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Trigonometric Widths and Best N-Term Approximations of the Generalized Periodic Besov Classes $B^{\Omega}_{p,\theta}$

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Abstract In this paper, we determine the estimates exact in order for the trigonometric widths and the best n-term trigonometric approximations of the generalized classes of periodic functions $B^{\Omega}_{p,\theta}$ in the space L_q for some values of parameters p, q.

Keywords trigonometric width; best n-term approximation; generalized Besov classes.

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1. Introduction

The aim of this paper is to study the two approximation characters by trigonometric polynomials, i.e., trigonometric widths and best n-term trigonometric approximations. As other approximation characters, the two approximation characters have been widely investigated and some estimates exact in order of many classes of functions have been obtained. For best n-term trigonometric approximation, the most general form was introduced by Stechkin [1] in the study of the convergence of orthogonal series. The first estimates of the best trigonometric approximations for certain specific functions of one variable were obtained by Ismagilov [2]. For more results, one can refer to the papers cited in this paper and the references therein.

It is well known that, for the Sobolev, the Hölder-Nikol'skii, and the Besov classes, the behavior of the Kolmogorov widths in the sense of weak asymptotic order coincides with the behavior of trigonometric widths in all cases where exact orders of these widths are established. Recently, the corresponding different extensions of the Hölder-Nikol'skii, and the Besov classes have been introduced by some researchers. In 1994, Pustovoitov [3] introduced the function class $H^{\Omega}_q(T^d)$. He first used a standard function $\Omega(\mathbf{t})$, a prototype of which is $\Omega(\mathbf{t})=\mathbf{t}^{\mathbf{r}}:=t_1^{r_1}\cdots t_d^{r_d}$ as a majorant function for the mixed modulus of smoothness of order l of functions $f\in L_q$ instead of the standard function $\mathbf{t}^{\mathbf{r}}$ and obtained the estimates of best approximations of classes

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H_q^Ω with some special $\Omega(t_1,\ldots,t_d)$. In 1997, Wang [4] introduced the Besov class $B^\Omega_{q,\theta}(T^d)$ by means of $\Omega(\mathbf{t})$, i.e., an extension of the Besov class $S^{\mathbf{r}}_{q,\theta}(T^d)$, which was introduced first by Amanov [5] and gave the asymptotic estimates for Kolmogorov n-width of this class under the condition $\Omega(\mathbf{t}) = \omega(t_1,\ldots,t_d)$. Similar to [3] and [4], Xu [6] introduced the generalized Besov class $B^\Omega_{p,\theta}(T^d)$ which is an extension of the usual Besov space $B^\alpha_{p,\theta}(T^d)$ with $\Omega(t) = t^\alpha$ and obtained the estimates exact in order for the Kolmogorov, the Gel'fand and the linear widths of the classes $B^\Omega_{p,\theta}(T^d)$ in the space $L_q(T^d)$ for $1 \le p,q \le \infty$. Stasyuk already studied the trigonometric widths and the best n-term approximations of the classes $B^\Omega_{q,\theta}(T^d)$ given by Wang [4] in the space L_q for some values of parameters p,q in [7–9], respectively. In this paper, we shall investigate the behavior of trigonometric widths and best n-term trigonometric approximations of the classes $B^\Omega_{p,\theta}(T^d)$ given by Xu [6].

Throughout this paper, we will use the following notations $\ll$ and $\asymp$. For two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ of positive real numbers, we write $a_n \ll b_n$ provided that $a_n \le cb_n$ for some $c>0$. If, furthermore, also $b_n \ll a_n$, then we write $a_n \asymp b_n$.

This paper is organized as follows: In Section 2, we recall some notations and definitions on trigonometric width and best n-term approximation. Our main results will be stated in this section. In Section 3, we will give the proofs of the main results.

2. Preliminary and main results

In this section, we first recall some notations and definitions which will be used in the formulation and proofs of the main results.

Let X be a normed linear space and A a subset of X, and X_n be an n-dimensional subspace of X. The quantity

$$E(A,X) = \sup_{f\in A} e(f, X_n)$$

is called the deviation of A from X_n, where $e(f,X_n) = \inf_{g\in X_n} \|f(\cdot) - g(\cdot)\|_X$. Thus $E(A,X_n)$ measures the extent to which the "worst element" of A can be approximated from X_n.

The n-width, in the sense of Kolmogorov, of A in X is given by

$$d_n(A,X) = \inf_{X_n} \sup_{f\in A} \inf_{g\in X_n} \|f(\cdot) - g(\cdot)\|_X,$$

where the leftmost infimum is taken over all subspaces $X_n \subset X$ of dimension $\le n$. A subspace X_n of X of dimension at most n for which

$$d_n(A,X) = E(A,X_n)$$

is called an optimal subspace for $d_n(A,X)$.

The trigonometric width of A in X is defined by

$$d_n^T(A,X) = \inf_{\Theta_n} \sup_{f\in A} \inf_{t(\Theta_n,\mathbf{x})} \|f(\cdot) - t(\Theta_n,\cdot)\|_X,$$

where $t(\Theta_n,\mathbf{x}) = \sum_{j=1}^n c_j e^{i(\mathbf{k}^j,\mathbf{x})}$, $\Theta_n = \{\mathbf{k}^j\}_{j=1}^n$ is an arbitrary collection of vectors $\mathbf{k}^j = (k_1^j,\ldots,k_d^j)$ from the integer lattice Z^d.

