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New Proof for Some Terminating Hypergeometric Series Identities

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Abstract The Abel's lemma on summation by parts is employed to evaluate terminating hypergeometric series. Several summation formulae are reviewed and some new identities are established.

Keywords Abel's lemma on summation by parts; classical hypergeometric series.

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1. Introduction

Following Bailey [1, §2.1] and Slater [10, §2.1], the generalized hypergeometric series is defined by

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \Big| \, z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}, $$

where we suppose that none of the denominator parameters is a nonpositive integer, so that the series is well defined. With the indeterminate x and nonnegative integer n, the shifted factorial reads as

$$ (x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1), \quad n = 1, 2, \ldots $$

and its fractional form is abbreviated with

$$ \left[\begin{matrix} \alpha, \beta, \ldots, \gamma \\ A, B, \ldots, C \end{matrix}\right]_n = \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}. $$

In order to facilitate the subsequent application, we reproduce Abel's lemma on summation by parts as follows. For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators $\triangledown$ and $\vartriangle$, respectively, by

$$ \triangledown \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \vartriangle \tau_k = \tau_k - \tau_{k+1} $$

where $\vartriangle$ is adopted for convenience, which differs from the usual one Δ only in the minus sign. Then Abel's lemma on summation by parts may be reformulated as

$$ \sum_{k \geq 0} A_k \vartriangle B_k = A_{-1} B_0 + \sum_{k \geq 0} B_k \triangledown A_k, $$

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provided that both series are terminating, i.e., there exists a natural number n such that $A_m = 0$ or $B_m = 0$ for all $m > n$.

Proof According to the definition of the forward difference, we have

$$\sum_{k\geq 0} A_k \triangle B_k = \sum_{k\geq 0} A_k \big\{B_k - B_{k+1}\big\} = \sum_{k\geq 0} A_k B_k - \sum_{k\geq 0} A_k B_{k+1}.$$

Replacing k by $k-1$ for the last sum, we get the following expression:

$$\sum_{k\geq 0} A_k \triangle B_k = A_{-1}B_0 + \sum_{k\geq 0} B_k \big\{A_k - A_{k-1}\big\} = A_{-1}B_0 + \sum_{k\geq 0} B_k \bigtriangledown A_k,$$

which proves the identity stated in the lemma. $\square$

For all the series treated in this paper, the extra term $A_{-1}B_0$ will vanish because of the presence of $k! = \Gamma(k+1)$ in the denominator of A_k, which forces $A_{-1} = 0$.

There exist many summation formulae [6, 8] for the classical hypergeometric series. Several fundamental ones have been recently reviewed in [2, 7] by this method. The purpose of this paper is to explore further application of Abel's lemma on summation by parts. Six classes of hypergeometric series will be examined and some new identities are established.

2. Terminating ${}_3F_2$-series

For the indeterminate a, c, denote by $\mathcal{V}(a,c)$ the hypergeometric series:

$$\mathcal{V}(a,c) := {}_3F_2\left[\begin{matrix}3a, 2/3-c, 1-3a\\ 1/2, 2-3c\end{matrix} \,\Big|\, \frac{3}{4}\right].$$

With the two sequences given by

$$\mathcal{A}_k = \left[\begin{matrix}1+3a, 2/3-a\\ 1, 1/2\end{matrix}\right]_k \left(\frac{3}{4}\right)^k \quad\text{and}\quad \mathcal{B}_k = \left[\begin{matrix}2/3-c, -1-3a\\ -1/3-a, 2-3c\end{matrix}\right]_k,$$

it is not hard to compute their differences

$$\bigtriangledown\mathcal{A}_k = \frac{(k-3a)(k-3a-1)}{3a(1+3a)}\left[\begin{matrix}3a, -1/3-a\\ 1, 1/2\end{matrix}\right]_k \left(\frac{3}{4}\right)^k;$$

$$\triangle\mathcal{B}_k = \frac{2k(1+a-c)}{(2-3c)(-1/3-a)}\left[\begin{matrix}2/3-c, -1-3a\\ 2/3-a, 3-3c\end{matrix}\right]_k.$$

