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Fenchel-Lagrange Duality and Saddle-Points for Constrained Vector Optimization

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Abstract The aim of this paper is to apply a perturbation approach to deal with Fenchel-Lagrange duality based on weak efficiency to a constrained vector optimization problem. Under the stability criterion, some relationships between the solutions of primal problem and the Fenchel-Lagrange duality are discussed. Moreover, under the same condition, two saddle-points theorems are proved.

Keywords Vector optimization; Fenchel-Lagrange duality; saddle-points; weak efficiency.

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1. Introduction

Conjugate duality provides a unified framework to duality in optimization and was fully developed in scalar optimization by Rockafellar [4, 5]. Conjugate duality was extended to vector optimization in finite dimensional spaces by Tanino and Sawaragi [6], and in infinite dimensional spaces by Postolica [12], and in partially ordered topological vector space based on weak efficiency by Tanino [10]. Moreover, in [10] Tanino obtained the weak and strong duality (i.e., stability criterion) assertions in vector optimization.

By considering some special perturbation functions, Wanka and Boţ [11] (see also Boţ et al. [2]) proposed three conjugate dual problems for a primal scalar optimization problem, namely the Lagrange, Fenchel and Fenchel-Lagrange dual problems. The relations between the optimal objective functions of these dual problems have been completely investigated. Inspired by the scalar case, Altangerel et al. [1] constructed three conjugate duality problems to a constrained vector optimization problem and obtained set-valued gap functions for the vector variational inequality by using the conjugate duality based on efficiency introduced in [3, 6]. However, so far, few authors intensively studied saddle-points theorem by using the conjugate duality based on weak efficiency introduced in [10].

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Motivated by the work reported in [1, 6–8, 10, 11], we define the Fenchel-Lagrange duality for a constrained vector optimization problem based on weak efficiency. Furthermore, under the stability criterion, we focus on discussing the relationships between the primal problem and the dual problem, and some saddle-points theorems.

The paper is organized as follows. In Section 2, we recall some notions and their properties. In Section 3, Fenchel-Lagrange dual problem based on weak efficiency for a constrained vector optimization problem is introduced. Moreover, under stability criterion, we discuss some relationships of the primal-dual problem. In Section 4, under stability criterion, we prove two saddle-points theorems.

2. Mathematical preliminaries

Let Y be a real topological vector space which is partially ordered by a pointed closed convex cone C with $\text{int}C \neq \emptyset$. For any $y_1, y_2 \in Y$, we use the following ordering relations:

$$y_1 > y_2 \Longleftrightarrow y_1 - y_2 \in \text{int}C, \quad y_1 \not> y_2 \Longleftrightarrow y_1 - y_2 \notin \text{int}C.$$

We add two imaginary points $+\infty$ and $-\infty$ to Y and denote the extended space by $\bar{Y}$. These two points are defined as the points which satisfy the following: For any $y \in Y$,

$$-\infty < y < +\infty, \ \ (\pm\infty) + y = y + (\pm\infty) = \pm\infty \ \text{ and } \ (\pm\infty) + (\pm\infty) = \pm\infty.$$

Assume that $-(\pm\infty) = \mp\infty$. The sum $+\infty - \infty$ is not considered since it can be avoided.

Given a set $Z \subset \bar{Y}$, we define the set $A(Z)$ of all points above Z, and the set $B(Z)$ of all points below Z by

$$A(Z) = \{y \in \bar{Y} \mid y > y' \quad \text{for some} \quad y' \in Z\}$$

and

$$B(Z) = \{y \in \bar{Y} \mid y < y' \quad \text{for some} \quad y' \in Z\},$$

respectively. Clearly, $A(Z) \subset Y \cup \{+\infty\}$, $B(Z) \subset Y \cup \{-\infty\}$ and $B(Z) = -A(-Z)$.