The best n-term trigonometric approximation of A in X is given by

$$e_n(A,X) = \sup_{f\in A} \inf_{\Theta_n} \inf_{P(\Theta_n,\mathbf{x})} \|f(\cdot) - P(\Theta_n,\cdot)\|_X,$$

where $P(\Theta_n,\mathbf{x})$ are polynomials of the form $P(\Theta_n,\mathbf{x}) = \sum_{j=1}^n c_j e^{i(\mathbf{k}^j,\mathbf{x})}$, $\Theta_n = \{\mathbf{k}^j\}_{j=1}^n$ is a system of vectors $\mathbf{k}^j = (k_1^j,\ldots,k_d^j)$ from the integer lattice Z^d, and c_j are arbitrary coefficients.

From the above definitions, it is obvious that

$$d_n(A,X) \le d_n^T(A,X); \quad e_n(A,X) \le d_n^T(A,X). \tag{1}$$

We now describe the classes $B_{p,\theta}^{\Omega}(T^d)$ introduced by Xu [6] that will be investigated in this paper. Let $L_p(T^d)$ $(1 \le p \le \infty)$ denote the normed space of measurable functions on $T^d = (-\pi,\pi]^d$, which is 2π-periodic with respect to each variable with the usual norm $\|\cdot\|_p$. Suppose that $k \in N$, and $\mathbf{h} \in R^d$. For each $f \in L_p(T^d)$,

$$\Delta_{\mathbf{h}}^k f(\mathbf{x}) = \sum_{l=0}^{k} (-1)^{l+k} \begin{pmatrix} k \\ l \end{pmatrix} f(\mathbf{x} + l\mathbf{h})$$

is the k^{th} difference of the function f at the point $\mathbf{x}$ with step $\mathbf{h}$. The order k modulus of smoothness $\Omega_k(f,t)_p$ of f is defined by

$$\Omega_k(f,t)_p := \sup_{|\mathbf{h}|\le t} \|\Delta_{\mathbf{h}}^k f\|_p.$$

Definition 1 *Let Ω denote a non-negative function on $R_+ = \{t : t \ge 0\}$. We say that $\Omega(t) \in \Phi_k^*$ if it satisfies:*

1) $\Omega(0) = 0$; $\Omega(t) > 0$ for any $t > 0$;

2) $\Omega(t)$ is continuous;

3) $\Omega(t)$ is almost increasing, i.e., for any two points t, τ with $0 \le t \le \tau$, we have $\Omega(t) \le C\Omega(\tau)$, where $C \ge 1$ is a constant independent of t and τ;

4) For any $n \in Z_+$, $\Omega(nt) \le Cn^k\Omega(t)$, where $k \ge 1$ is a fixed positive integer, $C > 0$ is a constant independent of n and t;

5) There exists $\alpha > 0$ such that $\Omega(t)/t^\alpha$ is almost increasing;

6) There exists β, $0 < \beta < k$, such that $\Omega(t)/t^\beta$ is almost decreasing, i.e., there exists $C > 0$ such that for any two points $0 < t \le \tau$ there always holds $\Omega(t)/t^\beta \ge C\Omega(\tau)/\tau^\beta$.

Definition 2 *Let $k \in N$, $\Omega(t) \in \Phi_k^*$, $1 \le \theta \le \infty$, and $1 \le p \le \infty$. We say $f \in B_{p,\theta}^{\Omega}(T^d)$ if f satisfies the following conditions:*

1) $f \in L_p(T^d)$;

2) $\|f\|_{b_{p,\theta}^{\Omega}(T^d)} < \infty$, where

$$\|f\|_{b_{p,\theta}^{\Omega}(T^d)} = \begin{cases} \{\int_0^{+\infty} (\frac{\Omega_k(f,t)_p}{\Omega(t)})^\theta \frac{\mathrm{d}t}{t}\}^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_{t>0} \frac{\Omega_k(f,t)_p}{\Omega(t)}, & \theta = \infty. \end{cases}$$

The space $B_{p,\theta}^{\Omega}(T^d)$ is a normed linear space with the norm

$$\|f\|_{B_{p,\theta}^{\Omega}(T^d)} := \|f\|_p + \|f\|_{b_{p,\theta}^{\Omega}(T^d)}.$$

When $\Omega(t)=t^\alpha$, $B^\Omega_{p,\theta}(T^d)$ is the usual Besov space $B^\alpha_{p,\theta}(T^d)$. Let

$$S^\Omega_{p,\theta}(T^d):=\{f\in L_p(T^d):\|f\|_{B^\Omega_{p,\theta}(T^d)}\le 1\}.$$

Now we are in a position to state our main results of this paper.

Theorem 1 *Suppose that $k\in N$, $\Omega(t)\in\Phi^*_k$, $1\le p,q,\theta\le\infty$, $\Omega(t)/t^\alpha$ is almost increasing and $\alpha>d$. Then*

$$d^T_n(S^\Omega_{p,\theta}(T^d),L_q(T^d))\asymp\begin{cases}\Omega(n^{-1/d}), & 1\le q\le p\le\infty;\\ \Omega(n^{-1/d})n^{1/p-1/2}, & 1\le p<2\le q<p/(p-1);\\ \Omega(n^{-1/d})n^{1/p-1/q}, & 1\le p\le q\le 2.\end{cases}$$

Theorem 2 *Suppose that $k\in N$, $\Omega(t)\in\Phi^*_k$, $1\le\theta\le\infty$, $\Omega(t)/t^\alpha$ is almost increasing and $\alpha>d$. Then*

$$e_n(S^\Omega_{p,\theta}(T^d),L_q(T^d))\asymp\begin{cases}\Omega(n^{-1/d}), & 2<q<p<\infty;\\ \Omega(n^{-1/d}), & 2<p\le q<\infty;\\ \Omega(n^{-1/d})n^{1/p-1/2}, & 1<p\le 2<q<\infty;\\ \Omega(n^{-1/d})n^{1/p-1/q}, & 1<p\le q\le 2.\end{cases}$$

3. Proofs of main results

In this section, we will give the proofs of our main results. To this end, we need the following notations and auxiliary lemmas.

