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The Influence of s-Conditional Permutability of Subgroups on the Structure of Finite Groups

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Abstract Let G be a finite group. Fix a prime divisor p of $|G|$ and a Sylow p-subgroup P of G, let d be the smallest generator number of P and $\mathcal{M}_d(P)$ denote a family of maximal subgroups $P_1, P_2, \dots, P_d$ of P satisfying $\bigcap_{i=1}^{d} P_i = \Phi(P)$, the Frattini subgroup of P. In this paper, we shall investigate the influence of s-conditional permutability of the members of some fixed $\mathcal{M}_d(P)$ on the structure of finite groups. Some new results are obtained and some known results are generalized.

Keywords finite groups; s-conditionally permutable groups; saturated formations; supersoluble groups; nilpotent groups.

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1. Introduction

Recall that a subgroup H of a group G is said to be permutable with a subgroup T of G if $HT = TH$. A subgroup H of a group G is called a permutable subgroup [1] (or quasinormal subgroup) [11] of G if H permutes with all subgroups of G. As a development, recently, Guo, Shum and Skiba [3–6] introduced the concept of X-permutable subgroup and X-semipermutable subgroup: Let X be a nonempty subset of G. A subgroup H is said to be X-permutable in G if for every subgroup T of G, there exists some $x \in X$ such that $HT^x = T^xH$. A subgroup H is said to be X-semipermutable in G if it is X-permutable with every subgroup T_1 of some supplement T of H in G. Later on, the following concepts were aslo introduced: A subgroup H is said to be s-conditionally permutable in G (see [8]) if for every Sylow subgroup T, there exists an element $x \in G$ such that $HT^x = T^xH$. A subgroup H is said to be SS-quasinormal [10] in G if there is a supplement B of H to G such that H permutes with every Sylow subgroup

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of B. Obviously, for any primary subgroup P of G, if P is SS-quasinormal in G, then P is s-condictionally permutable, but the converse is not true. By using the above ideas, a series of interesting results have been obtained [3–6, 8, 10].

The purpose of this paper is to go further into the influence of s-condictionally permutable subgroups on the structure of finite groups. Some new results are obtained and some known results are generalized.

Throughout this paper, all groups considered are finite and G denotes a group. The terminology and notations are standard, as in [2] and [7].

2. Preliminaries

In this section, we give the related concepts and some basic results which are needed in this paper.

Definition 2.1 ([9]) *Let d be the smallest generator number of a p-group P and $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$ be a set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$.*

Such subset $\mathcal{M}_d(P)$ is not unique for a fixed P in general [9].

Recall that a class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homorphic images and subdirect products. A formation $\mathfrak{F}$ is called saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. It is well known that the class of all supersoluble groups is a saturated formation.

Lemma 2.1 ([8, Lemma 2.1]) *Let H be an s-conditionally permutable subgroup of G. Then:*

1) If $H \leqslant K \trianglelefteq G$, then H is s-conditionally permutable in K.

2) If $N \lhd G$, then HN/N is s-conditionally permutable in G/N.

3) H^g is s-conditionally permutable in G for each element g of G.

The following result is well known.

Lemma 2.2 *Suppose that P is a Sylow subgroup of G. If $P \lhd \lhd G$, then $P \trianglelefteq G$.*

Lemma 2.3 ([7, IV Theorem 4.7]) *If P is a Sylow p-subgroup of a group G for some $p \in \pi(G)$ and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.4 ([2, Theorem 1.8.17]) *Let N be a non-trivial normal subgroup of a group G. If $N \bigcap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is a direct product of some abelian minimal normal subgroups of G.*

Lemma 2.5 ([7, III Lemma 3.3])

1) If $N \trianglelefteq G$, $U \leqslant G$ and $N \leqslant \Phi(U)$, then $N \leqslant \Phi(G)$.

2) If $M \trianglelefteq G$, then $\Phi(M) \leqslant \Phi(G)$.

