightarrow & ((T_1, T), (X, T)). \end{array} \quad (2.1)$$

By (1) both g_2, g_3 are isomorphisms. From Diagram (2.1), we have g_1 is an isomorphism. That is,

$$\operatorname{Hom}_{\operatorname{End}({}_{\Lambda}T)^{\mathrm{op}}}(\operatorname{Hom}_{\Lambda}(Y, T), \operatorname{Hom}_{\Lambda}(X, T)) \cong \operatorname{Hom}_{\Lambda}(X, Y)$$

for any $X \in \operatorname{mod} \Lambda$ and any $Y \in \operatorname{add}\check{} T$. □

Now we can prove the following results.

Lemma 2.2 *Assume that T is a selforthogonal module. Then for any $X \in {}^{\perp}T$ and any $Y \in \operatorname{add}\check{} T$, we have an isomorphism*

$$\operatorname{Ext}^i_{\operatorname{End}(T)^{\mathrm{op}}}(\operatorname{Hom}_{\Lambda}(Y, T), \operatorname{Hom}_{\Lambda}(X, T)) \cong \operatorname{Ext}^i_{\Lambda}(X, Y)$$

for all $i \geq 1$.

Proof Since $Y \in \operatorname{add}\check{} T$, there exists an exact sequence

$$0 \to Y \to T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \to T_{n-1} \xrightarrow{f_n} T_n \to 0$$

with $T_i \in \operatorname{add} T$. It is easy to know $\operatorname{Im} f_i \in {}^{\perp}T$ for $i = 1, 2, \ldots, n$. Then the sequence $0 \to \operatorname{Hom}_{\Lambda}(\operatorname{Im} f_1, T) \to \operatorname{Hom}_{\Lambda}(T_0, T) \to \operatorname{Hom}_{\Lambda}(Y, T) \to 0$ is exact. Note that $\operatorname{Hom}_{\Lambda}(T_0, T)$ is $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$-projective. Thus

$$\operatorname{Ext}^1_{\operatorname{End}(T)^{\mathrm{op}}}(\operatorname{Hom}_{\Lambda}(T_0, T), \operatorname{Hom}_{\Lambda}(X, T)) = 0.$$

Since $T_0, \operatorname{Im} f_1 \in \operatorname{add}\check{} T$, both g_1 and g_2 in the commutative diagram

$$\begin{array}{ccccccc} \operatorname{Hom}_{\Lambda}(X, T_0) & \longrightarrow & \operatorname{Hom}_{\Lambda}(X, \operatorname{Im} f_1) & \longrightarrow & \operatorname{Ext}^1_{\Lambda}(X, Y) & \longrightarrow & 0 \\ \downarrow g_1 & & \downarrow g_2 & & \downarrow & & \\ ((T_0, T), (X, T)) & \longrightarrow & ((\operatorname{Im} f_1, T), (X, T)) & \longrightarrow & \operatorname{Ext}^1_{\operatorname{End}(T)^{\mathrm{op}}}((Y, T), (X, T)) & \longrightarrow & 0 \end{array}$$

are isomorphisms by Lemma 2.1. It follows that

$$\mathrm{Ext}^1_{\mathrm{End}(T)^{\mathrm{op}}}((Y,T),(X,T)) \cong \mathrm{Ext}^1_\Lambda(X,Y).$$

Now the result follows from a dimension shifting. □

Corollary 2.3 *Assume that T is a selforthogonal module. Then for any $C \in \mathrm{add}\check{}\,T$, we have*

$$\mathrm{Ext}^i_{\mathrm{End}(T)^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(C,T),T) = 0.$$

Proof Let $X = \Lambda \in {}^\perp T$ and $C = Y \in \mathrm{add}\check{}\,T$. By Lemma 2.2, we have

$$\mathrm{Ext}^i_{\mathrm{End}(T)^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(C,T),\mathrm{Hom}_\Lambda(\Lambda,T)) \cong \mathrm{Ext}^i_\Lambda(\Lambda,C) = 0. \quad \square$$

3. Tilting bimodules

We first recall the notion of tilting pairs.

Definition 3.1 ([2, Section 2]) *A pair (C,T) is called a tilting pair if the following conditions hold:*

(1) C is selforthogonal; (2) T is selforthogonal; (3) $T \in \mathrm{add}\hat{}\,C$ and (4) $C \in \mathrm{add}\check{}\,T$.