Let

$$V_m(t)=1+2\sum_{k=1}^m\cos kt+2\sum_{k=m+1}^{2m}((2m-k)/m)\cos kt$$

be the de la Vallée Poussin kernel. Then, the multi-dimensional de la Vallée Poussin kernel is defined by

$$V_m(\mathbf{x}):=\prod_{j=1}^d V_m(x_j)$$

for $m\in N$. For functions f on T^d, we consider the convolution operator $V_mf:=f*V_m$ defining the de la Vallée Poussin sum of f. The differences of successive de la Vallée Poussin sums are defined by

$$\Phi_0f:=V_1f,\ \Phi_sf:=V_{2^s}f-V_{2^{s-1}}f,\quad s=1,2,\dots.$$

For a vector $\mathbf{s}=(s_1,\dots,s_d)$ with nonnegative integer coordinates, we associate the set $\rho(\mathbf{s})$ of vectors $\mathbf{k}$ with integer coordinates

$$\rho(\mathbf{s})=\{\mathbf{k}=(k_1,\dots,k_d):[2^{s_j-1}]\le|k_j|<2^{s_j},\ j=1,\dots,d\}.$$

Below, we present several statements, which will be used in the proofs of our main results.

Lemma 1 ([6]) *If $k\in N$, $\Omega(t)\in\Phi^*_k$, $1\le p,\ \theta\le\infty$, and $f\in B^\Omega_{p,\theta}(T^d)$, then f can be represented* *in the form of a series*

$$f = \sum_{s=0}^{\infty} \Phi_s f$$

converging to it in the sense of $L_p(T^d)$, *and*

$$\|f\|_{B^{\Omega}_{p,\theta}(T^d)} \asymp \begin{cases} \{\sum\limits_{s\in Z_+} (\frac{\|\Phi_s f\|_p}{\Omega(2^{-s})})^\theta\}^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_{s\in Z_+} \frac{\|\Phi_s f\|_p}{\Omega(2^{-s})}, & \theta = \infty. \end{cases}$$

Lemma 2 ([6]) *Suppose that* $k \in N$, $\Omega(t) \in \Phi^*_k$, $1 \le p, q, \theta \le \infty$, $\Omega(t)/t^\alpha$, *is almost increasing, and* $\alpha > d\max(3\frac{1}{2} - \frac{3}{p}, \frac{1}{2} + \frac{3}{q}, \frac{1}{p} - \frac{1}{q})$. *Then*

$$d_n(S^{\Omega}_{p,\theta}(T^d), L_q(T^d)) \asymp \begin{cases} \Omega(n^{-1/d}), & 1 \le q \le p \le \infty; \\ \Omega(n^{-1/d}), & 2 \le p \le q \le \infty; \\ \Omega(n^{-1/d})n^{1/p-1/2}, & 1 \le p \le 2 \le q \le \infty; \\ \Omega(n^{-1/d})n^{1/p-1/q}, & 1 \le p \le q \le 2. \end{cases}$$

Lemma 3 ([10]) *Assume that* $2 \le q < \infty$. *Then for each trigonometric polynomial*

$$P(\Omega_M; \mathbf{x}) = \sum_{j=1}^{M} e^{i(\mathbf{k}^j, \mathbf{x})}$$

and each $N \le M$ *there exists a trigonometric polynomial* $P(\Omega_N; \mathbf{x})$ *containing at most* N *terms such that*

$$\|P(\Omega_M; \mathbf{x}) - P(\Omega_N; \mathbf{x})\|_q \ll MN^{-1/2};$$

in addition $\Omega_N \subset \Omega_M$ *and all the coefficients of* $P(\Omega_N; \mathbf{x})$ *are equal and have the estimate* MN^{-1}.

Let $t(\Omega_{N_\mathbf{s}}; \mathbf{x})$ be the trigonometric polynomial approximating the "block" $t_\mathbf{s}(\mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})}$ in accordance with Lemma 3. Consider the linear operator $T_\mathbf{s}$ acting on $f(\mathbf{x})$ by the formula

$$(T_\mathbf{s} f)(\mathbf{x}) = f(\mathbf{x}) * (\sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})} - t(\Omega_{N_\mathbf{s}}; \mathbf{x})).$$

Lemma 4 ([11]) *Assume that* $1 < p < 2 < q < p/(p-1)$. *Then the norm of the operator* $T_\mathbf{s}$ *from* L_p *into* L_q ($\|T_\mathbf{s}\|_{p\to q}$) *has the following estimate:*

$$\|T_\mathbf{s}\|_{p\to q} = \sup_{\|f\|_p \le 1} \|T_\mathbf{s} f\|_q \ll 2^{(\mathbf{s},\mathbf{1})} N_\mathbf{s}^{-(1/2+1/p')},$$

where $p' = p/(p-1)$.

Lemma 5 ([12]) *Let* $2 < q < \infty$. *For any trigonometric polynomial* $P(\Theta_N, \cdot)$ *that contains at most* N *harmonics and for any* $M < N$, *there exists a trigonometric polynomial* $P(\Theta_M, \cdot)$ *that has at most* M *nonzero coefficients and is such that*

$$\|P(\Theta_N, \cdot) - P(\Theta_M, \cdot)\|_q \le C\sqrt{\frac{N}{M}}\|P(\Theta_N, \cdot)\|_2$$

and furthermore, $\Theta_M \subset \Theta_N$ *and* $C > 0$.

