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On Small Time Large Deviation Principle for Diffusion Processes on Hilbert Spaces under Non-Lipschitzian Condition

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Abstract Under the non-Lipschitzian condition, a small time large deviation principle of diffusion processes on Hilbert spaces is established. The operator theory and Gronwall inequality play an important role.

Keywords small time large deviation principle; stochastic evolution equation; non-Lipschitzian condition; rate function; Itô formula.

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1. Introduction

In [1, 2], small time asymptotics were obtained for the standard Ornstein-Uhlenbeck process on classical Wiener space and general Ornstein-Uhlenbeck process with unbounded linear drifts. Zhang [3] established the small time large deviation principle and the small time asymptotics for diffusion processes on Hilbert spaces under the Lipschitzian condition. In this paper, we further extend a small time large deviation principle of Zhang [3] to the case of the non-Lipschitzian condition. For the proof of the conclusion, our idea is to construct a family of positive increasing function $(\Phi_\rho)_{\rho>0}$ on $\mathbf{R}_+$ so that the Gronwall inequality can be applied. In fact, this idea is also taken in [4–6].

Let H be a separable Hilbert space and E be another separable Hilbert space such that H is imbedded in E densely and continuously and imbedding is Hilbert-Schmidt. Let μ be a mean zero Gaussian measure on $(E, \mathcal{B}(E))$ with the reproducing kernel space H, where $\mathcal{B}(E)$ denotes the Borel σ-field. The (H, E, μ) is an abstract Wiener space in the sense of Gross. More generally, to cover solutions of stochastic evolution equations, let A be a self-adjoint operator on H. The associated semigroup is denoted by $T_t = e^{-tA}$. Define $H_0 = D(\sqrt{A})$ with inner product $\langle h_1, h_2\rangle_{H_0} = \langle \sqrt{A}h_1, \sqrt{A}h_2\rangle_H$. In next section, we introduce the small time large deviation

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principle for diffusion processes on Hilbert spaces. Finally, the proof of main theorem is given in Section 3.

2. A small time large deviation principle

In this section, we introduce a small time large principle for solutions of a class of stochastic equations under the non-Lipschitzian condition. This corresponds not only to small noise but also small drift perturbation where an unbounded operator A and unbounded drifts are involved.

Let $W_t,\ t \geq 0$ be an E-valued Brownian motion with the reproducing Hilbert space H_0 defined on some probability space $(\Omega, \mathcal{F}_t, P)$. Let $L_{(2)}(H_0, H)$ denote the set of all Hilbert-Schmidt operators from H_0 into H with the Hilbert-Schmidt norm $\|\cdot\|_{(2)}$. Given $x \in E$, we consider the following stochastic evolution equation

$$u_t = x - \int_0^t Au_s \mathrm{d}s + \int_0^t b(u_s)\mathrm{d}s + \int_0^t \sigma(u_s)\mathrm{d}W_s. \tag{1}$$

In general, $u_t,\ t > 0$, will not belong to the domain of A and Eq.(1) is interpreted in the following sense

$$u_t = T_t x + \int_0^t T_{(t-s)}(b(u_s))\mathrm{d}s + \int_0^t T_{(t-s)}\sigma(u_s)\mathrm{d}W_s.$$

In what follows, we assume

(I) $b : E \to E, \sigma : E \to L_{(2)}(H_0, H)$ satisfy the non-Lipschitzian condition

$$|b(x) - b(y)|_E \leq c_2|x-y|_E r(|x-y|_E^2), \quad \|\sigma(x) - \sigma(y)\|_{(2)}^2 \leq c_1|x-y|_E^2 r(|x-y|_E^2),$$

where $r : (0,1) \to \mathbf{R}_+$, is a $\mathcal{C}^1$-function satisfying the conditions

(i) $\lim_{\eta\to 0} r(\eta) = +\infty$, $\eta r(\eta)$ is an increasing function and $\lim_{\eta \to 0} \eta r(\eta) = 0$;

(ii) $\lim_{\eta\to 0} \frac{\eta r'(\eta)}{r(\eta)} = 0$;

(iii) Define $\psi_\theta(a) = \int_0^a \frac{\mathrm{d}s}{sr(s)+\theta}$, $\forall a, \theta \geq 0$. It follows that

$$\psi_0(a) = +\infty,\ \lim_{\theta\to 0} \theta^2\psi_\theta(a) = +\infty,\ a > 0.$$

(II) $|b(x)|_E \leq c_2 + c_3|x|_E$, $\sup_x \|\sigma(x)\|_{(2)} \leq M$, where c_1, c_2, c_3 and M are constants.

