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A Note on Shift-Invariant Spaces Admitting a Single Generator

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Abstract In 2005, Garcia, Perez-Villala and Portal gave the regular and irregular sampling formulas in shift invariant space V_φ via a linear operator T between $L^2(0,1)$ and $L^2(R)$. In this paper, in terms of bases for $L^2(0,\alpha)$, two sampling theorems for $\alpha\mathbb{Z}$-shift invariant spaces with a single generator are obtained.

Keywords shift-invariant spaces; sampling; Riesz bases.

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1. Introduction

Before proceeding, we introduce some notations and notions. Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of all integers and the set of all positive integers, respectively. Given $\alpha > 0$. For an α-periodic measurable function f, define

$$\|f\|_0 = \text{essinf}_{t\in(0,\,\alpha)}|f(t)| \text{ and } \|f\|_\infty = \text{esssup}_{t\in(0,\,\alpha)}|f(t)|.$$

We denote by $\ell_0(\mathbb{Z})$ the set of all finitely supported sequences. For $f \in L^1(\mathbb{R})$, define its Fourier transform by

$$\hat{f}(\cdot) = \int_{\mathbb{R}} \mathrm{d}x f(x)e^{-2\pi ix\cdot}.$$

The Fourier transforms of the functions in $L^2(\mathbb{R})$ are understood as the unitary extension of the above. For an infinite matrix $M = \{m_{n,k}\}_{n,k\in\mathbb{Z}}$ defining a bounded operator in $\ell^2(\mathbb{Z})$, we write

$$\|M\|_2 := \sup_{\|c\|_{\ell^2(\mathbb{Z})}=1} \|Mc\|_{\ell^2(\mathbb{Z})}.$$

In [6], in terms of Riesz bases in $L^2(0,\,1)$, the authors investigated sampling in integer-shift invariant subspaces generated by a single function in $L^2(\mathbb{R})$. Inspired by their work, this paper addresses sampling in $\alpha\mathbb{Z}$-shift invariant subspaces generated by a single function in $L^2(\mathbb{R})$. Given

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$\phi \in L^2(\mathbb{R})$. Let $\{ \varphi(\cdot - \alpha n) : n \in \mathbb{Z} \}$ be a Riesz basis for its closed linear span V_φ^α:

$$V_\varphi^\alpha := \overline{\mathrm{Span}\{ \varphi(\cdot - \alpha n) : n \in \mathbb{Z} \}}, \tag{1.1}$$

i.e., there exist $0 < C_1 \leq C_2 < +\infty$ such that

$$C_1 \sum_{n \in \mathbb{Z}} |a_n|^2 \leq \| \sum_{n \in \mathbb{Z}} a_n \varphi(\cdot - \alpha n) \|_2^2 \leq C_2 \sum_{n \in \mathbb{Z}} |a_n|^2$$

for $a \in \ell^2(\mathbb{Z})$. This paper addresses sampling in V_φ^α. Let us begin with the following proposition.

Proposition 1.1 *Given $1 < n_0 \in \mathbb{N}$ and $x_0 \in \mathbb{R}$. Let θ be a measurable function supported on $(x_0, x_0 + n_0)$ such that, for some $0 < A \leq B < \infty$, $A \leq |\theta(\cdot)| \leq B$ a.e., on $(x_0, x_0 + n_0)$. Define φ_j via its Fourier transform by*

$$\hat{\varphi}_j(\cdot) = \theta(\cdot) e^{\frac{-2\pi i j \cdot}{n_0}}$$

for $j \in \{ 0, 1, 2, \ldots, n_0 - 1 \}$. Assume that τ is an n_0-periodic function such that, for some $0 < C \leq D < \infty$, $C \leq |\tau(\cdot)| \leq D$ a.e., on $\mathbb{R}$, and that φ is defined via its Fourier transform by $\hat{\varphi}(\cdot) = \tau(\cdot)\hat{\varphi}_0(\cdot)$. Then $\{\varphi(\cdot - \frac{k}{n_0}) : k \in \mathbb{Z}\}$ is a Riesz basis for $\overline{\mathrm{Span}\{ \varphi_j(\cdot - k) : 0 \leq j \leq n_0 - 1, k \in \mathbb{Z} \}}$.

