The Influence of $s$-Conditional Permutability of Subgroups on the Structure of Finite Groups

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Abstract Let $G$ be a finite group. Fix a prime divisor $p$ of $|G|$ and a Sylow $p$-subgroup $P$ of $G$, let $d$ be the smallest generator number of $P$ and $M_d(P)$ denote a family of maximal subgroups $P_1, P_2, \ldots, P_d$ of $P$ satisfying $\bigcap_{i=1}^d P_i = \Phi(P)$, the Frattini subgroup of $P$. In this paper, we shall investigate the influence of $s$-conditional permutability of the members of some fixed $M_d(P)$ on the structure of finite groups. Some new results are obtained and some known results are generalized.

Keywords finite groups; $s$-conditionally permutable groups; saturated formations; supersoluble groups; nilpotent groups.

Document code A
MR(2010) Subject Classification 20D10; 20D20; 20D25
Chinese Library Classification O151.2

1. Introduction

Recall that a subgroup $H$ of a group $G$ is said to be permutable with a subgroup $T$ of $G$ if $HT = TH$. A subgroup $H$ of a group $G$ is called a permutable subgroup [1] (or quasinormal subgroup) [11] of $G$ if $H$ permutes with all subgroups of $G$. As a development, recently, Guo, Shum and Skiba [3–6] introduced the concept of $X$-permutable subgroup and $X$-semipermutable subgroup: Let $X$ be a nonempty subset of $G$. A subgroup $H$ is said to be $X$-permutable in $G$ if for every $x \in X$ such that $HT^x = T^xH$. A subgroup $H$ is said to be $X$-semipermutable in $G$ if it is $X$-permutable with every subgroup $T_1$ of some supplement $T$ of $H$ in $G$. Later on, the following concepts were also introduced: A subgroup $H$ is said to be $s$-conditionally permutable in $G$ (see [8]) if for every Sylow subgroup $T$, there exists an element $x \in G$ such that $HT^x = T^xH$. A subgroup $H$ is said to be $SS$-quasinormal [10] in $G$ if there is a supplement $B$ of $H$ to $G$ such that $H$ permutes with every Sylow subgroup $T$ of $B$.
of $B$. Obviously, for any primary subgroup $P$ of $G$, if $P$ is $SS$-quasinormal in $G$, then $P$ is $s$-conditionally permutable, but the converse is not true. By using the above ideas, a series of interesting results have been obtained [3–6,8,10].

The purpose of this paper is to go further into the influence of $s$-conditionally permutable subgroups on the structure of finite groups. Some new results are obtained and some known results are generalized.

Throughout this paper, all groups considered are finite and $G$ denotes a group. The terminology and notations are standard, as in [2] and [7].

2. Preliminaries

In this section, we give the related concepts and some basic results which are needed in this paper.

**Definition 2.1** ([9]) Let $d$ be the smallest generator number of a $p$-group $P$ and $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$ be a set of maximal subgroups of $P$ such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$.

Such subset $\mathcal{M}_d(P)$ is not unique for a fixed $P$ in general [9].

Recall that a class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is called saturated if it contains every group $G$ with $G/\Phi(G) \in \mathfrak{F}$. It is well known that the class of all supersoluble groups is a saturated formation.

**Lemma 2.1** ([8, Lemma 2.1]) Let $H$ be an $s$-conditionally permutable subgroup of $G$. Then:

1) If $H \leq K \leq G$, then $H$ is $s$-conditionally permutable in $K$.
2) If $N \triangleleft G$, then $HN/N$ is $s$-conditionally permutable in $G/N$.
3) $H^g$ is $s$-conditionally permutable in $G$ for each element $g$ of $G$.

The following result is well known.

**Lemma 2.2** Suppose that $P$ is a Sylow subgroup of $G$. If $P \triangleleft \triangleleft G$, then $P \triangleleft \triangleleft G$.