Applying Abel's lemma on summation by parts, we can manipulate the $\mathcal{V}(a,c)$-series

$$\mathcal{V}(a,c) = \sum_{k\geq 0} \mathcal{B}_k \bigtriangledown \mathcal{A}_k = \sum_{k\geq 0} \mathcal{A}_k \triangle \mathcal{B}_k = \frac{(1+a-c)(1+3a)}{(1-c)} \sum_{k\geq 0} \left[\begin{matrix}2+3a, \frac{5}{3}-c, -3a\\ 1, \frac{3}{2}, 4-3c\end{matrix}\right]_k \left(\frac{3}{4}\right)^k,$$

which reads as the recurrence relation

$$\mathcal{V}(a,c) = \frac{(1+a-c)(1+3a)}{(1-c)}\mathcal{V}'(a,c), \tag{1}$$

where the $\mathcal{V}'(a,c)$ in the above identity is denoted by the following hypergeometric series:

$$\mathcal{V}'(a,c) := {}_3F_2\left[\begin{matrix}2+3a, 5/3-c, -3a\\ 3/2, 4-3c\end{matrix} \;\Big|\; \frac{3}{4}\right].$$

Giving the other two sequences

$$\mathcal{C}_k = \left[\begin{matrix}3+3a, 2/3-a\\ 1, 3/2\end{matrix}\right]_k \left(\frac{3}{4}\right)^k \quad \text{and} \quad \mathcal{D}_k = \left[\begin{matrix}5/3-c, -2-3a\\ -1/3-a, 4-3c\end{matrix}\right]_k,$$

and applying Abel's lemma on summation by parts, we can manipulate the $\mathcal{V}'(a,c)$-series

$$\mathcal{V}'(a,c) = \sum_{k\geq 0} \mathcal{D}_k \bigtriangledown \mathcal{C}_k = \sum_{k\geq 0} \mathcal{C}_k \triangle \mathcal{D}_k = \frac{(2+a-c)}{(1/3+a)(3c-4)} \sum_{k\geq 0} \left[\begin{matrix}3+3a, \frac{5}{3}-c, -2-3a\\ 1, \frac{1}{2}, 5-3c\end{matrix}\right]_k \left(\frac{3}{4}\right)^k,$$

which reads as the recurrence relation

$$\mathcal{V}'(a,c) = \frac{(2+a-c)}{(-1/3-a)(4-3c)}\, \mathcal{V}(a+1, c-1). \tag{2}$$

Combining the two relations (1) and (2), we get the following relation about $\mathcal{V}(a,c)$

$$\mathcal{V}(a,c) = \frac{-(1+a-c)(2+a-c)}{(1-c)(4/3-c)}\, \mathcal{V}(a+1, c-1).$$

Iterating this process m-times, we derive the following transformation formula.

Theorem 1 (Transformation) *With m be positive integer, we have*

$$\mathcal{V}(a,c) = \frac{(-1)^m (1+a-c)_{2m}}{(1-c)_m (4/3-c)_m}\, \mathcal{V}(a+m, c-m).$$

Letting $a = -n$ in Theorem 1 and noting that $\mathcal{V}(0,c) = 1$, we obtain the following formula,

Corollary 2 (Gessel [8, Eq.15.1a]) *With n be positive integer, we obtain*

$${}_3F_2\left[\begin{matrix}-3n, 2/3-c, 1+3n\\ 1/2, 2-3c\end{matrix} \;\Big|\; \frac{3}{4}\right] = \frac{(c)_n}{(4/3-c)_n}.$$

Next, taking $c = n + 2/3$ in Theorem 1 and noting that $\mathcal{V}(a, 2/3) = 1$, we get

Corollary 3 *With n be positive integer, we get*

$${}_3F_2\left[\begin{matrix}3a, -n, 1-3a\\ 1/2, -3n\end{matrix} \;\Big|\; \frac{3}{4}\right] = \frac{(1/3+a)_n (2/3-a)_n}{(1/3)_n (2/3)_n}.$$

Combining the two identities (1) and (2), differently, we get the recurrence relation for $\mathcal{V}'(a,c)$

$$\mathcal{V}'(a,c) = \frac{-(2+a-c)(3+a-c)(4/3+a)}{(1/3+a)(2-c)(4/3-c)} \mathcal{V}'(a+1, c-1).$$

Iterating this process m-times, we find the following transformation formula.