Proof It is easy to check that, for $d \in \ell_0(\mathbb{Z})$,

$$\sum_{l \in \mathbb{Z}} d_l \varphi_0(\cdot - \frac{l}{n_0}) = \sum_{j=0}^{n_0 - 1} \sum_{k \in \mathbb{Z}} d_{n_0 k + j} \varphi_0(\cdot - \frac{n_0 k + j}{n_0}) = \sum_{j=0}^{n_0 - 1} \sum_{k \in \mathbb{Z}} d_{n_0 k + j} \varphi_j(\cdot - k).$$

Write $V(\varphi_0) = \overline{\mathrm{Span}\{ \varphi_0(\cdot - \frac{k}{n_0}) : k \in \mathbb{Z} \}}$. To prove the proposition, it suffices to prove that $\{\varphi(\cdot - \frac{k}{n_0}) : k \in \mathbb{Z}\}$ is a Riesz basis for $V(\varphi_0)$. For $c \in \ell_0(\mathbb{Z})$, we have

$$\| \sum_{k \in \mathbb{Z}} c_k \varphi(\cdot - \frac{k}{n_0}) \|^2 = \int_{(0,\, n_0)} \mathrm{d}\xi | \sum_{k \in \mathbb{Z}} c_k e^{\frac{-2\pi i k \xi}{n_0}} |^2 \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + n_0 k)|^2,$$

$$\int_{(0,\, n_0)} \mathrm{d}\xi | \sum_{k \in \mathbb{Z}\mathbb{Z}} c_k e^{\frac{-2\pi i k \xi}{n_0}} |^2 = n_0 \sum_{k \in \mathbb{Z}\mathbb{Z}} |c_k|^2.$$

It follows that $\{\varphi(\cdot - \frac{k}{n_0}) : k \in \mathbb{Z}\}$ is a Riesz basis for $V(\varphi) = \overline{\mathrm{Span}\{ \varphi(\cdot - \frac{k}{n_0}) : k \in \mathbb{Z} \}}$ if and only if $\sum_{k \in \mathbb{Z}\mathbb{Z}} |\hat{\varphi}(\cdot + n_0 k)|^2$ is of positive bound from below and above. Observing that $C \leq |\tau(\cdot)| \leq D$, we have $V(\varphi) = V(\varphi_0)$, and that $\sum_{k \in \mathbb{Z}} |\hat{\varphi}(\cdot + n_0 k)|^2$ is of positive bound from below and above if and only if $\sum_{k \in \mathbb{Z}} |\hat{\varphi}_0(\cdot + n_0 k)|^2$ is of positive bound from below and above. However, by the definition of φ_0, $\sum_{k \in \mathbb{Z}} |\hat{\varphi}_0(\cdot + n_0 k)|^2$ is positively bounded from below and above. The proposition therefore follows. □

Generally speaking, sampling in integer-shift invariant subspaces generated by more than one function is not as easy as in invariant subspaces generated by one function. Proposition 1.1 shows that, under some hypotheses, an integer-shift invariant subspace generated by more than one function can be transformed into an $\alpha\mathbb{Z}$-shift invariant subspace generated by one function for some $\alpha > 0$. It is why we are interested in sampling in $\alpha\mathbb{Z}$-shift invariant subspaces generated by one function. The fundamentals of sampling in shift invariant subspaces can be found in [6, 8].

There are many references in this area [1–5]. We will investigate sampling theorems of the form

$$f(\cdot)=\sum_{n\in\mathbb{Z}} f(t_n)\varphi(\cdot-\alpha n)$$

for $f\in V_\varphi^\alpha$, where $\{\ \varphi(\cdot-\alpha n):\ n\in\mathbb{Z}\ \}$ is a Riesz basis for V_φ^α, α is a given positive number. The case of $t_n=a+\alpha n$ with a given $0\le a<\alpha$ is called regular sampling, and the case of $t_n=a+\alpha n+\delta_n$ with $\{\ \delta_n\ \}$ being a sequence in $(-\alpha,\alpha))$ is called irregular sampling. In Section 2, we will give some necessary lemmas. Section 3 will be devoted to regular and irregular sampling theorems.

2. Some necessary lemmas

Now, we will show some supported lemmas.