**Lemma 2.3** ([7, IV Theorem 4.7]) If $P$ is a Sylow $p$-subgroup of a group $G$ for some $p \in \pi(G)$ and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then $N$ is $p$-nilpotent.

**Lemma 2.4** ([2, Theorem 1.8.17]) Let $N$ be a non-trivial normal subgroup of a group $G$. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of $N$ is a direct product of some abelian minimal normal subgroups of $G$.

**Lemma 2.5** ([7, III Lemma 3.3])

1) If $N \trianglelefteq G$, $U \trianglelefteq G$ and $N \leq \Phi(U)$, then $N \leq \Phi(G)$.
2) If $M \trianglelefteq G$, then $\Phi(M) \leq \Phi(G)$.

3. Main results

**Theorem 3.1** Let $G$ be a $p$-soluble group and $P$ a Sylow $p$-subgroup of $G$. Suppose that every
The influence of $s$-conditional permutability of subgroups on the structure of finite groups

Suppose that the assertion is false and let $G$ be a counterexample of minimal order. We proceed with our proof as follows:

1. $O_p'(G) = 1$ and $\Phi(O_p(G)) = 1$

Assume that $O_p'(G) \neq 1$. Then, obviously, $PO_p'(G)/O_p'(G)$ is a Sylow $p$-subgroup of $G/O_p'(G)$ and $G/O_p'(G)$ is $p$-soluble. Let $P_1 \in \mathcal{M}_d(P)$. Since

$$|PO_p'(G)/O_p'(G) : P_1O_p'(G)/O_p'(G)| = |PO_p'(G) : P_1O_p'(G)| = p,$$

$P_1O_p'(G)/O_p'(G)$ is a maximal subgroup of $PO_p'(G)/O_p'(G)$. Since $P_1$ is $s$-conditionally permutable in $G$, by Lemma 2.1, $P_1O_p'(G)/O_p'(G)$ is $s$-conditionally permutable in $G/O_p'(G)$. Thus, the hypothesis holds for $G/O_p'(G)$. By the choice of $G$, $G/O_p'(G)$ is $p$-supersoluble. It follows that $G$ is $p$-supersoluble, a contradiction.

Now assume that $\Phi(O_p(G)) \neq 1$. By the same way, we see that the hypothesis holds for $G/\Phi(O_p(G))$. The minimal choice of $G$ implies that $G/\Phi(O_p(G))$ is $p$-supersoluble. Since the class of all $p$-supersoluble groups is a saturated formation, we obtain that $G$ is $p$-supersoluble, a contradiction.

2. $O_p(G) = R_1 \times \cdots \times R_r$, where $R_i (i = 1, \ldots, r)$ is a minimal normal subgroup of order $p$ of $G$.

Since $G$ is $p$-soluble and $O_p'(G) = 1$, we have $O_p(G) \neq 1$. Let $N$ be an arbitrary minimal normal subgroup of $G$ contained in $O_p(G)$. If $N \leqslant \Phi(P)$, then by Lemma 2.1, we see that the quotient group $G/N$ satisfies the hypothesis. The minimal choice of $G$ implies that $G/N$ is $p$-supersoluble and consequently $G$ is $p$-supersoluble, a contradiction. Thus $N \notin \Phi(P)$. Since $\Phi(P) = \bigcap_{i=1}^{d} P_i$, where $P_i \in \mathcal{M}_d(P)$, without loss of generality, we may assume that $N \notin P_1$. Let $N_1 = N \cap P_1$. Then $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$. Hence, $N_1$ is a maximal subgroup of $N$. Since $P_1$ is $s$-conditionally permutable in $G$, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $P_1Q \triangleleft G$ and so $N_1 = N \cap P_1 = N \cap P_1Q \triangleleft P_1Q$.