Lemma 6 ([13], The Hausdorff-Young theorem) *Let $1 < p \le 2$. Then for any $f \in L_p$,*

$$(\sum_{\mathbf{k}} |\hat{f}(\mathbf{k})|^{p'})^{1/p'} \le \|f\|_p.$$

If a sequence $\{c_{\mathbf{k}}\}$ is such that $\sum_{\mathbf{k}} |c_{\mathbf{k}}|^p < \infty$, then there exists a function $f \in L_{p'}$ for which $\hat{f}(\mathbf{k}) = c_{\mathbf{k}}$ and

$$\|f\|_{p'} \le (\sum_{\mathbf{k}} |\hat{f}(\mathbf{k})|^p)^{1/p},$$

where $\hat{f}(\mathbf{k})$ are Fourier coefficients of f and $1/p + 1/p' = 1$.

With the help of the above auxiliary lemmas, we can establish our main results. First, let us prove Theorem 1.

Proof of Theorem 1 We begin with the estimates of the lower bounds. From the relation (1) and the corresponding estimates of the Kolmogorov widths $d_n(S^{\Omega}_{p,\theta}(T^d), L_q(T^d))$, i.e., Lemma 2, we can obtain the required lower bounds.

Now we pass to the estimates of the upper bounds. We will prove the upper bounds respectively.

Firstly, for the cases $1 \le q \le p \le \infty$ and $1 \le p \le q \le 2$, we can obtain the required upper bounds following the proof of the linear widths given in [6].

Secondly, we prove the upper bounds for $1 \le p < 2 \le q < p/(p-1)$.

Let $f(\mathbf{x})$ be an arbitrary function in the class $S^{\Omega}_{p,\theta}(T^d)$. By Lemma 1, we can represent it in the following form:

$$f(\mathbf{x}) = \sum_{s=0}^{\infty} \Phi_s f(\mathbf{x}).$$

Let n be fixed. Taking into account the condition $2^{md} \asymp n$, we choose $m \in N$. Set $\beta = (\alpha - d(1/p - 1/2))/(\alpha - d(1/p - 1/q))$. For each s satisfying $m \le s < \beta m$, we associate the quantity

$$N_s = [\Omega^{-1}(2^{-m})\Omega(2^{-s})2^{sd}] + 1,$$

where $[a]$ is the integer part of a. It is easy to see that the quantities N_s, $m \le s < \beta m$ satisfy the estimate

$$\sum_{m \le s < \beta m} N_s \ll 2^{md} \ll n.$$

We consider the approximation of the function $f(\mathbf{x})$ by the polynomial

$$t(\mathbf{x}) = \sum_{0 \le s < m} \Phi_s f(\mathbf{x}) + \sum_{m \le s < \beta m} (t(\Omega_{N_s}; \mathbf{x}) + t(\Omega_{N_{s+1}}; \mathbf{x})) * \Phi_s f(\mathbf{x})$$

where $t(\Omega_{N_s}; \mathbf{x})$, $t(\Omega_{N_{s+1}}; \mathbf{x})$ are the trigonometric polynomials approximating the "blocks" $t_s(\mathbf{x}) = \sum_{\mathbf{k} \in \rho(s)} e^{i(\mathbf{k},\mathbf{x})}$ and $t_{s+1}(\mathbf{x}) = \sum_{\mathbf{k} \in \rho(s+1)} e^{i(\mathbf{k},\mathbf{x})}$ in accordance with Lemma 3 and

$$\rho(s) = \{\mathbf{k} = (k_1, \ldots, k_d) \in Z^d : 2^{s-1} \le |k_j| < 2^s, \quad j = 1, \ldots, d\}.$$

We can verify that $t(\mathbf{x})$ brings about the required estimate of the approximation for $f(\mathbf{x})$. In fact, we have

$$\begin{aligned}&\|f(\mathbf{x})-t(\mathbf{x})\|_q\\ &\le \|\sum_{m\le s<\beta m}(\Phi_s f(\mathbf{x})-\Phi_s f(\mathbf{x})*(t(\Omega_{N_s};\mathbf{x})+t(\Omega_{N_{s+1}};\mathbf{x})))\|_q+\|\sum_{s\ge\beta m}\Phi_s f(\mathbf{x})\|_q\\ &:= J_1+J_2.\end{aligned}$$

In order to continue the upper estimates, we need to estimate J_1 and J_2 separately.

First, let us estimate J_2. For $1<\theta<\infty$, we get

$$\begin{aligned}J_2&\le\sum_{s\ge\beta m}\|\Phi_s f(\mathbf{x})\|_q \ll \sum_{s\ge\beta m}\|\Phi_s f(\mathbf{x})\|_p 2^{sd(1/p-1/q)}\\ &\ll \frac{\Omega(2^{-m})}{2^{-\alpha m}}(\sum_{s\ge\beta m}2^{(sd(1/p-1/q)-\alpha s)\theta'})^{1/\theta'}(\sum_{s\ge\beta m}(\frac{\|\Phi_s f(\mathbf{x})\|_p}{\Omega(2^{-s})})^\theta)^{1/\theta}\\ &\ll \frac{\Omega(2^{-m})}{2^{-\alpha m}}2^{(d(1/p-1/q)-\alpha)\beta m},\end{aligned}\tag{2}$$

which together with $\beta=(\alpha-d(1/p-1/2))/(\alpha-d(1/p-1/q))$ and $2^{md}\asymp n$ implies

$$J_2\ll\Omega(2^{-m})2^{md(1/p-1/2)}\ll\Omega(n^{-1/d})n^{1/p-1/2}.$$

For $\theta=1,\infty$, the estimate (2) is also true.