Lemma 2.1 *Given $\alpha>0$ and $\varphi\in L^2(\mathbb{R})$. Assume that φ is a continuous function satisfying $|\varphi(\cdot)|\le\frac{C}{(1+|\cdot|)^\beta}$ on $\mathbb{R}$ for some $C>0$ and some $\beta>\frac{1}{2}$. Then $\sum_{n\in\mathbb{Z}}a_n\varphi(\cdot-\alpha n)$ is continuous on $\mathbb{R}$ for every $a\in\ell^2(\mathbb{Z})$.*

Proof Note that φ is continuous. It suffices to prove that $\sum_{n\in\mathbb{Z}}|a_n\varphi(\cdot-\alpha n)|$ converges uniformly on an arbitrary set $[-M\alpha,\,M\alpha]$ with $M>0$. For $n_2>n_1>2M$, $t\in[-M\alpha,\,M\alpha]$,

$$\begin{aligned}|\sum_{n_1\le|n|\le n_2}a_n\varphi(t-\alpha n)|&\le(\sum_{n_1\le|n|\le n_2}|a_n|^2)^{\frac{1}{2}}(\sum_{n_1\le|n|\le n_2}|\varphi(t-\alpha n)|^2)^{\frac{1}{2}}\\&\le C(\sum_{n_1\le|n|\le n_2}|a_n|^2)^{\frac{1}{2}}(\sum_{n_1\le|n|\le n_2}|\frac{1}{(1+\frac{\alpha}{2}|n|)^{2\beta}}|)^{\frac{1}{2}}\longrightarrow 0\end{aligned}$$

as $n_1\longrightarrow\infty$. It follows that $\sum_{n\in\mathbb{Z}}a_n\varphi(\cdot-\alpha n)$ converges uniformly on $[-M\alpha,\,M\alpha]$. The proof is completed. □

Lemma 2.2 *Let F be a measurable function on $(0,\alpha)$. Then $\{F(\cdot)e^{2\pi in\frac{\cdot}{\alpha}}:n\in\mathbb{Z}\}$ is a Riesz basis for $L^2(0,\alpha)$ if and only if $0<\|F\|_0\le\|F\|_\infty<\infty$.*

Proof Necessity. Suppose $\{F(\cdot)e^{2\pi in\frac{\cdot}{\alpha}}:n\in\mathbb{Z}\}$ is a Riesz basis for $L^2(0,\alpha)$. Then, there exist $0<A\le B<+\infty$ such that

$$A\sum_{n\in\mathbb{Z}}|c_n|^2\le\int_{(0,\alpha)}\mathrm{d}x|\sum_{n\in\mathbb{Z}}c_nF(x)e^{2\pi in\frac{x}{\alpha}}|^2\le B\sum_{n\in\mathbb{Z}}|c_n|^2,$$

equivalently,

$$\begin{aligned}A\int_{(0,\alpha)}\mathrm{d}x|\sum_{n\in\mathbb{Z}}c_n\alpha^{-\frac{1}{2}}e^{2\pi in\frac{x}{\alpha}}|^2&\le\int_{(0,\alpha)}\mathrm{d}x|F(x)|^2|\sum_{n\in\mathbb{Z}}c_ne^{2\pi in\frac{x}{\alpha}}|^2\\&\le B\int_{(0,\alpha)}\mathrm{d}x|\sum_{n\in\mathbb{Z}}c_n\alpha^{-\frac{1}{2}}e^{2\pi in\frac{x}{\alpha}}|^2.\end{aligned}$$

It follows that $\frac{A}{\alpha}\le|F(\cdot)|\le\frac{B}{\alpha}$ a.e., on $(0,\alpha)$.

Sufficiency. Suppose that $0<\|F\|_0\le\|F\|_\infty<\infty$. Define $T(L^2(0,\alpha)\to L^2(0,\alpha))$ by $T(f)=\alpha^{\frac{1}{2}}F(\cdot)f$. Then T is a bounded invertible operator. Note that $\{\alpha^{-\frac{1}{2}}e^{2\pi in\frac{\cdot}{\alpha}}:n\in\mathbb{Z}\}$ is an orthonormal basis for $L^2(0,\alpha)$ and $T\alpha^{-\frac{1}{2}}e^{2\pi in\frac{\cdot}{\alpha}} = F(\cdot)e^{2\pi in\frac{\cdot}{\alpha}}$. It follows that $\{F(\cdot)e^{2\pi in\frac{\cdot}{\alpha}}\}$ is a Riesz basis for $L^2(0,\alpha)$. □

Lemma 2.3 *Given $\alpha > 0$ and $\varphi \in L^2(\mathbb{R})$. Assume that $\{\varphi(\cdot - \alpha n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_φ^α, where V_φ^α is defined as in (1.1). Define*

$$T : L^2(0,\alpha) \to V_\varphi^\alpha \quad \text{by} \quad Tf(\cdot) = \sum_{n\in\mathbb{Z}} \langle f(\cdot), \alpha^{-\frac{1}{2}}e^{2\pi in\frac{\cdot}{\alpha}}\rangle \varphi(\cdot - \alpha n).$$

Then T is a bounded and invertible operator.