3. Main results

Theorem 3.1 *Let G be a p-soluble group and P a Sylow p-subgroup of G. Suppose that every* *member of some fixed $\mathcal{M}_d(P)$ is s-conditionally permutable in G, then G is p-supersoluble.*

Proof Suppose that the assertion is false and let G be a counterexample of minimal order. We proceed with our proof as follows:

(1) $O_{p'}(G) = 1$ and $\Phi(O_p(G)) = 1$

Assume that $O_{p'}(G) \neq 1$. Then, obviously, $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$ and $G/O_{p'}(G)$ is p-soluble. Let $P_1 \in \mathcal{M}_d(P)$. Since

$$|PO_{p'}(G)/O_{p'}(G) : P_1O_{p'}(G)/O_{p'}(G)| = |PO_{p'}(G) : P_1O_{p'}(G)| = p,$$

$P_1O_{p'}(G)/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Since P_1 is s-conditionally permutable in G, by Lemma 2.1, $P_1O_{p'}(G)/O_{p'}(G)$ is s-conditionally permutable in $G/O_{p'}(G)$. Thus, the hypothesis holds for $G/O_{p'}(G)$. By the choice of G, $G/O_{p'}(G)$ is p-supersoluble. It follows that G is p-supersoluble, a contradiction.

Now assume that $\Phi(O_p(G)) \neq 1$. By the same way, we see that the hypothesis holds for $G/\Phi(O_p(G))$. The minimal choice of G implies that $G/\Phi(O_p(G))$ is p-supersoluble. Since the class of all p-supersoluble groups is a saturated formation, we obtain that G is p-supersoluble, a contradiction.

(2) $O_p(G) = R_1 \times \cdots \times R_r$, where R_i $(i = 1, \dots, r)$ is a minimal normal subgroup of order p of G.

Since G is p-soluble and $O_{p'}(G) = 1$, we have $O_p(G) \neq 1$. Let N be an arbitrary minimal normal subgroup of G contained in $O_p(G)$. If $N \leqslant \Phi(P)$, then by Lemma 2.1, we see that the quotient group G/N satisfies the hypothesis. The minimal choice of G implies that G/N is p-supersoluble and consequently G is p-supersouble, a contradiction. Thus $N \nleqslant \Phi(P)$. Since $\Phi(P) = \bigcap_{i=1}^{d} P_i$, where $P_i \in \mathcal{M}_d(P)$, without loss of generality, we may assume that $N \nleqslant P_1$. Let $N_1 = N \cap P_1$. Then $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$. Hence, N_1 is a maximal subgroup of N. Since P_1 is s-conditionally permutable in G, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $P_1Q \leqslant G$ and so $N_1 = N \cap P_1 = N \cap P_1Q \trianglelefteq P_1Q$. It follows that $Q \leqslant N_G(N_1)$. On the other hand, $N = N \cap P_1 \trianglelefteq P$. Therefore $N_1 \trianglelefteq G$. But since N is the minimal normal subgroup of G, $N_1 = 1$ and N is a cyclic subgroup of order p. Hence $N \cap P_1 = 1$. By Huppert [7, I.17.4], there exists a subgroup M of G such that $G = NM$ and $N \cap M = 1$. Obviously, $N \nleqslant \Phi(G)$. This induces that $O_p(G) \cap \Phi(G) = 1$. Thus by using Lemma 2.4, we obtain that $O_p(G) = R_1 \times \cdots \times R_r$, where R_i $(i = 1, \dots, r)$ is a minimal normal subgroup of order p of G.

(3) The final contradiction.

Since $G/C_G(R_i)$ is isomorphic with some subgroup of $\mathrm{Aut}(R_i)$ and $|\mathrm{Aut}(R_i)| = p - 1$, $G/C_G(O_p(G)) = G/(\cap_{i=1}^{r} C_G(R_i))$ is p-supersoluble. On the other hand, since G is p-soluble and $O_{p'}(G) = 1$, $C_G(O_p(G)) \leqslant O_p(G)$ by [2, Theorem 1.8.18]. Thus $G/O_p(G)$ is p-supersoluble. Now the claim (2) implies that G is p-supersoluble. The final contradiction completes the proof. □

As immediate corollaries of Theorem 3.1, we have the following:

Corollary 3.1.1 *Let G be a soluble group. If every member of some fixed $\mathcal{M}_d(P)$ is s-*

conditionally permutable in G, for each prime p in $\pi(G)$ and a Sylow p-subgroup P of G, then G is supersoluble.