Next, we proceed to estimate J_1. For this purpose, for each $s\in N$ satisfying $m\le s<\beta m$ we consider the linear operator T_s acting on $f(\mathbf{x})$ by the formula

$$(T_s f)(\mathbf{x})=f(\mathbf{x})*(\sum_{\mathbf{k}\in\rho(s)\cup\rho(s+1)}e^{i(\mathbf{k},\mathbf{x})}-t(\Omega_{N_s};\mathbf{x})-t(\Omega_{N_{s+1}};\mathbf{x})).$$

We will divide two cases $p>1$ and $p=1$ to estimate J_1.

In the case $p>1$ and $1<\theta<\infty$, according to Lemma 4 and $2^{md}\asymp n$, we can obtain

$$\begin{aligned}J_1&\le\sum_{m\le s<\beta m}\|\Phi_s f(\mathbf{x})-\Phi_s f(\mathbf{x})*(t(\Omega_{N_s};\mathbf{x})+t(\Omega_{N_{s+1}};\mathbf{x}))\|_q\\ &=\sum_{m\le s<\beta m}\|\Phi_s f(\mathbf{x})*(\sum_{\mathbf{k}\in\rho(s)\cup\rho(s+1)}e^{i(\mathbf{k},\mathbf{x})}-t(\Omega_{N_s};\mathbf{x})-t(\Omega_{N_{s+1}};\mathbf{x}))\|_q\\ &\ll\sum_{m\le s<\beta m}\|T_s\|_{p\to q}\|\Phi_s f(\mathbf{x})\|_p\\ &\ll\sum_{m\le s<\beta m}2^{sd}N_s^{-(1/2+1/p')}\|\Phi_s f(\mathbf{x})\|_p\\ &\ll\Omega^{1/2+1/p'}(2^{-m})\sum_{m\le s<\beta m}2^{sd(1/p-1/2)}\Omega^{1/p-1/2}(2^{-s})\frac{\|\Phi_s f(\mathbf{x})\|_p}{\Omega(2^{-s})}\\ &\ll\Omega(2^{-m})2^{md(1/p-1/2)}\ll\Omega(n^{-1/d})n^{1/p-1/2}.\end{aligned}\tag{3}$$

For $\theta=1,\infty$, the estimate (3) also holds.

In the case $p=1$, let p_0 be a quantity such that $1<p_0<2$, which we specify below. Then for J_1 we can write (see (3))

$$\begin{aligned} J_1 &\leq \sum_{m\leq s<\beta m} \|\Phi_s f(\mathbf{x}) * (\sum_{\mathbf{k}\in\rho(s)\cup\rho(s+1)} e^{i(\mathbf{k},\mathbf{x})} - t(\Omega_{N_s};\mathbf{x}) - t(\Omega_{N_{s+1}};\mathbf{x}))\|_q \\ &\ll \sum_{m\leq s<\beta m} \|T_s\|_{p_0\to q}\|\Phi_s f(\mathbf{x})\|_{p_0} \\ &\ll \sum_{m\leq s<\beta m} 2^{sd} N_s^{-(1/2+1/p_0')}\|\Phi_s f(\mathbf{x})\|_{p_0} \\ &\ll \sum_{m\leq s<\beta m} 2^{sd} N_s^{-(1/2+1/p_0')} 2^{sd(1-1/p_0)}\|\Phi_s f(\mathbf{x})\|_1 \\ &\ll \frac{\Omega(2^{-m})}{2^{-\alpha m(1/2-1/p_0')}} \sum_{m\leq s<\beta m} 2^{s[d/2-\alpha(1/2-1/p_0')]}\frac{\|\Phi_s f(\mathbf{x})\|_1}{\Omega(2^{-s})}. \end{aligned}$$

We now choose p_0 such that $d/2-\alpha(1/2-1/p_0')<0$. Clearly, there exists such $p_0\in(1,2)$ by the hypothesis of Theorem 1. Then we have

$$J_1 \ll \Omega(2^{-m})2^{md/2} \ll \Omega(n^{-1/d})n^{1/2}.$$

The required upper estimates are established. This completes the proof of Theorem 1. □

In the following, we give the proof of Theorem 2.

Proof of Theorem 2 We start with the estimates of the upper bounds. For $1<p\leq q\leq 2$, according to the relation (1) and Theorem 1, we have

$$e_n(S^{\Omega}_{p,\theta}(T^d), L_q(T^d)) \ll \Omega(n^{-1/d})n^{1/p-1/q}.$$

For the remaining three cases, by the relation

$$e_n(S^{\Omega}_{p,\theta}(T^d), L_q(T^d)) \leq e_n(S^{\Omega}_{2,\theta}(T^d), L_q(T^d)), \quad p\geq 2,$$

note that it suffices to prove the upper bounds for $1<p\leq 2<q<\infty$.

Let $f(\mathbf{x})$ be an arbitrary function from the class $S^{\Omega}_{p,\theta}(T^d)$. We represent it in the following form:

$$f(\mathbf{x}) = \sum_{s<m}\Phi_s f(\mathbf{x}) + \sum_{m\leq s<\beta m}\Phi_s f(\mathbf{x}) + \sum_{s\geq\beta m}\Phi_s f(\mathbf{x}), \tag{4}$$

where $\beta>1$ is a certain real number (we fix it below).