Theorem 4 (Transformation) *With m be positive integer, we have*

$$\mathcal{V}'(a,c) = \frac{(-1)^m (2+a-c)_{2m} (4/3+a)_m}{(1/3+a)_m (2-c)_m (4/3-c)_m} \mathcal{V}'(a+m, c-m).$$

Taking $a = -n - 2/3$ in Theorem 4, we recover the terminating form of Gessel's formula.

Corollary 5 (Gessel [8, Eq.15.2a]) *With n be positive integer, we obtain*

$$ {}_3F_2\left[\begin{matrix}-3n, 2/3-c, 2+3n\\ 3/2, 1-3c\end{matrix} \,\Big|\, \frac{3}{4}\right] = \left[\begin{matrix}c-1/3, 1/3\\ 2-c, 4/3\end{matrix}\right]_n. $$

This formula was first obtained by Gessel [8] through the WZ-method. Another useful method to obtain hypergeometric series identities is the inversion technique. The reader can refer to [3–5,9] for more information about that approach.

3. Terminating ${}_4F_3$-series

For the indeterminate b, c, denote by $\mathcal{U}(b,c)$ the hypergeometric series:

$$ \mathcal{U}(b,c) := {}_4F_3\left[\begin{matrix}c, 1+4c/3, 1/2+c, 1-2b\\ 4c/3, 1+2c, 1/2+b+2c\end{matrix} \,\Big|\, 4\right]. $$

With the two sequences given by

$$ \mathcal{A}_k = \left[\begin{matrix}1+c\\ 1\end{matrix}\right]_k 4^k \quad \text{and} \quad \mathcal{B}_k = \left[\begin{matrix}1/2+c, 1-2b\\ 1+2c, 1/2+b+2c\end{matrix}\right]_k, $$

it is not hard to compute their differences

$$ \triangledown \mathcal{A}_k = \frac{3k+4c}{4c}\left[\begin{matrix}c\\ 1\end{matrix}\right]_k 4^k \quad \text{and} \quad \vartriangle \mathcal{B}_k = \frac{(3k+2+4c)(2b+2c)}{(1+2c)(1+2b+4c)}\left[\begin{matrix}1/2+c, 1-2b\\ 2+2c, 3/2+b+2c\end{matrix}\right]_k. $$

Applying Abel's lemma on summation by parts, we can manipulate the $\mathcal{U}(b,c)$-series

$$ \mathcal{U}(b,c) = \sum_{k\ge 0} \mathcal{B}_k \triangledown \mathcal{A}_k = \sum_{k\ge 0} \mathcal{A}_k \vartriangle \mathcal{B}_k = \sum_{k\ge 0} \frac{(3k+2+4c)(2b+2c)}{(2+4c)(1/2+b+2c)}\left[\begin{matrix}1+c, 1/2+c, 1-2b\\ 1, 2+2c, 3/2+b+2c\end{matrix}\right]_k 4^k, $$

which reads as the recurrence relation

$$ \mathcal{U}(b,c) = \frac{2b+2c}{1/2+b+2c} \times \mathcal{U}(b, c+1/2). $$

Iterating this process m-times, we get the following transformation formula.

Theorem 6 (Transformation) *With m be positive integer, we have*

$$ \mathcal{U}(b,c) = \left[\begin{matrix}2b+2c\\ 1/2+b+2c\end{matrix}\right]_m \times \mathcal{U}(b, c+m/2). $$

Letting $c = -n$ with $n = m + \delta/2$ and $\delta = 0, 1$ in Theorem 6, we get the summation formula.