Corollary 3.1.2 ([8, Lemma 4.1]) *Let G be a p-soluble group. If every maximal subgroup of every Sylow p-subgroup of G is s-conditionally permutable in G, then G is p-supersoluble.*

Corollary 3.1.3 ([10, Theorem 1.3]) *Let G be a p-soluble group and P a Sylow p-subgroup of G. Suppose that every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-supersoluble.*

Following [13], a subgroup H of a group G is said to be s-semipermutable in G if for every prime p with $(p, |H|) = 1$, H permutes with every Sylow p-subgroup of G.

Corollary 3.1.4 *Let G be a p-soluble group and P a Sylow p-subgroup of G. Suppose that every member of some fixed $\mathcal{M}_d(P)$ is s-semipermutable in G, then G is p-supersoluble.*

Theorem 3.2 *Let G be a p-soluble group and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of some fixed $\mathcal{M}_d(P)$ is s-conditionally permutable in G, then G is p-nilpotent.*

Proof Suppose that the theorem is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$ and by [2, Lemma 3.6.10] $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. Let $P_1 \in \mathcal{M}_d(P)$. Obviously, $P_1O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)$. By the hypothesis, P_1 is s-condictionally permutable in G. Then by Lemma 2.1 we see that $PO_{p'}(G)/O_{p'}(G)$ is s-conditionally permutable in $G/O_{p'}(G)$. Thus the hypothesis holds for $G/O_{p'}(G)$. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and consequently G is p-nilpotent, a contradiction.

(2) $O_p(G) = R_1 \times \cdots \times R_r$, where R_i $(i = 1, \dots, r)$ is a minimal normal subgroup of G of order p (see the proof (2) of Theorem 3.1).

(3) The final contradiction.

Since $G/C_G(R_i)$ is an abelian group of exponent $p-1$, $P \leqslant \bigcap_{i=1}^{r} C_G(R_i) = C_G(O_p(G))$ by (2). Moreover, by (1) and [2, Theorem 1.8.18], $C_G(O_p(G)) \leqslant O_p(G)$. Hence $P = O_p(G)$ and therefore $G = N_G(P)$ is p-nilpotent. The final contradiction completes the proof. □

Corollary 3.2.1 *Let p be a prime dividing the order of G and H a p-soluble normal subgroup of G such that G/H is p-nilpotent. Suppose that P is a Sylow p-subgroup of H. If $N_G(P)$ is p-nilpotent and every member in $\mathcal{M}_d(P)$ is s-conditionally permutable in G, then G is p-nilpotent.*

Proof Since $N_H(P) \leqslant N_G(P)$, $N_H(P)$ is p-nilpotent. By Lemma 2.1(1), every member in $\mathcal{M}_d(P)$ is s-conditionally permutable in H. Hence by Theorem 3.2, H is p-nilpotent. Let N be the normal Hall p'-subgroup of H. Then $N \trianglelefteq G$. We claim that G/N (with repect to H/N) satisfies the hypothesis of the corollary. In fact, $H/N \trianglelefteq G/N$, $(G/N)/(H/N) \cong G/N$ is p-

nilpotent and $N_{G/N}(NP/N) = N_G(P)N/N$ is p-nilpotent. Let P_1N/N be a maximal subgroup of PN/N, where $P_1 \in \mathcal{M}_d(P)$. Since P_1 is s-conditionally permutable in G, P_1N/N is s-conditionally permutable in G/N by Lemma 2.1. Hence our claim holds. If $N \neq 1$, then G/N is p-nilpotent by induction. It follows that G is p-nilpotent. If $N = 1$, then $H = P$ is a p-group. In this case, $G = N_G(P)$ is p-nilpotent. This completes the proof. □

Theorem 3.3 *Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal Hall subgroup H of G such that $G/H \in \mathfrak{F}$ and for every Sylow subgroup P of H, every member of $\mathcal{M}_d(P)$ is s-conditionally permutable in G.*

Proof The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Let q be the largest prime divisor of $|H|$ and Q be a sylow q-subgroup of H. Then:

(1) $Q \trianglelefteq G$.