Definition 2.1 ([10]) *(i) A point $\hat{y} \in \bar{Y}$ is said to be a maximal point of $Z \subset \bar{Y}$ if $\hat{y} \in Z$ and $\hat{y} \notin B(Z)$, that is, if $\hat{y} \in Z$ and there is no $y' \in Z$ such that $\hat{y} < y'$. The set of all maximal points of Z is called the maximum of Z and is denoted by* Max$Z$. *The minimum of Z,* Min$Z$, *is defined analogously.*

(ii) A point $\hat{y} \in \bar{Y}$ is said to be a supremal point of $Z \subset \bar{Y}$ if $\hat{y} \notin B(Z)$ and $B(\{\hat{y}\}) \subset B(Z)$, that is, if there is no $y \in Z$ such that $\hat{y} < y$ and if the relation $y' < \hat{y}$ implies the existence of some $y \in Z$ such that $y' < y$. The set of all supremal points of Z is called the supremum of Z and is denoted by Sup$Z$. *The infimum of Z,* Inf$Z$, *is defined analogously.*

Proposition 2.1 ([10]) *(i) For $Z \subset \bar{Y}$, $A(Z) = A(\text{Inf}Z)$ and $B(Z) = B(\text{Sup}Z)$.*

(ii) Let $Z_1 \subset \bar{Y}$ and $Z_2 \subset \bar{Y}$. Then

$$\text{Sup} \bigcup_{x \in X} [Z_1 + Z_2] = \text{Sup} \bigcup_{x \in X} [Z_1 + \text{Sup}Z_2],$$

where the sum $+\infty - \infty$ is assumed not to occur.

From Corollary 4.3 in [10], we have the following proposition.

Proposition 2.2 *If W is a set-valued map from X to $\bar{Y}$, then*

$$\operatorname{Sup} \bigcup_{x \in X} W(x) = \operatorname{Sup} \bigcup_{x \in X} \operatorname{Sup} W(x).$$

Let X be another real topological vector space and let $L(X, Y)$ be the space of all linear continuous operators from X to Y. For $x \in X$ and $T \in L(X, Y)$, Tx represents an element in Y.

Definition 2.2 ([10]) *Let f be a vector-valued map from X to $\bar{Y}$.*

(i) A set-valued mapping $f^* : L(X, Y) \to 2^{\bar{Y}}$ defined by

$$f^*(T) = \operatorname{Sup} \bigcup_{x \in X} [Tx - f(x)], \quad \text{for} \quad T \in L(X, Y)$$

is called the conjugate mapping of f.

(ii) A set-valued mapping $f^{**} : X \to 2^{\bar{Y}}$ defined by

$$f^{**}(x) = \operatorname{Sup} \bigcup_{T \in L(X,Y)} [Tx - f^*(T)], \quad \text{for} \quad x \in X$$

is called the biconjugate mapping of f.

Definition 2.3 *Let $W : X \to \bar{Y}$ be a set-valued mapping. Let $\hat{x} \in X$ and $\hat{y} \in W(\hat{x})$. An operator $T \in L(X, Y)$ is called a subgradient of W at $(\hat{x}; \hat{y})$ if*

$$T\hat{x} - \hat{y} \in \operatorname{Max} \bigcup_{x \in X} [Tx - W(x)].$$

The set of all subgradients of W at $(\hat{x}; \hat{y})$ is called the subdifferential of W at $(\hat{x}; \hat{y})$ and is denoted by $\partial W(\hat{x}; \hat{y})$. If $\partial W(\hat{x}; \hat{y}) \neq \emptyset$ for every $\hat{y} \in W(\hat{x})$, then W is said to be subdifferentiable at $\hat{x}$.

According to [7], we have the following definition especially to the vector-valued mapping $f : X \to \bar{Y}$.