It follows that $Q \leqslant N_G(N_1)$. On the other hand, $N = N \cap P_1 \triangleleft P$. Therefore $N_1 \leqslant G$. But since $N$ is the minimal normal subgroup of $G$, $N_1 = 1$ and $N$ is a cyclic subgroup of order $p$. Hence $N \cap P_1 = 1$. By Huppert [7, I.17.4], there exists a subgroup $M$ of $G$ such that $G = NM$ and $N \cap M = 1$. Obviously, $N \notin \Phi(G)$. This induces that $O_p(G) \cap \Phi(G) = 1$. Thus by using Lemma 2.4, we obtain that $O_p(G) = R_1 \times \cdots \times R_r$, where $R_i (i = 1, \ldots, r)$ is a minimal normal subgroup of order $p$ of $G$.

3. The final contradiction.

Since $G/C_G(R_i)$ is isomorphic with some subgroup of $\text{Aut}(R_i)$ and $|\text{Aut}(R_i)| = p - 1$, $G/C_G(O_p(G)) = G/(\bigcap_{i=1}^{r} C_G(R_i))$ is $p$-supersoluble. On the other hand, since $G$ is $p$-soluble and $O_p'(G) = 1$, $C_G(O_p(G)) \leqslant O_p(G)$ by [2, Theorem 1.8.18]. Thus $G/O_p(G)$ is $p$-supersoluble. Now the claim (2) implies that $G$ is $p$-supersoluble. The final contradiction completes the proof. □

As immediate corollaries of Theorem 3.1, we have the following:

Corollary 3.1.1 Let $G$ be a soluble group. If every member of some fixed $\mathcal{M}_d(P)$ is $s$-
conditionally permutable in $G$, for each prime $p$ in $\pi(G)$ and a Sylow $p$-subgroup $P$ of $G$, then $G$ is supersoluble.

Corollary 3.1.2 ([8, Lemma 4.1]) Let $G$ be a $p$-soluble group. If every maximal subgroup of every Sylow $p$-subgroup of $G$ is $s$-conditionally permutably in $G$, then $G$ is $p$-supersoluble.

Corollary 3.1.3 ([10, Theorem 1.3]) Let $G$ be a $p$-soluble group and $P$ a Sylow $p$-subgroup of $G$. Suppose that every member of some fixed $\mathcal{M}_d(P)$ is $SS$-quasinormal in $G$, then $G$ is $p$-supersoluble.

Following [13], a subgroup $H$ of a group $G$ is said to be $s$-semipermutable in $G$ if for every prime $p$ with $(p, |H|) = 1$, $H$ permutes with every Sylow $p$-subgroup of $G$.

Corollary 3.1.4 Let $G$ be a $p$-soluble group and $P$ a Sylow $p$-subgroup of $G$. Suppose that every member of some fixed $\mathcal{M}_d(P)$ is $s$-semipermutable in $G$, then $G$ is $p$-supersoluble.

Theorem 3.2 Let $G$ be a $p$-soluble group and $P$ a Sylow $p$-subgroup of $G$. If $N_G(P)$ is $p$-nilpotent and every member of some fixed $\mathcal{M}_d(P)$ is $s$-conditionally permutably in $G$, then $G$ is $p$-nilpotent.

Proof Suppose that the theorem is false and let $G$ be a counterexample of minimal order. Then:

1. $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow $p$-subgroup of $G/O_{p'}(G)$ and by [2, Lemma 3.6.10] $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is $p$-nilpotent. Let $P_1 \in \mathcal{M}_d(P)$. Obviously, $P_1O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)$. By the hypothesis, $P_1$ is $s$-conditionally permutably in $G$. Then by Lemma 2.1 we see that $PO_{p'}(G)/O_{p'}(G)$ is $s$-conditionally permutably in $G/O_{p'}(G)$. Thus the hypothesis holds for $G/O_{p'}(G)$. The minimal choice of $G$ implies that $G/O_{p'}(G)$ is $p$-nilpotent and consequently $G$ is $p$-nilpotent, a contradiction.

2. $O_p(G) = R_1 \times \cdots \times R_r$, where $R_i$ $(i = 1, \ldots, r)$ is a minimal normal subgroup of $G$ of order $p$ (see the proof (2) of Theorem 3.1).