By Lemma 2.1(1), every member of $\mathcal{M}_d(P)$ is s-conditionally permutable in H. Hence by Corollary 3.1.1, H is supersoluble. Then, since q is the largest prime divisor of $|H|$, $Q \trianglelefteq H$. Since Q char $H \trianglelefteq G$, $Q \trianglelefteq G$.

(2) $\Phi(Q) = 1$.

Since $Q \trianglelefteq G$, $\Phi(Q) \subseteq \Phi(G)$. Obviously, $G/\Phi(Q)$ satisfies the hypothesis. If $\Phi(Q) \neq 1$, then $G/\Phi(Q) \in \mathfrak{F}$. Then, since $\mathfrak{F}$ is a saturated formation, we have $G \in \mathfrak{F}$, a contradiction. Therefore $\Phi(Q) = 1$.

(3) Every minimal normal subgroup of G contained in Q is of order q.

Let N be an arbitrary minimal normal subgroup of G contained in Q. Since $N \nleqslant \Phi(Q)$, we can, without loss of generality, assume that $N \nleqslant Q_1$, where $Q_1 \in \mathcal{M}_d(Q)$. Let $N_1 = N \bigcap Q_1$. Then $|N : N_1| = |N : N \bigcap Q_1| = |NQ_1 : Q_1| = |Q : Q_1| = q$. Hence N_1 is the maximal subgroup of N and so $N_1 \trianglelefteq N$. Since Q_1 is s-conditionally permutable in G, for any $p \in \pi(G)$ with $p \neq q$, there exists a Sylow p-subgroup P of G such that $Q_1P \leqslant G$. Thus, $N_1 = N \cap Q_1 \leqslant N \cap Q_1P \trianglelefteq Q_1P$. It follows that $Q_1 \leqslant N_G(N_1)$ and $P \leqslant N_G(N_1)$. Consequently, $Q = NQ_1 \leqslant N_G(N_1)$. Since H is the Hall subgroup of G by hypothesis, Q is also a Sylow q-subgroup of G. This shows that $N_1 \trianglelefteq G$ and so $N_1 = 1$. Hence N is a cyclic subgroup of prime order q. It is easy to see that $N \nsubseteq \Phi(G)$ and so $Q \cap \Phi(G) = 1$. Therefore, by Lemma 2.4, $Q = R_1 \times \cdots \times R_r$, where R_i $(i = 1, \ldots, r)$ is the minimal normal subgroup of G of order q.

(4) The final contradiction.

It is easy to see that G/Q satisfies the hypothesis. The minimal choice of G implies that $G/Q \in \mathfrak{F}$. By (3), we see that every chief factor of G contained in Q is $\mathfrak{U}$-center. Since $\mathfrak{U} \subseteq \mathfrak{F}$, by [2, Lemma 3.1.6 and Lemma 3.18], we obtain that $G \in \mathfrak{F}$. The final contradiction completes the proof. □

Theorem 3.4 *Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$*

and, for every Sylow p-subgroup P of $F(H)$ satisfying $(|G : F(H)|, p) = 1$, every member of $\mathcal{M}_d(P)$ is s-conditionally permutable in G.

Proof The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is not true and let G be a counterexample of minimal order. Let P be an arbitrary Sylow p-subgroup of $F(H)$. Then P char $F(H) \trianglelefteq G$ and so $P \trianglelefteq G$. Since $\Phi(P)$ char $P \trianglelefteq G$, $\Phi(P) \trianglelefteq G$. We now proceed with our proof as follows:

(1) $\Phi(P) = 1$.