Proof Define $T_1 : L^2(0,\alpha) \to \ell^2(\mathbb{Z})$ by $T_1 f = \langle f(\cdot), \alpha^{-\frac{1}{2}}e^{2\pi in\frac{\cdot}{\alpha}}\rangle_{n\in\mathbb{Z}}$ for $f \in L^2(0,\alpha)$, and $T_2 : \ell^2(\mathbb{Z}) \to V_\varphi^\alpha$ by $T_2 c = \sum_{n\in\mathbb{Z}} c_n \varphi(\cdot - \alpha n)$. Then it is easy to check that both T_1 and T_2 are bounded and invertible. Also observing $T = T_2 T_1$ leads to this Lemma. □

Lemma 2.4 *Let $F(\cdot) = \sum_{k\in\mathbb{Z}} a_k e^{-2\pi ik\frac{\cdot}{\alpha}} \in L^2(0,\alpha)$ satisfy that $0 \le \|F(\cdot)\|_0 \le \|F(\cdot)\|_\infty \le \infty$, and let $\{F_n\}_{n\in\mathbb{Z}}$ be a sequence of functions in $L^2(0,\alpha)$ with Fourier expansions $F_n(\cdot) = \sum_{k\in\mathbb{Z}} a_k(n) e^{-2\pi ik\frac{x}{\alpha}}$. Define the infinite matrix $D = \{d_{n,k}\}_{n,k\in\mathbb{Z}}$ by $d_{n,k} := a_{n-k}(n) - a_{n-k}$, $n,k \in \mathbb{Z}$. Assume that $\|D\|_2 < \alpha^{\frac{1}{2}}\|F(\cdot)\|_0$. Then the sequence $\{F_n(\cdot)e^{2\pi in\frac{\cdot}{\alpha}}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,\alpha)$.*

Proof To this end we use the following proposition, which can be found in [5, p. 354]:

Let $\mathcal{H}$ be a Hilbert space, and let $\{f_k\}_{k=1}^\infty$ be a Riesz basis for $\mathcal{H}$ with Riesz bounds C_1 and C_2. Assume that $\{g_k\}_{k=1}^\infty$ is a sequence in $\mathcal{H}$, and that there exists a constant $R < C_1$ such that

$$\sum_{k=1}^\infty |\langle f_k - g_k, f\rangle|^2 \le R\|f\|^2$$

for $f \in \mathcal{H}$. Then $\{g_k\}_{k=1}^\infty$ is a Riesz basis for $\mathcal{H}$.

By Lemma 2.2, $\{F(\cdot)e^{2\pi in\frac{\cdot}{\alpha}}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,\alpha)$ with framebounds $\alpha\|F\|_0$ and $\alpha\|F\|_\infty$. For $f(\cdot) = \sum_{j\in\mathbb{Z}} \overline{c_j} e^{2\pi ij\frac{\cdot}{\alpha}}$ in $L^2(0,\alpha)$, it is easy to check that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\langle F_n(\cdot)e^{2\pi in\frac{\cdot}{\alpha}} - F(\cdot)e^{2\pi in\frac{\cdot}{\alpha}}, f\rangle|^2 &= \alpha^2 \sum_{n\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} (a_{n-k}(n) - a_{n-k})c_k|^2 \\ &= \alpha^2\|Dc\|^2_{\ell^2(\mathbb{Z})} \le \|D\|_2^2 \alpha^2 \|c\|^2_{\ell^2(\mathbb{Z})} = \|D\|_2^2\|f\|^2.\end{aligned}$$

Also observing that $\|D\|_2 < \alpha^{\frac{1}{2}}\|F(\cdot)\|_0$ leads to the lemma. □

3. Sampling theorems in V_φ^α

We are in a position to give the main results.