3. The final contradiction.

Since $G/C_G(R_i)$ is an abelian group of exponent $p - 1$, $P \subseteq \bigcap_{i=1}^r C_G(R_i) = C_G(O_p(G))$ by (2). Moreover, by (1) and [2, Theorem 1.8.18], $C_G(O_p(G)) \leq O_p(G)$. Hence $P = O_p(G)$ and therefore $G = N_G(P)$ is $p$-nilpotent. The final contradiction completes the proof. □

Corollary 3.2.1 Let $p$ be a prime dividing the order of $G$ and $H$ a $p$-soluble normal subgroup of $G$ such that $G/H$ is $p$-nilpotent. Suppose that $P$ is a Sylow $p$-subgroup of $H$. If $N_G(P)$ is $p$-nilpotent and every member in $\mathcal{M}_d(P)$ is $s$-conditionally permutably in $G$, then $G$ is $p$-nilpotent.

Proof Since $N_H(P) \leq N_G(P)$, $N_H(P)$ is $p$-nilpotent. By Lemma 2.1(1), every member in $\mathcal{M}_d(P)$ is $s$-conditionally permutably in $H$. Hence by Theorem 3.2, $H$ is $p$-nilpotent. Let $N$ be the normal Hall $p'$-subgroup of $H$. Then $N \unlhd G$. We claim that $G/N$ (with respect to $H/N$) satisfies the hypothesis of the corollary. In fact, $H/N \unlhd G/N$, $(G/N)/(H/N) \cong G/N$ is $p$-
The influence of s-conditional permutability of subgroups on the structure of finite groups

nilpotent and $N_{G/N}(NP/N) = N_G(P)N/N$ is $p$-nilpotent. Let $P_1N/N$ be a maximal subgroup of $PN/N$, where $P_1 \in \mathcal{M}_d(P)$. Since $P_1$ is $s$-conditionally permutable in $G$, $P_1N/N$ is $s$-conditionally permutable in $G/N$ by Lemma 2.1. Hence our claim holds. If $N \neq 1$, then $G/N$ is $p$-nilpotent by induction. It follows that $G$ is $p$-nilpotent. If $N = 1$, then $H = P$ is a $p$-group. In this case, $G = N_G(P)$ is $p$-nilpotent. This completes the proof. □

**Theorem 3.3** Let $\mathfrak{F}$ be a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal Hall subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$ and for every Sylow subgroup $P$ of $H$, every member of $\mathcal{M}_d(P)$ is $s$-conditionally permutable in $G$.

**Proof** The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let $G$ be a counterexample of minimal order. Let $q$ be the largest prime divisor of $|H|$ and $Q$ be a Sylow $q$-subgroup of $H$. Then:

1. $Q \trianglelefteq G$.
   By Lemma 2.1(1), every member of $\mathcal{M}_d(P)$ is $s$-conditionally permutable in $H$. Hence by Corollary 3.1.1, $H$ is supersoluble. Then, since $q$ is the largest prime divisor of $|H|$, $Q \trianglelefteq H$. Since $Q$ char $H \trianglelefteq G$, $Q \subseteq G$.

2. $\Phi(Q) = 1$.
   Since $Q \trianglelefteq G$, $\Phi(Q) \subseteq \Phi(G)$. Obviously, $G/\Phi(Q)$ satisfies the hypothesis. If $\Phi(Q) \neq 1$, then $G/\Phi(Q) \in \mathfrak{F}$. Then, since $\mathfrak{F}$ is a saturated formation, we have $G \in \mathfrak{F}$, a contradiction. Therefore $\Phi(Q) = 1$.