Definition 2.4 *A vector-valued mapping $f : X \to Y \cup \{+\infty\}$ is said to be C-convex, if for any $\lambda \in [0, 1]$ and $x_1, x_2 \in X$,*

$$\lambda f(x_1) \cap Y + (1 - \lambda) f(x_2) \cap Y \in f(\lambda x_1 + (1 - \lambda) x_2) + C.$$

3. A Fenchel-Lagrange dual problem

Let X be a real topological vector space, Y and Z be two real partially ordered topological vector spaces, $C \subset Y$ and $D \subset Z$ be two pointed closed convex cones with nonempty interiors. Let $f : X \to Y \cup \{+\infty\}$ and $g : X \to Z$ be two vector-valued mappings with $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\} \neq \emptyset$. Let $E \subset X$ be a nonempty set and $E \subset \operatorname{dom} f$. Consider the following constrained vector optimization problem:

$$\text{(P)} \qquad \min_{x \in S} f(x), \ \text{ where } S := \{x \in E \mid g(x) \in -D\}.$$

In the following, we suppose always that the feasible set $S \neq \emptyset$. Solving this problem means to find the set $\text{Inf}(P) = \text{Inf}\{f(x) \mid x \in S\}$ or the set $\text{Min}(P) = \text{Min}\{f(x) \mid x \in S\}$.

In order to introduce the Fenchel-Lagrange dual form of (P). So we introduce the perturbation function as follows: $\Phi_{FL} : X \times X \times Z \to Y \cup \{+\infty\}$ be a vector-valued mapping defined by

$$\Phi_{FL}(x,p,q) = \begin{cases} f(x+p), & \text{if } x \in E,\ g(x) \in -(D+q), \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation parameters $p \in X$ and $q \in Z$. Obviously, $\Phi_{FL}(x,0,0) = f(x)$, for all $x \in E, g(x) \in -D$. Now we consider the conjugate mapping of Φ_{FL}:

$$\begin{aligned}\Phi^*_{FL}(T,\Gamma,\Lambda) &= \text{Sup}\{Tx + \Gamma p + \Lambda q - \Phi_{FL}(x,p,q) \mid x \in X, p \in X, q \in Z\} \\ &= \text{Sup}\{Tx + \Gamma p + \Lambda q - f(x+p) \mid x \in E, g(x) \in -(D+q), p \in X, q \in Z\},\end{aligned}$$

for $T \in L(X,Y)$, $\Gamma \in L(X,Y)$ and $\Lambda \in L(Z,Y)$. Let $r = x + p \in X$ and $s = g(x) + q \in -D$. Then, by Proposition 2.1(ii), we obtain that

$$\begin{aligned}-\Phi^*_{FL}(0,\Gamma,\Lambda) &= -\text{Sup}\{\Gamma(r-x) + \Lambda(s - g(x)) - f(r) \mid x \in E, r \in X, s \in -D\} \\ &= \text{Inf}\{f(r) - \Gamma r + \Gamma x + \Lambda g(x) - \Lambda s \mid x \in E, r \in X, s \in -D\} \\ &= \text{Inf}\{\{f(r) - \Gamma r \mid r \in X\} + \{\Gamma x + \Lambda g(x) \mid x \in E\} + \{\Lambda s \mid s \in D\}\} \\ &= \text{Inf}\{\text{Inf}\{f(r) - \Gamma r \mid r \in X\} + \{\Gamma x + \Lambda g(x) \mid x \in E\} + \{\Lambda s \mid s \in D\}\} \\ &= \text{Inf}\{-f^*(\Gamma) + \{\Gamma x + \Lambda g(x) \mid x \in E\} + \{\Lambda s \mid s \in D\}\}.\end{aligned}$$

We define the Fenchel-Lagrange dual problem to (P) as

$$(D_{FL}) \qquad \max_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} \text{Inf}\{-f^*(\Gamma) + \{\Gamma x + \Lambda g(x) \mid x \in E\} + \{\Lambda s \mid s \in D\}\}.$$

The dual problem (D_{FL}) can be understood as a problem to obtain the set

$$\begin{aligned}\text{Sup}(D_{FL}) &= \text{Sup} \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} [-\Phi^*_{FL}(0,\Gamma,\Lambda)] \\ &= \text{Sup} \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} \text{Inf}\{-f^*(\Gamma) + \{\Gamma x + \Lambda g(x) \mid x \in E\} + \{\Lambda s \mid s \in D\}\}.\end{aligned}$$

From [10, Proposition 5.1], [10, Theorem 5.1] and [7, Theorem 3.1], one can state the weak and strong duality assertions as follows.