Theorem 3.1 *Given $\alpha > 0$, $0 \le a < \alpha$ and $\varphi \in L^2(\mathbb{R})$. Assume that φ is a continuous function satisfying $|\varphi(\cdot)| \le \frac{C}{(1+|\cdot|)^\beta}$ for some $C > 0$ and some $\beta > \frac{1}{2}$, where V_φ^α is defined as in (1.1). Assume further that $\{\varphi(\cdot - \alpha n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_φ^α. Define that $\widetilde{K}_a(\cdot) = \alpha^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} \overline{\varphi(a - \alpha n)} e^{2\pi in\frac{\cdot}{\alpha}}$. Then the following conditions are equivalent:*

(1) $0 < \|\widetilde{K}_a\|_0 \le \|\widetilde{K}_a\|_\infty < +\infty$;

(2) There exists $S_a \in V_\varphi^\alpha$ such that $\{S_a(\cdot - \alpha n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_φ^α and

$$f(\cdot) = \alpha^{-1} \sum_{n\in\mathbb{Z}} f(a+\alpha n) S_a(\cdot - \alpha n).$$

In this case, $S_a = T(\frac{1}{\widetilde{K}_a})$, where T is defined in Lemma 2.3.

Proof We first prove that (1) implies (2). Define $S_a = T(\frac{1}{\widetilde{K}_a})$. By Lemma 2.2, both $\{\frac{\alpha^{-1}}{\widetilde{K}_a(\cdot)} e^{2\pi i n \frac{\cdot}{\alpha}} : n \in \mathbb{Z}\}$ and $\{\widetilde{K}_a(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} : n \in \mathbb{Z}\}$ are Riesz basis for $L^2(0,\alpha)$, and it is obvious that they are mutually dual. Also observing that T is bounded and invertible, we have

$$T^{-1} f = \alpha^{-1} \sum_{n\in\mathbb{Z}} \langle T^{-1} f, \widetilde{K}_a(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} \rangle \frac{1}{\widetilde{K}_a(\cdot)} e^{2\pi i n \frac{\cdot}{\alpha}}, \quad f \in V_\varphi^\alpha. \tag{3.1}$$

By the definition of T, $Tg = \langle g(\cdot), \widetilde{K}_t(\cdot)\rangle$ for $g \in L^2(0,\alpha)$. It follows that

$$Tg(t+\alpha n) = \langle g(\cdot), \widetilde{K}_t(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} \rangle = T(e^{-2\pi i n \frac{\cdot}{\alpha}} g(\cdot))(t). \tag{3.2}$$

Put $g = T^{-1} f$. Then $\langle T^{-1} f, \widetilde{K}_a(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} \rangle = f(a+\alpha n)$ for $n \in \mathbb{Z}$. So, it follows from (3.1) and (3.2) that

$$f(\cdot) = \alpha^{-1} \sum_{n\in\mathbb{Z}} f(a+\alpha n) S_a(\cdot - \alpha n), \quad f \in V_\varphi^\alpha.$$

Substituting $g(\cdot) = \frac{1}{\widetilde{K}_a(\cdot)}$ into (3.2), we have $S_a(t-\alpha n) = T(\frac{e^{2\pi i n \frac{\cdot}{\alpha}}}{\widetilde{K}_a(\cdot)})(t)$ for $n \in \mathbb{Z}$. Also observing that $\{\frac{e^{2\pi i n \frac{\cdot}{\alpha}}}{\widetilde{K}_a(\cdot)} : n \in \mathbb{Z}\}$ is a Riesz basis for V_φ^α by Lemma 2.3, we have $\{S_a(\cdot - \alpha n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_φ^α.

Now we prove that (2) implies (1). Write $h = T^{-1} S_a$. For $F \in L^2(0,\alpha)$, we have $TF(\cdot) = \alpha^{-1} \sum_{n\in\mathbb{Z}} TF(a+\alpha n) S_a(\cdot - \alpha n)$. So by (3.2),

$$TF(\cdot) = \alpha^{-1} \sum_{n\in\mathbb{Z}} \langle F(\cdot), \widetilde{K}_a(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} \rangle T(h(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}}).$$

It follows that

$$F(\cdot) = \alpha^{-1} \sum_{n\in\mathbb{Z}} \langle F(\cdot), \widetilde{K}_a(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} \rangle h(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}}, \quad F \in L^2(0,\alpha). \tag{3.3}$$