Similarly to the discussion of Fei [6], we can prove the existence and uniqueness of Eq. (1) under non-Lipschitzian condition. Let $\varepsilon > 0$. It is easy to see that the process $u_{\varepsilon t}$ coincides in law with the solution of the following equation

$$u_t^\varepsilon = T_{\varepsilon t}x + \varepsilon\int_0^t T_{\varepsilon(t-s)}(b(u_s^\varepsilon))\mathrm{d}s + \varepsilon^{1/2}\int_0^t T_{\varepsilon(t-s)}\sigma(u_s^\varepsilon)\mathrm{d}W_s.$$

Let μ_ε^x be the law of $u_\cdot^\varepsilon$ on $C([0,1] \to E)$ by

$$I(f) = \inf_{h\in\Gamma_f}\{\frac{1}{2}\int_0^1 |\dot{h}(t)|_{H_0}^2 \mathrm{d}t\},$$

where

$$\Gamma_f = \{h \in C([0,1] \to H_0);\ h \text{ is absolutly continuous and such that } f(t) = x + \int_0^t \sigma(f(s))\dot{h}(s)\mathrm{d}s, 0 \leq t \leq 1\}.$$

We state the main result in this paper.

Theorem *Assume that the coefficients of Eq. (1) satisfy the conditions (I) and (II). Then μ_ε^x satisfies a large deviation principle with the rate function $I(\cdot)$, that is:*

(1) For any closed set F,

$$\lim_{\varepsilon\to 0}\sup_{x_n\to x}\varepsilon\log\mu_\varepsilon^{x_n}(F)\le -\inf_{f\in F}(I(f)).$$

(2) For any open set G,

$$\lim_{\varepsilon\to 0}\inf_{x_n\to x}\varepsilon\log\mu_\varepsilon^{x_n}(G)\ge -\inf_{f\in G}(I(f)).$$

3. The proof of the small time large deviation principle

While proceeding, we provide the several lemmas for completing the proof of the small time large deviation principle under the non-Lipschitzian condition.

The Proof of Theorem Let ν_ε be the law of solution $v_\cdot^\varepsilon$ of the following stochastic equation

$$v_t^\varepsilon = x+\varepsilon^{1/2}\int_0^t\sigma(v_s^\varepsilon)\mathrm{d}W_s,\quad t\ge 0.$$

Then it is known (see, Da Prato and Zabczyk [7]) that ν_ε satisfies a large deviation principle on $C([0,1]\to E)$ with the rate function $I(\cdot)$. Thus, by Theorem 4.2.13 in [8], it suffices to show that the two families $\{\mu_\varepsilon\}$, $\{\nu_\varepsilon\}$ of probability measures are so-called exponentially equivalent. That is, the following proposition holds:

Proposition *Assume that the coefficients of Eq. (1) satisfy the conditions (I) and (II). For any $\delta>0$, we have*

$$\lim_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|u_t^\varepsilon-v_t^\varepsilon|_E>\delta)=-\infty.$$

Proof Let

$$Y_t^\varepsilon=(T_{\varepsilon t}x-x)+\varepsilon\int_0^t T_{\varepsilon(t-s)}(b(u_s^\varepsilon))\mathrm{d}s+\varepsilon^{1/2}\int_0^t(T_{(t-s)}-I)\sigma(u_s^\varepsilon)\mathrm{d}W_s,$$

$$Z_t^\varepsilon=\varepsilon^{1/2}\int_0^t(\sigma(u_s^\varepsilon)-\sigma(v_s^\varepsilon))\mathrm{d}W_s.$$

We have $u_t^\varepsilon-v_t^\varepsilon=Y_t^\varepsilon+Z_t^\varepsilon$. We need the following two lemmas for the proof of Proposition.

Lemma 1 *Let $\delta>0$. Then*

$$\lim_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|Y_t^\varepsilon|_E>\delta)=-\infty.$$

Proof Following the discussions of Lemmas 3.3 and 3.5 in [3], we can easily obtain the claim of Lemma 1.