Propostion 3.1 (Weak duality for (D_{FL})) *For any $x \in S$, $\Gamma \in L(X,Y)$ and $\Lambda \in L(Z,Y)$, $f(x) \notin B(-\Phi^*_{FL}(0,\Gamma,\Lambda))$.*

Theorem 3.1 (Strong duality for (D_{FL})) *If the primal problem (P) is stable with respect to Φ_{FL} (i.e., the value mapping $W_{FL}(p,q) := \text{Inf}\{\Phi_{FL}(x,p,q) \mid x \in X\}$ is subdifferentiable at $(0_X, 0_Z)$), then* $\text{Min}(P) = \text{Inf}(P) = \text{Sup}(D_{FL}) = \text{Max}(D_{FL})$.

Every $\hat{x} \in S$ satisfying the relationship $f(\hat{x}) \in \text{Min}(P)$ is called a solution of the problem (P). Every $(\hat{\Gamma}, \hat{\Lambda}) \in L(X,Y) \times L(Z,Y)$ satisfying the relationship $-\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda}) \cap \text{Max}(D_{FL}) \neq \emptyset$ is called a solution of the problem (D_{FL}).

In the following, we shall discuss the relationships between the solutions of (P) and (D_{FL}).

Theorem 3.2 *Suppose that the problem (P) is stable with respect to Φ_{FL}. If $\hat{x}$ is a solution of (P), then there exists $\hat{\Gamma} \in L(X,Y)$ and $\hat{\Lambda} \in L(Z,Y)$ such that $(\hat{\Gamma},\hat{\Lambda})$ is a solution of (D_{FL}) with $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$.*

Proof Since (P) is stable with respect to Φ_{FL} and $\hat{x}$ is a solution of (P), by Theorem 3.1, we have

$$\begin{aligned} f(\hat{x}) &\in \mathrm{Min}(P) \subset \mathrm{Max}(D_{FL}) \\ &= \mathrm{Max} \bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L(Z,Y)}} [-\Phi^*_{FL}(0,\Gamma,\Lambda)] \subset \bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L(Z,Y)}} [-\Phi^*_{FL}(0,\Gamma,\Lambda)], \end{aligned}$$

and there exist $\hat{\Gamma} \in L(X,Y)$ and $\hat{\Lambda} \in L(Z,Y)$ such that $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$.

Next, we shall prove $(\hat{\Gamma},\hat{\Lambda})$ is a solution of (D_{FL}). Assume that $(\hat{\Gamma},\hat{\Lambda})$ is not a solution of (D_{FL}). Since $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$, $f(\hat{x}) \notin \mathrm{Max}\bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L(Z,Y)}}[-\Phi^*_{FL}(0,\Gamma,\Lambda)]$. Hence, there exist $\Gamma_1 \in L(X,Y)$, $\Lambda_1 \in L(Z,Y)$ and $y_1 \in -\Phi^*_{FL}(0,\Gamma_1,\Lambda_1)$ such that $f(\hat{x}) < y_1$. This shows that $f(\hat{x}) \in B(-\Phi^*_{FL}(0,\Gamma_1,\Lambda_1))$, which contradicts Proposition 3.1. □

Theorem 3.3 *If $(\hat{x},\hat{\Gamma},\hat{\Lambda}) \in S \times L(X,Y) \times L(Z,Y)$ satisfies $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$, then $\hat{x}$ is a solution of (P) and $(\hat{\Gamma},\hat{\Lambda})$ is a solution of (D_{FL}).*

Proof Assume that $\hat{x}$ is not a solution of (P), then $f(\hat{x}) \notin \mathrm{Min}(P) = \mathrm{Min}\{f(x) \mid x \in S\}$. Hence, there exists $x_1 \in S$ such that $f(x_1) < f(\hat{x})$. It follows from $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$ that $f(x_1) \in B(-\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda}))$, which contradicts Proposition 3.1.