Corollary 7 *With n be positive integer, we obtain*

$$ {}_4F_3\left[\begin{matrix}-n, 1-4n/3, 1/2-n, 1-2b\\ -4n/3, 1-2n, 1/2+b-2n\end{matrix} \,\Big|\, 4\right] = \begin{cases} \dfrac{(1-2b)_{2m}}{(1/2-b)_{2m}}, & \delta = 0;\\[2mm] \dfrac{(2-2b)_{2m}}{(3/2-b)_{2m}}, & \delta = 1.\end{cases} $$

The case $\delta = 0$ was first obtained by Gessel [8, Eq.29.1a] through the WZ-method.

4. Terminating ${}_5F_4$-series

(I) For the indeterminate c, denote by $\Omega(c)$ the hypergeometric series:

$$\Omega(c) := {}_5F_4\left[\begin{matrix}-3/2, 1+6c/11, 1/2+c/2, c/2, 4+2c\\ 6c/11, (2+2c)/3, (3+2c)/3, (4+2c)/3\end{matrix}\,\Big|\,\frac{16}{27}\right].$$

With the two sequences given by

$$\mathcal{A}_k = \left[\begin{matrix}1+c/2, 5+2c\\ 1, (4+2c)/3\end{matrix}\right]_k \quad \text{and} \quad \mathcal{B}_k = \left[\begin{matrix}1/2+c/2, -3/2\\ 1+2c/3, (2+2c)/3\end{matrix}\right]_k \left(\frac{16}{27}\right)^k,$$

it is not hard to compute their differences

$$\triangledown\mathcal{A}_k = \frac{11k+6c}{6c}\left[\begin{matrix}c/2, 4+2c\\ 1, (4+2c)/3\end{matrix}\right]_k;$$

$$\vartriangle\mathcal{B}_k = \frac{(11k+6c+6)(k+2c+5)}{(6c+6)(3+2c)}\left[\begin{matrix}1/2+c/2, -3/2\\ 2+2c/3, (5+2c)/3\end{matrix}\right]_k \left(\frac{16}{27}\right)^k.$$

Applying Abel's lemma on summation by parts, we can manipulate the $\Omega(c)$-series

$$\begin{aligned}\Omega(c) &= \sum_{k\geq 0}\mathcal{B}_k \triangledown \mathcal{A}_k = \sum_{k\geq 0}\mathcal{A}_k \vartriangle\mathcal{B}_k\\ &= \frac{(2c+5)}{(2c+3)}\sum_{k\geq 0}\frac{11k+6c+6}{6c+6}\left[\begin{matrix}-3/2, 1/2+c/2, 1+c/2, 6+2c\\ 1, (5+2c)/3, (6+2c)/3, (4+2c)/3\end{matrix}\right]_k \left(\frac{16}{27}\right)^k,\end{aligned}$$

which reads as the recurrence relation

$$\Omega(c) = \frac{(2c+5)}{(2c+3)} \times \Omega(c+1).$$

Iterating this process m-times, we derive the following transformation formula.

Theorem 8 (Transformation) *With m be positive integer, we obtain*

$$\Omega(c) = \frac{(c+5/2)_m}{(c+3/2)_m} \times \Omega(c+m).$$

Letting $c = -n$ in Theorem 8, we get the following summation formula.

Corollary 9 (Gessel [8, Eq.12.2a]) *With n be positive integer, we get*

$${}_5F_4\left[\begin{matrix}-3/2, 1-6n/11, 1/2-n/2, -n/2, 4-2n\\ -6n/11, (2-2n)/3, (3-2n)/3, (4-2n)/3\end{matrix}\,\Big|\,\frac{16}{27}\right] = \frac{1}{6}\frac{(-3/2)_n}{(-1/2)_n}, \quad n \geq 3.$$

(II) For the indeterminate c, denote by $\Phi(c)$ the hypergeometric series:

$$\Phi(c) := {}_5F_4\left[\begin{matrix}3/4, 1+3c/11, (7+4c)/16, (4c-1)/16, c\\ 3c/11, (3+2c)/6, (5+2c)/6, (7+2c)/6\end{matrix}\,\Big|\,\frac{16}{27}\right].$$