Let $n\in N$ be fixed. We choose m such that $2^{md}\asymp n$. For each $s\in N$ satisfying $m\leq s<\beta m$ we associate the quantity

$$N_s = [\Omega^{-1}(2^{-m})\Omega(2^{-s})2^{sd}]+1,$$

where $[a]$ is the integer part of a and show that

$$\sum_{m\leq s<\beta m} N_s \ll 2^{md} \ll n.$$

We consider the approximation of the function $f(\mathbf{x})$ by the polynomial

$$P(\Theta_n,\mathbf{x}) = \sum_{s<m}\Phi_s f(\mathbf{x}) + \sum_{m\leq s<\beta m} P(\Theta_{N_s},\mathbf{x}), \tag{5}$$

in which, according to Lemma 5, for every $\Phi_s f(\mathbf{x})$, there exists a polynomial $P(\Theta_{N_s}, \mathbf{x})$ such that

$$\|\Phi_s f(\mathbf{x}) - P(\Theta_{N_s}, \mathbf{x})\|_q \ll (\frac{2^{sd}}{N_s})^{1/2} \|\Phi_s f(\mathbf{x})\|_2.$$

Taking into account (4) and (5), we have

$$\begin{aligned}\|f(\mathbf{x}) - P(\Theta_n, \mathbf{x})\|_q &= \|\sum_{s \geq m} \Phi_s f(\mathbf{x}) - \sum_{m \leq s < \beta m} P(\Theta_{N_s}, \mathbf{x})\|_q \\ &\leq \|\sum_{m \leq s < \beta m} (\Phi_s f(\mathbf{x}) - P(\Theta_{N_s}, \mathbf{x}))\|_q + \|\sum_{s \geq \beta m} \Phi_s f(\mathbf{x})\|_q \\ &:= I_1 + I_2. \end{aligned} \tag{6}$$

Further, we estimate each term in (6) separately. For the second term, taking $\beta = (\alpha - d(1/p - 1/2))/(\alpha - d(1/p - 1/q))$, just as the estimate of J_2 in Theorem 1, we have

$$I_2 \ll \Omega(2^{-m}) 2^{md(1/p-1/2)} \ll \Omega(n^{-1/d}) n^{1/p-1/2}.$$

Now let us estimate I_1. According to Lemma 5, the Nikolskii inequality of different metrics for $\Phi_s f(\mathbf{x})$ and $2^{md} \asymp n$, we get

$$\begin{aligned} I_1 &\leq \sum_{m \leq s < \beta m} \|\Phi_s f(\mathbf{x}) - P(\Theta_{N_s}, \mathbf{x})\|_q \\ &\ll \frac{\Omega(2^{-m})}{2^{-\alpha m/2}} \sum_{m \leq s < \beta m} 2^{s[d(1/p-1/2)-\alpha/2]} \frac{\|\Phi_s f(\mathbf{x})\|_p}{\Omega(2^{-s})} \\ &\ll \Omega(2^{-m}) 2^{md(1/p-1/2)} \ll \Omega(n^{-1/d}) n^{1/p-1/2}. \end{aligned}$$

In the proof of the lower bounds, we use the following duality relation ([14, p. 42]):

$$e_n(f)_q = \inf_{\Theta_n} \sup_{\substack{P \in L^{\perp}(\Theta_n) \\ \|P\|_{q'} \leq 1}} \int_{T^d} f(\mathbf{x}) P(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{7}$$

where $L^{\perp}(\Theta_n)$ is the set of functions orthogonal to the subspace of trigonometric polynomials with "indices" of harmonics from the set $\Theta_n = \{\mathbf{k}^j\}_{j=1}^n$ and $1/q + 1/q' = 1$. We will divide our consideration into three cases.

Firstly, for $1 < p \leq 2 < q < \infty$, note that it suffices to establish the lower bounds in the case $q = 2$. In this case, we need to construct a special function which belongs to the class $S^{\Omega}_{p,\theta}(T^d)$ and a concrete function $P(\mathbf{x})$ which satisfies the conditions of the right-side of relation (7). For this purpose, we need the following notations.

For fixed $n \in N$, we can choose $m \in N$ such that for the number of elements of the set

$$\rho(m) = \{\mathbf{k} = (k_1, \ldots, k_d) \in Z^d : 2^{m-1} \leq |k_j| < 2^m, \quad j = 1, \ldots, d\},$$

we have $|\rho(m)| \geq 2n$ and $|\rho(m)| = 2^{md} \asymp n$. Consider the polynomial

$$D_m(\mathbf{x}) = \sum_{\mathbf{k} \in \rho(m)} e^{i(\mathbf{k}, \mathbf{x})}.$$

It is known that the convolution operator V_m (see [13, p. 92]) is bounded, i.e.,

$$\|V_m\|_{p \to p} \leq 3^d, \quad 1 \leq p \leq \infty.$$

From this, it is easy to find that the operators Φ_s, $s \in Z_+$ are also bounded.