We say that (C,T) is a n-tilting pair (or a tilting pair of dimension n) if (C,T) is a tilting pair such that $\dim_{\mathrm{add}\,C}(T) \le n$.

Let Λ be an Artin algebra over a commutative Artin ring R, that is, Λ is an Artin R-algebra. Denote the Artin algebra duality $\mathrm{Hom}_R(-,E(R/J(R)))$ by $\mathbb{D}$, where $J(R)$ is Jacobson radical of R and $E(R/J(R))$ is the injective envelope of $R/J(R)$. Note that $\mathbb{D}(\Lambda)$ is a finitely generated two-sided injective cogenerator [6, Section 3.2].

Lemma 3.2 *Assume that $C,T \in \mathrm{mod}\,\Lambda$. Then the following conditions hold:*

(1) T is a n-tilting module if and only if (Λ,T) is an n-tilting pair;

(2) C is a n-cotilting module if and only if $(C,\mathbb{D}(\Lambda))$ is a n-tilting pair.

Proof We only prove (2). $\Rightarrow$. Assume that C is an n-cotilting module. Then there is an injective cogenerator I such that $I \in (\mathrm{add}\hat{}\,C)_n$. By [7, Lemma 2.1], we know that $(\mathrm{add}\hat{}\,C)_n = \{X \in {}_C\mathcal{X} | \mathrm{Ext}^{n+1}_\Lambda(X,C^\perp) = 0\}$ is closed under extensions and direct summands. Note that $\mathbb{D}(\Lambda) \in \mathrm{add}\,I$. We have $\mathbb{D}(\Lambda) \in (\mathrm{add}\hat{}\,C)_n$. On the other hand, since $\mathrm{id}_\Lambda C \le n$, there exists an exact sequence

$$0 \to C \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \to 0$$

with I_i injective. Since $\mathbb{D}(\Lambda)$ is an injective cogenerator, we have that $I_i \in \mathrm{add}\,\mathbb{D}(\Lambda)$. Hence $C \in (\mathrm{add}\,\check{\mathbb{D}}(\Lambda))_n$. It is now easy to check that $(C,\mathbb{D}(\Lambda))$ is a n-tilting pair.

$\Leftarrow$. Assume that $(C,\mathbb{D}(\Lambda))$ is a n-tilting pair. Since $\mathbb{D}(\Lambda)$ is an injective cogenerator, we have that $\mathrm{id}_\Lambda C \le n$ from $C \in (\mathrm{add}\,\check{\mathbb{D}}(\Lambda))_n$. By [2, Proposition 2.3] we also have $\mathbb{D}(\Lambda) \in (\mathrm{add}\hat{}\,C)_n$. But $(\mathrm{add}\hat{}\,C)_n$ is closed under extensions and direct summands. So we have that $E \in (\mathrm{add}\hat{}\,C)_n$ for every injective cogenerator E. Therefore, C is a n-cotilting module. □

Now we can obtain a tilting bimodule from a given tilting pair.

Theorem 3.3 *Assume that (C,T) is a n-tilting pair. Then $\mathrm{Hom}_\Lambda(C,T)$ is a tilting left $\mathrm{End}_\Lambda(C)$-right $\mathrm{End}_\Lambda(T)$-bimodule of projective dimension $\le n$ on both sides.*

Proof Since $C \in \check{\mathrm{add}}T$, it follows immediately from the proof of Lemma 2.2 that there exists a projective resolution of right $\mathrm{End}_\Lambda(T)$-modules $\mathrm{Hom}_\Lambda(C,T)$ of form

$$0 \to (T_n,T) \to (T_{n-1},T) \to \cdots \to (T_1,T) \to (T_0,T) \to (C,T) \to 0$$

with $T_i \in \mathrm{add}T$. So we have $\mathrm{pd}_{\mathrm{End}_\Lambda(T)}\mathrm{Hom}_\Lambda(C,T) \le n$.