With the two sequences given by

$$\mathcal{A}_k = \left[\begin{matrix}1+c, (15+4c)/16\\ 1, (7+2c)/6\end{matrix}\right]_k \quad \text{and} \quad \mathcal{B}_k = \left[\begin{matrix}(4c+7)/16, 3/4\\ (3+2c)/6, (5+2c)/6\end{matrix}\right]_k \left(\frac{16}{27}\right)^k,$$

it is not hard to compute their differences

$$\triangledown \mathcal{A}_k = \frac{11k+3c}{3c}\begin{bmatrix}1+c,(4c-1)/16\\1,(7+2c)/6\end{bmatrix}_k;$$

$$\triangle \mathcal{B}_k = \frac{4(11k+6+3c)(1+c+k)}{3(3+2c)(5+2c)}\begin{bmatrix}(4c+7)/16,3/4\\(9+2c)/6,(11+2c)/6\end{bmatrix}_k\Big(\frac{16}{27}\Big)^k.$$

Applying Abel's lemma on summation by parts, we can manipulate the $\Phi(c)$-series

$$\Phi(c) = \sum_{k\geq 0}\mathcal{B}_k \triangledown \mathcal{A}_k = \sum_{k\geq 0}\mathcal{A}_k \triangle \mathcal{B}_k = \frac{4(1+c)(2+c)}{(3+2c)(5+2c)}\times$$

$$\sum_{k\geq 0}\frac{11k+6+3c}{6+3c}\begin{bmatrix}3/4,(7+4c)/16,(4c+15)/16,2+c\\1,(9+2c)/6,(11+2c)/6,(7+2c)/6\end{bmatrix}_k\Big(\frac{16}{27}\Big)^k,$$

which reads as the recurrence relation

$$\Phi(c) = \frac{(1+c)(2+c)}{(3/2+c)(5/2+c)}\,\Phi(c+2).$$

Iterating this process m-times, we have the following transformation formula.

Theorem 10 (Transformation) *With m be positive integer, we obtain*

$$\Phi(c) = \frac{(1+c)_{2m}}{(3/2+c)_{2m}}\times\Phi(c+2m).$$

Putting $c=-n$ with $n=2m+\delta$ and $\delta=0,1$ in Theorem 10 and noting that $\Phi(\delta)=0$, we get the following terminating hypergeometric series identity.

Corollary 11 *With n be positive integer, we get*

$${}_5F_4\left[\begin{matrix}3/4,1-3n/11,(7-4n)/16,(-4n-1)/16,-n\\-3n/11,(3-2n)/6,(5-2n)/6,(7-2n)/6\end{matrix}\,\Big|\,\frac{16}{27}\right]=0.$$

Next, taking $c=-2n+1/4$ in Theorem 10, we recover another form of Gessel's identity.

Corollary 12 (Gessel [8, Eq.12.3a]) *With n be positive integer, we have*

$${}_5F_4\left[\begin{matrix}3/4,1+\frac{3-24n}{44},1/4-2n,(1-n)/2,-n/2\\\frac{3-24n}{44},(7-8n)/12,(11-8n)/12,(15-8n)/12\end{matrix}\,\Big|\,\frac{16}{27}\right]=\begin{bmatrix}3/8,-1/8\\1/8,-3/8\end{bmatrix}_n.$$

(III) For the indeterminate b,c, denote by $\mathcal{H}(b,c)$ the hypergeometric series:

$$\mathcal{H}(b,c) := {}_5F_4\left[\begin{matrix}c,1+3c/2,1/2+c,3b,-3b\\3c/2,1/2,1-b+2c,1+b+2c\end{matrix}\,\Big|\,1\right].$$