Since $\{S_a(\cdot - \alpha n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_φ^α, by (3.2) and Lemma 2.3, $\{h(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} : n \in \mathbb{Z}\}$ is a Riesz basis for $L^2(0,\alpha)$. It together with (3.3) implies that $\{\alpha^{-1} \widetilde{K}_a(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} : n \in \mathbb{Z}\}$ is also a Riesz basis dual to $\{h(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} : n \in \mathbb{Z}\}$ for $L^2(0,\alpha)$. Then, by Lemma 2.2, $0 < \|\widetilde{K}_a(\cdot)\|_0 \le \|\widetilde{K}_a(\cdot)\|_\infty < \infty$. It is obvious that $\{\frac{1}{\widetilde{K}_a(\cdot)} e^{2\pi i n \frac{\cdot}{\alpha}} : n \in \mathbb{Z}\}$ is the dual of $\{\alpha^{-1} \widetilde{K}_a(\cdot) e^{2\pi i n \frac{\cdot}{\alpha}} : n \in \mathbb{Z}\}$. We therefore have $h(\cdot) = \frac{1}{\widetilde{K}_a(\cdot)}$, and thus $S_a = T(\frac{1}{\widetilde{K}_a(\cdot)})$. The proof is completed. □

Theorem 3.2 *Given $\alpha > 0$, $0 \le a < \alpha$ and $\varphi \in L^2(\mathbb{R})$. Assume that φ is a continuous function satisfying $|\varphi(\cdot)| \le \frac{C}{(1+|\cdot|)^\beta}$ for some $c > 0$ and some $\beta > \frac{1}{2}$, where V_φ^α is defined as in (1.1), that $\{\varphi(\cdot - \alpha n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_φ^α, and that $\Delta = \{\delta_n\}_{n\in\mathbb{Z}}$ is a sequence in $(-\alpha, \alpha)$ such that the infinite matrix $D_\Delta = \{d_{n,k}\}_{n\in\mathbb{Z}}$ whose entries are given by*

$$d_{n,k} := \overline{\varphi(a+\alpha(n-k)+\delta_n)} - \overline{\varphi(a+\alpha(n-k))}, \quad n,k \in \mathbb{Z},$$

satisfies $\|D_\Delta\|_2 < \alpha^{\frac{1}{2}}\|\widetilde{K}_a\|_0$. *Then there exists a Riesz basis* $\{S_n\}_{n\in\mathbb{Z}}$ *for* V_φ^α *such that*

$$f(\cdot) = \sum_{n\in\mathbb{Z}} f(a + \alpha n + \delta_n) S_n(t)$$

for $f \in V_\varphi^\alpha$.

Proof Applying Theorem 3.1 to

$$\widetilde{K}_a(\cdot) = \sum_{k\in\mathbb{Z}} \overline{\varphi(a+\alpha k)}\alpha^{-\frac{1}{2}} e^{-2\pi i k \frac{\cdot}{\alpha}}$$

and

$$\widetilde{K}_{a+\delta_n}(x) = \sum_{k\in\mathbb{Z}} \overline{\varphi(a+\alpha k+\delta_n)}\alpha^{-\frac{1}{2}} e^{-2\pi i k \frac{\cdot}{\alpha}}, \quad n \in \mathbb{Z},$$

we obtain that $\{\widetilde{K}_{a+\delta_n}(\cdot)e^{-2\pi i n \frac{\cdot}{\alpha}}\}_{n\in\mathbb{Z}} = \{\widetilde{K}_{a+\alpha n+\delta_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,\alpha)$. Denote by $\{\widetilde{G}_n\}_{n\in\mathbb{Z}}$ its dual Riesz basis. By Lemma 2.3, $\{S_n := T(\widetilde{G}_n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ^α. Now, given $f \in V_\varphi^\alpha$, we expand the function $F = T^{-1}(f) \in L^2(0,\alpha)$ with respect to $\{\widetilde{G}_n\}_{n\in\mathbb{Z}}$. Thus,

$$F = \sum_{n\in\mathbb{Z}} \langle F, \widetilde{K}_{a+\alpha n+\delta_n}\rangle_{L^2(0,\alpha)} = \sum_{n\in\mathbb{Z}} f(a+\alpha n+\delta_n)\widetilde{G}_n \in L^2(0,\alpha).$$

Applying the operator T, we get $f = \sum_{n\in\mathbb{Z}} f(a+\alpha n+\delta_n)T(\widetilde{G}_n)$ in $L^2(\mathbb{R})$. □

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