Assume that $\Phi(P) \neq 1$. Obviously, $(G/\Phi(P))/(H/\Phi(P)) \cong G/H \in \mathfrak{F}$. Let $F(H/\Phi(P)) = T/\Phi(P)$. Then $F(H) \subseteq T$. On the other hand, since $\Phi(P) \subseteq \Phi(G)$, T is nilpotent by [12, Theorem IV 3.7]. It follows that $T \subseteq F(H)$ and so $T = F(H)$. Since $\Phi(P) = \bigcap_{i=1}^{d} P_i$, where $P_i \in \mathcal{M}_d(P)$, $P_i/\Phi(P)$ is a maximal subgroup of $P/\Phi(P)$. Obviously, $\mathcal{M}_d(P/\Phi(P))$=$\{P_1/\Phi(P), \ldots, P_d/\Phi(P)\}$ and $(|G/\Phi(P) : F(H/\Phi(P))|, p) = (|G/\Phi(P) : F(H)/\Phi(P)|, p) = (|G : F(H)|, p) = 1$. Since P_i is s-conditionally permutable in G by hypothesis, by Lemma 2.1, $P_i/\Phi(P)$ is s-conditionally permutable in $G/\Phi(P)$. Let $Q_1\Phi(P)/\Phi(P)$ be a maximal subgroup of the Sylow q-subgroup $Q\Phi(P)/\Phi(P)$ of $F(H)/\Phi(P) = F(H/\Phi(P))$, where $q \neq p$, Q is a Sylow q-subgroup of $F(H)$ and $Q_1 \in \mathcal{M}_t(Q)$. By the hypothesis, Q_1 is s-conditionally permutable in G. Hence by Lemma 2.1, $Q\Phi(P)/\Phi(P) = Q_1\Phi(P)/\Phi(P)$ is s-conditionally permutable in $G/\Phi(P)$. The minimal choice of G implies that $G/\Phi(P) \in \mathfrak{F}$. Then, since $\mathfrak{F}$ is a saturated formation, we obtain that $G \in \mathfrak{F}$, a contradiction.

(2) Every minimal normal subgroup of G contained in P is of order p.

Let N be an arbitrary minimal normal subgroup of G contained in P. Since $\Phi(P) = 1$, $N \nleqslant \Phi(P)$. Without loss of generality, we may assume that $N \nleqslant P_1$, where $P_1 \in \mathcal{M}_d(P)$. Let $N_1 = N \cap P_1$. Since $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$, N_1 is a maximal subgroup of N and so $N_1 \trianglelefteq N$. Since P_1 is s-conditionally permutable in G, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q such that $P_1Q \leqslant G$. Hence $N_1 = N \cap P_1 \leqslant N \cap P_1Q \trianglelefteq P_1Q$. It follows that $P_1 \leqslant N_G(N_1)$ and $Q \leqslant N_G(N_1)$. Consequently, $P = NP_1 \leqslant N_G(N_1)$. Since $(|G : F(H)|, p) = 1$, P is also a Sylow p-subgroup of G. This shows that $N_1 \trianglelefteq G$. Since N is a minimal normal subgroup of G, $N_1 = 1$ and thereby N is a cyclic subgroup of order p.

(3) The final contradiction.

By (2), we know that $F(H) = R_1 \times \cdots \times R_s$, where R_i $(i = 1, \ldots, s)$ is a minimal normal subgroup of order p of G. Since $G/C_G(R_i) \cong \mathrm{Aut}(R_i)$, $G/C_G(R_i)$ is cyclic. Thus, $G/(\cap_{i=1}^{s} C_G(R_i)) \in \mathfrak{F}$. Because $\cap_{i=1}^{s} C_G(R_i) = C_G(F(H))$, we have $G/C_G(F(H)) \in \mathfrak{F}$. Therefore, $G/C_H(F(H)) = G/(H \cap C_G(F(H))) \in \mathfrak{F}$. Since $F(H)$ is an abelian group, we know that $F(H) \subseteq C_H(F(H))$. On the other hand, we have $C_H(F(H)) \subseteq F(H)$ for H is soluble. Hence, $F(H) = C_H(F(H))$. So $G/F(H) = G/C_H(F(H)) \in \mathfrak{F}$. Thus by Theorem 3.3, $G \in \mathfrak{F}$. The final contradiction completes the proof. □

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