On the other hand, since $T \in \hat{\mathrm{add}}\,C$, there exists an exact sequence

$$0 \to C_n \overset{f_n}{\to} C_{n-1} \overset{f_{n-1}}{\to} \cdots \to C_1 \overset{f_1}{\to} C_0 \overset{f_0}{\to} T \to 0 \tag{3.1}$$

with $C_i \in \mathrm{add}\,C$. Since C is selforthogonal, by a dimension shifting to this sequence, we know that $\mathrm{Im}f_i \in C^\perp$, in particular, $T \in C^\perp$. Since T is selforthogonal, applying $\mathrm{Hom}_\Lambda(-,T)$ to the sequence above, we have that $\mathrm{Im}f_i \in {}^\perp T$ and then the sequence of right $\mathrm{End}_\Lambda(T)$-module

$$0 \to (T,T) \to (C_0,T) \to (C_1,T) \to \cdots \to (C_{n-1},T) \to (C_n,T) \to 0$$

is exact. That is, $\mathrm{End}_\Lambda(T)_{\mathrm{End}_\Lambda(T)} \in \mathrm{add}\,(\check{C},T)$.

Moreover, since $C \in {}^\perp T$, $C \in \check{\mathrm{add}}T$ in a tilting pair (C,T), by Lemma 2.2 we have

$$\mathrm{Ext}^i_{\mathrm{End}_\Lambda(T)^{\mathrm{op}}}((C,T),(C,T)) \cong \mathrm{Ext}^i_\Lambda(C,C) = 0.$$

This means $\mathrm{Hom}_\Lambda(C,T)$ is selforthogonal as a right $\mathrm{End}_\Lambda(T)$-module. Therefore $\mathrm{Hom}_\Lambda(C,T)$ is a tilting module of projective dimension $\le n$ as a right $\mathrm{End}_\Lambda(T)$-module.

Finally, by [2, Proposition 2.3], $\mathrm{Hom}_\Lambda(C,T)$ is also a tilting module of projective dimension $\le n$ as a left $\mathrm{End}_\Lambda(C)$-module. □

Corollary 3.4 ([3, Theorem 1.5]) *Assume that ${}_\Lambda T$ is a tilting module of projective dimension $\le n$. Then $T_{\mathrm{End}_\Lambda(T)}$ is a tilting module of projective dimension $\le n$.*

Proof Setting $C = \Lambda$ in Theorem 3.3, we get that

$$\mathrm{Hom}_\Lambda(C,T)_{\mathrm{End}_\Lambda(T)} = \mathrm{Hom}_\Lambda(\Lambda,T)_{\mathrm{End}_\Lambda(T)} \cong T_{\mathrm{End}_\Lambda(T)}$$

is a tilting module of projective dimension $\le n$. □

Corollary 3.5 *Assume that ${}_\Lambda C$ is a n-cotilting module. Then $\mathbb{D}(C)$ is n-tilting as a right $\mathrm{End}_\Lambda(\mathbb{D}(\Lambda))$-module, i.e., a right Λ-module.*

Proof Since ${}_\Lambda C$ is a n-cotilting module, by Lemma 3.2, $(C,\mathbb{D}(\Lambda))$ is a n-tilting pair. Then by adjoint isomorphism [8, Theorem 2.11] we have

$$\mathrm{Hom}_\Lambda(C,\mathbb{D}(\Lambda)) \cong \mathrm{Hom}_R(\Lambda \otimes_\Lambda C, E(R/J(R))) \cong \mathrm{Hom}_R(C, E(R/J(R))) = \mathbb{D}(C).$$

By Theorem 3.3, we have that $\mathbb{D}(C) \cong \mathrm{Hom}_\Lambda(C,\mathbb{D}(\Lambda))$ is n-tilting as a right $\mathrm{End}_\Lambda(\mathbb{D}(\mathbb{C}))$-module. Meanwhile, we have that

$$\mathrm{End}_\Lambda(\mathbb{D}(\Lambda)) \cong \mathrm{Hom}_\Lambda(\mathrm{Hom}_R(\Lambda, E(R/J(R))), \mathrm{Hom}_R(\Lambda, E(R/J(R))))$$

$$\cong \mathrm{Hom}_R(\Lambda \otimes_\Lambda \mathrm{Hom}_R(\Lambda, E(R/J(R))), E(R/J(R))) \cong \mathbb{D}^2(\Lambda) \cong \Lambda.$$

Hence $\mathbb{D}(C)$ is a right n-tilting Λ-module. □

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