Assume that $(\hat{\Gamma},\hat{\Lambda})$ is not a solution of (D_{FL}). Since $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$, we have

$$f(\hat{x}) \notin \mathrm{Max}(D_{FL}) = \mathrm{Max} \bigcup_{\Gamma\in L(X,Y),\Lambda\in L(Z,Y)} [-\Phi^*_{FL}(0,\Gamma,\Lambda)].$$

Hence, there exist $\Gamma_1 \in L(X,Y)$, $\Lambda_1 \in L(Z,Y)$ and $y_1 \in -\Phi^*_{FL}(0,\Gamma_1,\Lambda_1)$ such that $f(\hat{x}) < y_1$. This shows that $f(\hat{x}) \in B(-\Phi^*_{FL}(0,\Gamma_1,\Lambda_1))$, which contradicts Proposition 3.1 again. □

Remark Under the condition of $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$, it is clear that $\hat{x}$ is a solution of (P) and $(\hat{\Gamma},\hat{\Lambda})$ is a solution of (D_{FL}). Thus, we only need to find $(\hat{x},\hat{\Gamma},\hat{\Lambda}) \in S \times L(X,Y) \times L(Z,Y)$ satisfying $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$ in order to obtain solutions of (P) and (D_{FL}).

4. Saddle-point theorems

In this section, we define the Lagrangian maps and their saddle points for the problem (P) and investigate their properties.

Definition 4.1 *The set-valued map $L : E \times L(X,Y) \times L(Z,Y) \to 2^{Y\cup\{+\infty\}}$, defined by*

$$L(x,\Gamma,\Lambda) = \mathrm{Inf}\{-f^*(\Gamma) + \Gamma x + \Lambda g(x) + \{\Lambda s \mid s \in D\}\}$$

is called the Lagrangian map of the problem (P) relative to the perturbation function Φ_{FL}.

From Proposition 2.1(ii), obviously, we have the following result.

Proposition 4.1 *For each* $\Gamma \in L(X,Y)$ *and* $\Lambda \in L(Z,Y)$,

$$\mathrm{Inf} \bigcup_{x\in E} L(x,\Gamma,\Lambda) = -\Phi^*_{FL}(0,\Gamma,\Lambda).$$

Definition 4.2 *A point* $(\hat{x},\hat{\Gamma},\hat{\Lambda}) \in S \times L(X,Y) \times L(Z,Y)$ *is called a saddle point of* $L(x,\Gamma,\Lambda)$ *if*

$$L(\hat{x},\hat{\Gamma},\hat{\Lambda}) \cap [\mathrm{Sup} \bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L(Z,Y)}} L(\hat{x},\Gamma,\Lambda)] \cap [\mathrm{Inf} \bigcup_{x\in E} L(x,\hat{\Gamma},\hat{\Lambda})] \neq \emptyset.$$

Theorem 4.1 *If* $(\hat{x},\hat{\Gamma},\hat{\Lambda}) \in S \times L(X,Y) \times L(Z,Y)$ *satisfies* $f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda})$, *then* $(\hat{x},\hat{\Gamma},\hat{\Lambda})$ *is a saddle point of* $L(x,\Gamma,\Lambda)$.

Proof By Propositions 4.1, we have

$$f(\hat{x}) \in -\Phi^*_{FL}(0,\hat{\Gamma},\hat{\Lambda}) = \mathrm{Inf} \bigcup_{x\in E} L(x,\hat{\Gamma},\hat{\Lambda}). \tag{1}$$