With the two sequences given by

$$\mathcal{A}_k = \begin{bmatrix}1+c,-1/2-3c\\1,1/2\end{bmatrix}_k \quad\text{and}\quad \mathcal{B}_k = \begin{bmatrix}1/2+c,3b,-3b\\-3/2-3c,1-b+2c,1+b+2c\end{bmatrix}_k,$$

it is not hard to compute their differences

$$\triangledown \mathcal{A}_k = \frac{2k+3c}{3c}\begin{bmatrix}c,-3/2-3c\\1,1/2\end{bmatrix}_k;$$

$$\triangle\mathcal{B}_k = \frac{(4k+3+6c)(1+2b+2c)(1-2b+2c)}{(3+6c)(1-b+2c)(1+b+2c)} \begin{bmatrix} 1/2+c, 3b, -3b \\ -1/2-3c, 2-b+2c, 2+b+2c \end{bmatrix}_k.$$

Applying Abel's lemma on summation by parts, we can manipulate the $\mathcal{H}(b,c)$-series

$$\mathcal{H}(b,c) = \sum_{k\geq 0} \mathcal{B}_k \bigtriangledown \mathcal{A}_k = \sum_{k\geq 0} \mathcal{A}_k \triangle \mathcal{B}_k = \frac{(1+2b+2c)(1-2b+2c)}{(1-b+2c)(1+b+2c)} \times$$

$$\sum_{k\geq 0} \frac{4k+3+6c}{3+6c} \begin{bmatrix} 1+c, 1/2+c, 3b, -3b \\ 1, 1/2, 2-b+2c, 2+b+2c \end{bmatrix}_k,$$

which reads as the recurrence relation

$$\mathcal{H}(b,c) = \frac{(1+2b+2c)(1-2b+2c)}{(1-b+2c)(1+b+2c)} \times \mathcal{H}(b, c+1/2).$$

Iterating this process m-times, we get the following transformation formula.

Theorem 13 (Transformation) *With n be positive integer, we obtain*

$$\mathcal{H}(b,c) = \begin{bmatrix} 1+2b+2c, 1-2b+2c \\ 1-b+2c, 1+b+2c \end{bmatrix}_m \times \mathcal{H}(b, c+m/2).$$

Letting $c=-n$ with $n=m+\delta/2$ and $\delta=0,1$ in Theorem 13, we obtain the following result.

Corollary 14 *With n be positive integer, we get*

$${}_5F_4 \left[\begin{matrix} -n, 1-3n/2, 1/2-n, 3b, -3b \\ -3n/2, 1/2, 1-b-2n, 1+b-2n \end{matrix} \middle| 1 \right] = \begin{cases} \begin{bmatrix} 2b, -2b \\ b, -b \end{bmatrix}_{2m}, & \delta = 0; \\ \begin{bmatrix} 1+2b, 1-2b \\ 1+b, 1-b \end{bmatrix}_{2m}, & \delta = 1. \end{cases}$$

The case $\delta=0$ was obtained by Gessel [8, Eq.23.5a] through the WZ-method.

(IV) For the indeterminate b,c, denote by $\mathcal{W}(b,c)$ the hypergeometric series:

$$\mathcal{W}(b,c) := {}_5F_4 \left[\begin{matrix} c, 1+2c/5, 1/2+c, 2b, -2b \\ 2c/5, 1+2c, 1-3b+c, 1+3b+c \end{matrix} \middle| -4 \right].$$

With the two sequences given by

$$\mathcal{A}_k = \begin{bmatrix} 1+c, 3/2+c \\ 1, 4+4c \end{bmatrix}_k \big(-4\big)^k \quad \text{and} \quad \mathcal{B}_k = \begin{bmatrix} 4+4c, 2b, -2b \\ 2+2c, 1-3b+c, 1+3b+c \end{bmatrix}_k,$$

it is not hard to compute their differences

$$\bigtriangledown\mathcal{A}_k = \frac{(5k+2c)(k+1+2c)}{2c(1+2c)} \begin{bmatrix} c, 1/2+c \\ 1, 4+4c \end{bmatrix}_k \big(-4\big)^k;$$

$$\triangle\mathcal{B}_k = \frac{(5k+2c+2)(1+b+c)(1-b+c)}{(2+2c)(1+3b+c)(1-3b+c)} \begin{bmatrix} 4+4c, 2b, -2b \\ 3+2c, 2-3b+c, 2+3b+c \end{bmatrix}_k.$$