Taking into account the relation [15, p. 214]

$$\|\sum_{k=n}^{l} \cos kx\|_p \asymp (l-n)^{1-1/p}, \quad \forall n, l \in N, \quad l > n, \quad p \in (1,\infty),$$

and $\|\Phi_s\|_{p\to p} \ll 3^d$, we get

$$\|D_m\|_{B^{\Omega}_{p,\theta}(T^d)} \asymp \{(\frac{\|\Phi_{m-1}D_m\|_p}{\Omega(2^{-(m-1)})})^\theta + (\frac{\|\Phi_m D_m\|_p}{\Omega(2^{-m})})^\theta\}^{1/\theta}$$
$$\ll \frac{\|D_m\|_p}{\Omega(2^{-m})} \ll 2^{md(1-1/p)}/\Omega(2^{-m}).$$

For $\theta = \infty$, we also have

$$\|D_m\|_{B^{\Omega}_{p,\infty}(T^d)} \ll 2^{md(1-1/p)}/\Omega(2^{-m}).$$

The above estimates imply that the function

$$f_0(\mathbf{x}) = C\Omega(2^{-m})2^{md(1/p-1)}D_m(\mathbf{x}), \quad C > 0,\ 1 \le \theta \le \infty, \tag{8}$$

belongs to the classes $S^{\Omega}_{p,\theta}(T^d)$.

Now we construct a function $P(\mathbf{x})$ that satisfies the conditions of the right-side of relation (7). Consider the polynomial

$$F(\mathbf{x}) = D_m(\mathbf{x}) - \sum_{\mathbf{k}^j\in\Theta_n}{}' e^{i(\mathbf{k}^j,\mathbf{x})},$$

where the prime means that the summation is carried out only over $\mathbf{k}^j \in \Theta_n$ that are contained in $\rho(m)$. If $2 < q < \infty$, then $1 < q' < 2$ and

$$\|F(\cdot)\|_{q'} \le \|F(\cdot)\|_2 \le \|D_m(\cdot)\|_2 + \|\sum_{\mathbf{k}^j\in\Theta_n}{}' e^{i(\mathbf{k}^j,\cdot)}\|_2.$$

Further, let us estimate each term in the last inequality. By virtue of the Parsevale equality, we have

$$\|\sum_{\mathbf{k}^j\in\Theta_n}{}' e^{i(\mathbf{k}^j,\cdot)}\|_2 = (\sum_{\mathbf{k}^j\in\Theta_n}{}' 1)^{1/2} \le \sqrt{n}$$

and

$$\|D_m(\cdot)\|_2 = |\rho(m)|^{1/2} \asymp 2^{md/2}.$$

Hence we get

$$\|F(\cdot)\|_{q'} \ll 2^{md/2} + \sqrt{n}.$$

Let

$$P(\mathbf{x}) = C_1(2^{md/2} + \sqrt{n})^{-1}F(\mathbf{x}), \quad C_1 > 0, \tag{9}$$

which satisfies the conditions $\|P(\cdot)\|_q' \le 1$, $P(\cdot) \in L^{\perp}(\Theta_n)$.

Substituting (8) and (9) in (7), we obtain the following relation:

$$e_n(S^{\Omega}_{p,\theta}(T^d), L_q(T^d)) \ge e_n(S^{\Omega}_{p,\theta}(T^d), L_2(T^d))$$

$$\gg \int_{T^d} \Omega(2^{-m}) 2^{md(1/p-1)} D_m(\mathbf{x}) \frac{D_m(\mathbf{x}) - {\sum_{\mathbf{k}^j\in\Theta_n}}' e^{i(\mathbf{k}^j,\mathbf{x})}}{2^{md/2}+\sqrt{n}} d\mathbf{x}$$
$$= \frac{\Omega(2^{-m})2^{md(1/p-1)}}{2^{md/2}+\sqrt{n}} (\int_{T^d} D_m^2(\mathbf{x})d\mathbf{x} + \int_{T^d} D_m(\mathbf{x}) {\sum_{\mathbf{k}^j\in\Theta_n}}' e^{i(\mathbf{k}^j,\mathbf{x})} d\mathbf{x})$$
$$\gg \Omega(2^{-m})2^{md(1/p-3/2)} (\|D_m(\cdot)\|_2^2 - \| {\sum_{\mathbf{k}^j\in\Theta_n}}' e^{i(\mathbf{k}^j,\cdot)}\|_2^2)$$
$$\gg \Omega(2^{-m})2^{md(1/p-3/2)}(2^{md}-n)$$
$$\gg \Omega(2^{-m})2^{md(1/p-1/2)} \gg \Omega(n^{-1/d})n^{1/p-1/2}.$$

We finish the proof of the lower bounds for $1 < p \le 2 < q < \infty$.

Secondly, for $2 < q < p < \infty$ and $2 < p \le q < \infty$, we deal with the lower estimates using the duality relation (7). The idea is similar to the above proof. In this case, we shall use the following Rudin-Shaprio [16, p. 155] to construct the functions we need.

For every $s_j \in N$, there exists a polynomial

$$R_{s_j}(x_j) = \sum_{k_j=2^{s_j-1}}^{2^{s_j}-1} \varepsilon_{k_j} e^{i(k_j,x_j)}, \quad \varepsilon_{k_j} = \pm 1,$$

such that

$$\|R_{s_j}(x_j)\|_\infty \ll 2^{s_j/2}. \tag{10}$$

For given $n \in N$, we choose m such that, for the number of elements of the set $F_m = \bigcup_{1\le s\le m}\rho(s)$, we have $|F_m| \ge 2n$ and $|F_m| \asymp 2^{md} \asymp n$. Consider the function

$$g(\mathbf{x}) = \sum_{1\le s\le m}\prod_{j=1}^d R_s(x_j).$$

For $1 \le \theta < \infty$, taking into account (10) and $\|\Phi_s\|_{p\to p} \ll 3^d$, we get