We first prove that $f(\hat{x}) \in L(\hat{x},\hat{\Gamma},\hat{\Lambda})$. It follows from (1), Proposition 2.2 and Proposition 2.1(ii) that

$$\begin{aligned} f(\hat{x}) &\in \mathrm{Inf} \bigcup_{x\in E} \mathrm{Inf}\{-f^*(\hat{\Gamma}) + \hat{\Gamma}x + \hat{\Lambda}g(x) + \{\hat{\Lambda}s \mid s\in D\}\} \\ &= \mathrm{Inf} \bigcup_{x\in E} \{-f^*(\hat{\Gamma}) + \hat{\Gamma}x + \hat{\Lambda}g(x) + \{\hat{\Lambda}s \mid s\in D\}\} \\ &= \mathrm{Inf}\{\{f(x) - \hat{\Gamma}x \mid x\in X\} + \{\hat{\Gamma}x + \hat{\Lambda}g(x) \mid x\in E\} + \{\hat{\Lambda}s \mid s\in D\}\}. \end{aligned}$$

Thus, we have

$$f(\hat{x}) \notin A(\{f(x) - \hat{\Gamma}x \mid x\in X\} + \{\hat{\Gamma}x + \hat{\Lambda}g(x) \mid x\in E\} + \{\hat{\Lambda}s \mid s\in D\}). \tag{2}$$

Note that $-g(\hat{x}) \in D$ and

$$f(\hat{x}) = f(\hat{x}) - \hat{\Gamma}\hat{x} + \hat{\Gamma}\hat{x} + \hat{\Lambda}g(\hat{x}) + \hat{\Lambda}(-g(\hat{x})).$$

Suppose

$$f(\hat{x}) \notin \mathrm{Min}\{\{f(x) - \hat{\Gamma}x \mid x\in X\} + \hat{\Gamma}\hat{x} + \hat{\Lambda}g(\hat{x}) + \{\hat{\Lambda}s \mid s\in D\}\}.$$

Then there exists $y_1 \in \{f(x) - \hat{\Gamma}x \mid x\in X\} + \hat{\Gamma}\hat{x} + \hat{\Lambda}g(\hat{x}) + \{\hat{\Lambda}s \mid s\in D\}$ such that $y_1 < f(\hat{x})$, which contradicts (2). Consequently, we obtain that

$$\begin{aligned} f(\hat{x}) &\in \mathrm{Min}\{\{f(x) - \hat{\Gamma}x \mid x\in X\} + \hat{\Gamma}\hat{x} + \hat{\Lambda}g(\hat{x}) + \{\hat{\Lambda}s \mid s\in D\}\} \\ &\subset \mathrm{Inf}\{\{f(x) - \hat{\Gamma}x \mid x\in X\} + \hat{\Gamma}\hat{x} + \hat{\Lambda}g(\hat{x}) + \{\hat{\Lambda}s \mid s\in D\}\} \\ &= \mathrm{Inf}\{\mathrm{Inf}\{f(x) - \hat{\Gamma}x \mid x\in X\} + \hat{\Gamma}\hat{x} + \hat{\Lambda}g(\hat{x}) + \{\hat{\Lambda}s \mid s\in D\}\} \\ &= \mathrm{Inf}\{-f^*(\hat{\Gamma}) + \hat{\Gamma}\hat{x} + \hat{\Lambda}g(\hat{x}) + \{\hat{\Lambda}s \mid s\in D\}\} \\ &= L(\hat{x},\hat{\Gamma},\hat{\Lambda}). \end{aligned}$$

Next, we prove that

$$f(\hat{x}) \in \operatorname{Sup} \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} L(\hat{x}, \Gamma, \Lambda).$$

Suppose it is false. Then

$$\begin{aligned} f(\hat{x}) \notin \operatorname{Max} & \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} L(\hat{x}, \Gamma, \Lambda) \\ = \operatorname{Max} & \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} \operatorname{Inf}\{-f^*(\Gamma) + \Gamma\hat{x} + \Lambda g(\hat{x}) + \{\Lambda s \mid s \in D\}\} \\ = \operatorname{Max} & \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} [\Gamma\hat{x} + \Lambda g(\hat{x}) + \operatorname{Inf}\{-f^*(\Gamma) + \{\Lambda s \mid s \in D\}\}]. \end{aligned}$$