Applying Abel's lemma on summation by parts, we can manipulate the $\mathcal{W}(b,c)$-series

$$\mathcal{W}(b,c) = \sum_{k\geq 0} \mathcal{B}_k \bigtriangledown \mathcal{A}_k = \sum_{k\geq 0} \mathcal{A}_k \triangle \mathcal{B}_k = \frac{(1+b+c)(1-b+c)}{(1+3b+c)(1-3b+c)} \times$$

$$\sum_{k\geq 0}\frac{5k+2+2c}{2+2c}\left[\begin{matrix}1+c,3/2+c,2b,-2b\\ 1,3+2c,2-3b+c,2+3b+c\end{matrix}\right]_k\big(-4\big)^k,$$

which reads as the recurrence relation

$$\mathcal{W}(b,c)=\frac{(1+b+c)(1-b+c)}{(1+3b+c)(1-3b+c)}\times\mathcal{W}(b,c+1).$$

Iterating this process m-times, we find the following transformation formula.

Theorem 15 (Transformation) *With n be positive integer, we get*

$$\mathcal{W}(b,c)=\left[\begin{matrix}1+b+c,1-b+c\\ 1-3b+c,1+3b+c\end{matrix}\right]_m\times\mathcal{W}(b,c+m).$$

Letting $c=-n$ with $n=m+\delta/2$ and $\delta=0,1$ in Theorem 15, we get the following identity.

Corollary 16 *With n be positive integer, we obtain*

$${}_5F_4\left[\begin{matrix}-n,1-\frac{2n}{5},1/2-n,2b,-2b\\ -\frac{2n}{5},1-3b-n,1-2n,1+3b-n\end{matrix}\;\Big|\;-4\right]=\begin{cases}\left[\begin{matrix}b, & -b\\ 3b, & -3b\end{matrix}\right]_m, & \delta=0;\\ \left[\begin{matrix}1/2+b,1/2-b\\ 1/2+3b,1/2-3b\end{matrix}\right]_m, & \delta=1.\end{cases}$$

The special case $\delta=0$ was obtained by Gessel [8, Eq.23.10a] through the WZ-method.

References

[1] BAILEY W N. *Generalized Hypergeometric Series* [M]. Cambridge University Press, Cambridge, 1935.

[2] CHEN V Y B, CHEN W Y C, GU N S S. *The Abel lemma and the q-Gosper algorithm* [J]. Math. Comp., 2008, **77**(262): 1057–1074.

[3] CHU Wenchang. *Inversion techniques and combinatorial identities* [J]. Boll. Un. Mat. Ital. B (7), 1993, **7**(4): 737–760.

[4] CHU Wenchang. *Inversion techniques and combinatorial identities–strange evaluations of hypergeometric series* [J]. Pure Math. Appl., 1993, **4**(4): 409–428.

[5] CHU Wenchang. *Inversion Techniques and Combinatorial Identities: A Quick Introduction to Hypergeometric Evaluations* [M]. Kluwer Acad. Publ., Dordrecht, 1994.

[6] CHU Wenchang. *Inversion techniques and combinatorial identities. A unified treatment for the ${}_7F_6$-series identities* [J]. Collect. Math., 1994, **45**(1): 13–43.

[7] CHU Wenchang. *Abel's method on summation by parts and hypergeometric series* [J]. J. Difference Equ. Appl., 2006, **12**(8): 783–798.

[8] GESSEL I M. *Finding identities with the WZ method* [J]. J. Symbolic Comput., 1995, **20**(5-6): 537–566.

[9] GOULD H W, HSU L C. *Some new inverse series relations* [J]. Duke Math. J., 1973, **40**: 885–891.

[10] SLATER L J. *Generalized Hypergeometric Functions* [M]. Cambridge University Press, Cambridge 1966.