$$\|g\|_{B^\Omega_{p,\theta}(T^d)} \asymp \{\sum_{1\le s\le m} (\frac{\|\Phi_s g\|_p}{\Omega(2^{-s})})^\theta\}^{1/\theta}$$
$$= \{\sum_{1\le s\le m} (\Omega^{-1}(2^{-s})\|\Phi_s(\prod_{j=1}^d R_s(x_j) + \prod_{j=1}^d R_{s+1}(x_j))\|_p)^\theta\}^{1/\theta}$$
$$\ll \Omega^{-1}(2^{-m})\{\sum_{1\le s\le m} (\|\prod_{j=1}^d R_s(x_j)\|_p + \|\prod_{j=1}^d R_{s+1}(x_j)\|_p)^\theta\}^{1/\theta}$$
$$\ll \Omega^{-1}(2^{-m})\{\sum_{1\le s\le m} 2^{sd\theta/2}\}^{1/\theta}$$
$$\ll \Omega^{-1}(2^{-m})2^{md/2}.$$

For $\theta = \infty$, we have

$$\|g\|_{B^\Omega_{p,\infty}(T^d)} \asymp \sup_{1\le s\le m}\frac{\|\Phi_s g\|_p}{\Omega(2^{-s})} = \sup_{1\le s\le m}\Omega^{-1}(2^{-s})\|\Phi_s(\prod_{j=1}^d R_s(x_j) + \prod_{j=1}^d R_{s+1}(x_j))\|_p$$

$$\ll \Omega^{-1}(2^{-m}) \sup_{1\le s\le m} \| \prod_{j=1}^{d} R_s(x_j)\|_p$$
$$\ll \Omega^{-1}(2^{-m}) 2^{md/2}.$$

Thus the function

$$f_1(\mathbf{x}) = C_2\Omega(2^{-m})2^{-md/2}g(\mathbf{x}), \quad C_2>0,\ 1\le\theta\le\infty, \tag{11}$$

belongs to the classes $S^{\Omega}_{p,\theta}(T^d)$.

We now construct a function $P(\mathbf{x})$. We set

$$F(\mathbf{x}) = g(\mathbf{x}) - \sum_{1\le s\le m}{}' \prod_{j=1}^{d} R_s(x_j),$$

where the prime means that the sum contains only the harmonics of the function $g(\mathbf{x})$ with "indices" from Θ_n.

Taking into account that $1<q'<2$, we get

$$\|F(\cdot)\|_{q'} \le \|F(\cdot)\|_2 \le \|g(\cdot)\|_2 + \| \sum_{1\le s\le m}{}' \prod_{j=1}^{d} R_s(x_j)\|_2 \ll 2^{md/2}+\sqrt{n}.$$

Then

$$P(\mathbf{x}) = C_3F(\mathbf{x})/(2^{md/2}+\sqrt{n}), \quad C_3>0, \tag{12}$$

satisfies the conditions of the right-side of relation (7).

Substituting (11) and (12) into (7), we obtain

$$e_n(S^{\Omega}_{p,\theta}(T^d), L_q(T^d)) \ge \frac{\Omega(2^{-m})2^{-md/2}}{2^{md/2}+\sqrt{n}}\Big(\int_{T^d} g^2(\mathbf{x})d\mathbf{x} - \int_{T^d} g(\mathbf{x}) \sum_{1\le s\le m}{}' \prod_{j=1}^{d} R_s(x_j)d\mathbf{x}\Big)$$
$$\gg \Omega(2^{-m})2^{-md}\cdot$$
$$\Big(\|g(\cdot)\|_2^2 - \int_{T^d} \sum_{\substack{1\le s\le m\\ \mathbf{k}^j\overline{\in}\Theta_n}} \prod_{j=1}^{d} R_s(x_j) \sum_{\substack{1\le s\le m\\ \mathbf{k}^j\in\Theta_n}} \prod_{j=1}^{d} R_s(x_j)d\mathbf{x} - \int_{T^d}\Big(\sum_{1\le s\le m}{}' \prod_{j=1}^{d} R_s(x_j)\Big)^2 d\mathbf{x}\Big)$$
$$\gg \Omega(2^{-m})2^{-md}(2^{md}-n) \gg \Omega(n^{-1/d}).$$

Thirdly, for $1<p\le q\le 2$, we will give the lower estimates by the definition of best n-term trigonometric approximation. By virtue of the function $f_0 = C\Omega(2^{-m})2^{md(1/p-1)}D_m(\mathbf{x}) \in S^{\Omega}_{p,\theta}(T^d)$ given above where $D_m(\mathbf{x}) = \sum_{\mathbf{k}\in\rho(m)} e^{i(\mathbf{k},\mathbf{x})}$ with $|\rho(m)|\ge 2n$, $|\rho(m)|\asymp 2^{md}\asymp n$ and Lemma 6, we can obtain

$$e_n(S^{\Omega}_{p,\theta}(T^d), L_q(T^d)) \ge e_n(f_0, L_q(T^d)) = \inf_{\Theta_n}\inf_{P(\Theta_n,x)} \|f_0(\cdot)-P(\Theta_n,\cdot)\|_q$$
$$\ge \inf_{\Theta_n}\inf_{P(\Theta_n,x)} \Big(\sum_k |\hat{f}_0(k)-\hat{P}(\Theta_n,k)|^{q'}\Big)^{1/q'}$$
$$\gg \Omega(2^{-m})2^{md(1/p-1)}2^{md/q'}$$
$$\gg \Omega(n^{-1/d})n^{1/p-1/q},$$

where $\hat{P}(\Theta_n, k)$ denote Fourier coefficients of $P(\Theta_n, x)$.

The proof of Theorem 2 is completed. $\square$

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