Since $f(\hat{x}) \in L(\hat{x}, \hat{\Gamma}, \hat{\Lambda}) \subset \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} L(\hat{x}, \Gamma, \Lambda)$, there exist $\bar{\Gamma} \in L(X,Y)$, $\bar{\Lambda} \in L(Z,Y)$ and $\bar{y} \in \operatorname{Inf}\{-f^*(\bar{\Gamma}) + \{\bar{\Lambda}s \mid s \in D\}\}$ such that $f(\hat{x}) < \bar{\Gamma}\hat{x} + \bar{\Lambda}g(\hat{x}) + \bar{y}$, i.e.,

$$\bar{y} > f(\hat{x}) - \bar{\Gamma}\hat{x} + \bar{\Lambda}(-g(\hat{x})).$$

Note that

$$f(\hat{x}) - \bar{\Gamma}\hat{x} + \bar{\Lambda}(-g(\hat{x})) \in \{f(x) - \bar{\Gamma}x \mid x \in X\} + \{\bar{\Lambda}s \mid s \in D\}.$$

Therefore,

$$\bar{y} \in A(\{f(x) - \bar{\Gamma}x \mid x \in X\} + \{\bar{\Lambda}s \mid s \in D\}). \tag{3}$$

On the other hand, it follows from Proposition 2.1(ii) that

$$\begin{aligned} \bar{y} &\in \operatorname{Inf}\{-f^*(\bar{\Gamma}) + \{\bar{\Lambda}s \mid s \in D\}\} \\ &= \operatorname{Inf}\{\{f(x) - \bar{\Gamma}x \mid x \in X\} + \{\bar{\Lambda}s \mid s \in D\}\}. \end{aligned}$$

Whence,

$$\bar{y} \notin A(\{f(x) - \bar{\Gamma}x \mid x \in X\} + \{\bar{\Lambda}s \mid s \in D\}),$$

which contradicts (3). Hence, $(\hat{x}, \hat{\Gamma}, \hat{\Lambda})$ is a saddle point of $L(x, \Gamma, \Lambda)$. □

From Theorem 4.1, we get readily the following result.

Theorem 4.2 *Assume the problem (P) is stable with respect to Φ_{FL}. If $\hat{x} \in S$ is a solution of (P), then there exists $(\hat{\Gamma}, \hat{\Lambda}) \in L(X,Y) \times L(Z,Y)$ such that $(\hat{x}, \hat{\Gamma}, \hat{\Lambda})$ is a saddle point of $L(x, \Gamma, \Lambda)$.*

Proof Since (P) is stable with respect to Φ_{FL} and $\hat{x} \in S$ is a solution of (P), then by Theorem 3.1,

$$f(\hat{x}) \in \operatorname{Min}(P) \subset \operatorname{Max}(D_{FL}) = \operatorname{Max} \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} [-\Phi^*_{FL}(0, \Gamma, \Lambda)] \subset \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L(Z,Y)}} [-\Phi^*_{FL}(0, \Gamma, \Lambda)].$$

Hence, there exist $\hat{\Gamma} \in L(X,Y)$ and $\hat{\Lambda} \in L(Z,Y)$ such that

$$f(\hat{x}) \in -\Phi^*_{FL}(0, \hat{\Gamma}, \hat{\Lambda}).$$

By Theorem 4.1, we get that $(\hat{x}, \hat{\Gamma}, \hat{\Lambda})$ is a saddle point of $L(X, \Gamma, \Lambda)$. $\square$

Remark 4.1 From Theorems 4.2, 4.1 and 3.2, we know that if $\hat{x} \in S$ is a solution of (P) and $(\hat{\Gamma}, \hat{\Lambda}) \in L(X, Y) \times L(Z, Y)$ is a solution of (D_{FL}) with $f(\hat{x}) \in -\Phi^*_{FL}(0, \hat{\Gamma}, \hat{\Lambda})$, then $(\hat{x}, \hat{\Gamma}, \hat{\Lambda})$ is a saddle point of $L(x, \Gamma, \Lambda)$. However, the converse may not hold.

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