Lemma 2 *Let $\delta>0$. Then*

$$\lim_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|Z_t^\varepsilon|_E>\delta)=-\infty.$$

Proof For $\rho > 0$, from the assumptions (i) and (ii) on r, we can introduce $\xi_t^\varepsilon = |Z_t^\varepsilon|_E^2$ and stopping times

$$\tau_1^\varepsilon = \inf\{t > 0; |Y_t^\varepsilon| > \rho\}, \quad \tau_2^\varepsilon = \inf\{t > 0; \xi_t^\varepsilon > \delta^2, r(\xi_t^\varepsilon) + \xi_t^\varepsilon r'(\xi_t^\varepsilon) < 0\},$$

and put $\tau = \tau_1^\varepsilon \wedge \tau_2^\varepsilon$. By Itô formula, we deduce

$$\xi_t^\epsilon = 2\varepsilon^{1/2} \int_0^t < Z_s^\varepsilon, (\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon))\mathrm{d}W_s > + \int_0^t \varepsilon \mathrm{tr}(\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon))(\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon))^*)\mathrm{d}s.$$

Let $\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon) = e^{\lambda\psi_\rho(\xi_{t\wedge\tau}^\varepsilon)}$, $\lambda > 0$. Thus we have

$$\begin{aligned}
\Phi'_\rho(\xi_{t\wedge\tau}^\varepsilon) &= \lambda\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon)\frac{1}{\xi_{t\wedge\tau}^\varepsilon r(\xi_{t\wedge\tau}^\varepsilon) + \rho} \le \frac{\lambda}{\rho}\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon),\\
\Phi''_\rho(\xi_{t\wedge\tau}^\varepsilon) &= \lambda\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon)\frac{1}{(\xi_{t\wedge\tau}^\varepsilon r(\xi_{t\wedge\tau}^\varepsilon) + \rho)^2}(\lambda - (r(\xi_{t\wedge\tau}^\varepsilon) + \xi_{t\wedge\tau}^\varepsilon r'(\xi_{t\wedge\tau}^\varepsilon)))\\
&\le \lambda^2\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon)\frac{1}{(\xi_{t\wedge\tau}^\varepsilon r(\xi_{t\wedge\tau}^\varepsilon) + \rho)^2} \le \frac{\lambda^2}{\rho^2}\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon). \qquad (2)
\end{aligned}$$

From $\xi_{1\wedge\tau}^\epsilon = \delta^2$, we have $\Phi(\xi_{1\wedge\tau}^\varepsilon) = e^{\lambda\psi_\rho(\delta^2)}$. Since

$$|u_{t\wedge\tau}^\varepsilon - v_{t\wedge\tau}^\varepsilon|_E^2 \le 2(|Y_{t\wedge\tau}^\varepsilon|_E^2 + |Z_{t\wedge\tau}^\varepsilon|_E^2) \le 2(\rho^2 + \xi_{t\wedge\tau}^\varepsilon) \le 2(\rho^2 + \delta^2),$$

by the condition (I), we have

$$\begin{aligned}
&\|\sigma(u_{t\wedge\tau}^\varepsilon) - \sigma(v_{t\wedge\tau}^\varepsilon)\|_{(2)}^2 \le c(|u_{t\wedge\tau}^\varepsilon - v_{t\wedge\tau}^\varepsilon|_E^2 r(|u_{t\wedge\tau}^\varepsilon - v_{t\wedge\tau}^\varepsilon|_E^2) + 1)\\
&\quad \le c(2(\rho^2 + \delta^2)r(2(\rho^2 + \delta^2)) + 1),\\
&\mathrm{tr}((Z_{t\wedge\tau}^\varepsilon \otimes Z_{t\wedge\tau}^\varepsilon)(\sigma(u_{t\wedge\tau}^\varepsilon) - \sigma(v_{t\wedge\tau}^\varepsilon))(\sigma(u_{t\wedge\tau}^\varepsilon) - \sigma(v_t^\varepsilon))^*)\\
&\quad \le |Z_{t\wedge\tau}^\varepsilon|_E^2 |u_{t\wedge\tau}^\varepsilon - v_{t\wedge\tau}^\varepsilon|_E^2 \le 2\xi_{t\wedge\tau}^\varepsilon(\rho^2 + \xi_{t\wedge\tau}^\varepsilon) \le 2\delta^2(\rho^2 + \delta^2). \qquad (3)
\end{aligned}$$

From Itô formula, it follows that

$$\begin{aligned}
\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon) =& 1 + 2\varepsilon^{1/2}\int_0^{t\wedge\tau} \Phi'_\rho(\xi_s^\varepsilon) < Z_s^\varepsilon, (\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon))\mathrm{d}W_s > +\\
&\varepsilon\int_0^{t\wedge\tau} \Phi'_\rho(\xi_s^\varepsilon)\|\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon)\|_{(2)}^2 \mathrm{d}s +\\
&2\varepsilon\int_0^{t\wedge\tau} \Phi''_\rho(\xi_s^\varepsilon)\mathrm{tr}((Z_s^\varepsilon \otimes Z_s^\varepsilon)(\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon))(\sigma(u_s^\varepsilon) - \sigma(v_s^\varepsilon))^*)\mathrm{d}s. \qquad (4)
\end{aligned}$$

Thus, taking the expectation for Eq.(4) together with (2) and (3), we deduce

$$\begin{aligned}
E[\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon)] &\le 1 + \lambda^2\varepsilon(\frac{2c(\rho^2 + \delta^2)r(2(\rho^2 + \delta^2)) + c}{\rho\lambda} + \frac{4\delta^2(\rho^2 + \delta^2)}{\rho^2})\int_0^{t\wedge\tau} E[\Phi_\rho(\xi_s^\varepsilon)]\mathrm{d}s\\
&= 1 + k(\rho)\lambda^2\varepsilon\int_0^{t\wedge\tau} E[\Phi_\rho(\xi_s^\varepsilon)]\mathrm{d}s,
\end{aligned}$$

where

$$k(\rho) = \frac{2c(\rho^2 + \delta^2)r(2(\rho^2 + \delta^2)) + c}{\rho\lambda} + \frac{4\delta^2(\rho^2 + \delta^2)}{\rho^2}.$$

Therefore, by the Gronwall inequality, we get

$$E[\Phi_\rho(\xi_{t\wedge\tau}^\varepsilon)] \le \exp(k(\rho)\lambda^2\varepsilon).$$

Consequently,

$$P(\sup_{0\le t\le 1}\xi_t^\varepsilon\ge\delta^2,\sup_{0\le t\le 1}|Y_t^\varepsilon|\le\rho)\exp(\lambda\psi_\rho(\delta^2))\le P(\tau_2^\varepsilon\le 1,\tau_1^\varepsilon>1)\exp(\lambda\psi_\rho(\delta^2))$$
$$\le E[\Phi_\rho(\xi^\varepsilon_{1\wedge\tau})]\le\exp(k(\rho)\lambda^2\varepsilon).$$

Hence, we have

$$P(\sup_{0\le t\le 1}|Z_t^\varepsilon|\ge\delta,\sup_{0\le t\le 1}|Y_t^\varepsilon|\le\rho)\le\exp(k(\rho)\lambda^2\varepsilon-\lambda\psi_\rho(\delta^2)).$$

Taking $\lambda=\frac{1}{\varepsilon}$, we get

$$\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|Z_t^\varepsilon|\ge\delta,\sup_{0\le t\le 1}|Y_t^\varepsilon|\le\rho)\le\frac{4\delta^2(\rho^2+\delta^2)}{\rho^2}-\psi_\rho(\delta^2).$$

Thus, from Lemma 1 we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|Z_t^\varepsilon|\ge\delta)\\&\le(\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|Z_t^\varepsilon|\ge\delta,\sup_{0\le t\le 1}|Y_t^\varepsilon|\le\rho))\\&\vee(\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|Y_t^\varepsilon|>\rho))\le\frac{4\delta^2(\rho^2+\delta^2)}{\rho^2}-\psi_\rho(\delta^2).\end{aligned}\tag{5}$$

Since $\lim_{\rho\to 0}\psi_\rho(\delta^2)=+\infty$, $\lim_{\rho\to 0}\rho^2\psi_\rho(\delta^2)=+\infty$, we have

$$\lim_{\rho\to 0}(\frac{4\delta^2(\rho^2+\delta^2)}{\rho^2}-\psi_\rho(\delta))=\lim_{\rho\to 0}\psi_\rho(\delta)(\frac{4\delta^2(\rho^2+\delta^2)}{\rho^2\psi_\rho(\delta^2)}-1)=-\infty.$$

Hence, setting $\rho\to 0$ in (5) gives

$$\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{0\le t\le 1}|Z_t^\varepsilon|\ge\delta)=-\infty.$$

Thus, the claim of Lemma 2 holds.

Finally, from Lemmas 1 and 2, we get the Proposition, and the proof of Theorem is completed. □

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