3. Every minimal normal subgroup of $G$ contained in $Q$ is of order $q$.
   Let $N$ be an arbitrary minimal normal subgroup of $G$ contained in $Q$. Since $N \not\subseteq \Phi(Q)$, we can, without loss of generality, assume that $N \not\subseteq Q_1$, where $Q_1 \in \mathcal{M}_d(Q)$. Let $N_1 = N \cap Q_1$. Then $|N : N_1| = |N : N \cap Q_1| = |NQ_1 : Q_1| = |Q : Q_1| = q$. Hence $N_1$ is the maximal subgroup of $N$ and so $N_1 \subseteq N$. Since $Q_1$ is $s$-conditionally permutable in $G$, for any $p \in \pi(G)$ with $p \neq q$, there exists a Sylow $p$-subgroup $P$ of $G$ such that $Q_1P \leq G$. Thus, $N_1 = N \cap Q_1 \leq N \cap Q_1P \leq Q_1P$. It follows that $Q_1 \leq N_G(N_1)$ and $P \leq N_G(N_1)$. Consequently, $Q = NQ_1 \leq N_G(N_1)$. Since $H$ is the Hall subgroup of $G$ by hypothesis, $Q$ is also a Sylow $q$-subgroup of $G$. This shows that $N_1 \subseteq G$ and so $N_1 = 1$. Hence $N$ is a cyclic subgroup of prime order $q$. It is easy to see that $N \not\subseteq \Phi(G)$ and so $Q \cap \Phi(G) = 1$. Therefore, by Lemma 2.4, $Q = R_1 \times \cdots \times R_r$, where $R_i$ ($i = 1, \ldots, r$) is the minimal normal subgroup of $G$ of order $q$.

4. The final contradiction.
   It is easy to see that $G/Q$ satisfies the hypothesis. The minimal choice of $G$ implies that $G/Q \in \mathfrak{F}$. By (3), we see that every chief factor of $G$ contained in $Q$ is $\mathcal{U}$-center. Since $\mathcal{U} \subseteq \mathfrak{F}$, by [2, Lemma 3.1.6 and Lemma 3.18], we obtain that $G \in \mathfrak{F}$. The final contradiction completes the proof. □

**Theorem 3.4** Let $\mathfrak{F}$ be a saturated formation containing the class $\mathcal{U}$ of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$.
and, for every Sylow $p$-subgroup $P$ of $F(H)$ satisfying $([G : F(H)], p) = 1$, every member of $\mathcal{M}_d(P)$ is $s$-conditionally permutable in $G$.

**Proof** The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is not true and let $G$ be a counterexample of minimal order. Let $P$ be an arbitrary Sylow $p$-subgroup of $F(H)$. Then $P \text{ char } H \trianglelefteq G$ and so $P \trianglelefteq G$. Since $\Phi(P) \text{ char } P \trianglelefteq G$, $\Phi(P) \trianglelefteq G$. We now proceed with our proof as follows:

(1) $\Phi(P) = 1$.

Assume that $\Phi(P) \neq 1$. Obviously, $(G/\Phi(P))/(H/\Phi(P)) \cong G/H \in \mathfrak{F}$. Let $F(H/\Phi(P)) = T/\Phi(P)$. Then $F(H) \subseteq T$. On the other hand, since $\Phi(P) \subseteq \Phi(G)$, $T$ is nilpotent by [12, Theorem IV 3.7]. It follows that $T \subseteq F(H)$ and so $T = F(H)$. Since $\Phi(P) = \bigcap_{i=1}^r P_i$, where $P_i \in \mathcal{M}_d(P)$, $P_i/\Phi(P)$ is a maximal subgroup of $P/\Phi(P)$. Obviously, $\mathcal{M}_d(P/\Phi(P)) = \{P_1/\Phi(P), \ldots, P_d/\Phi(P)\}$ and $(|G/\Phi(P) : F(H/\Phi(P))|, p) = (|G/\Phi(P) : F(H/\Phi(P))|, p) = (|G : F(H)|, p) = 1$. Since $P_i$ is $s$-conditionally permutable in $G$ by hypothesis, by Lemma 2.1, $P_i/\Phi(P)$ is $s$-conditionally permutable in $G/\Phi(P)$. Let $Q_1\Phi(P)/\Phi(P)$ be a maximal subgroup of the Sylow $q$-subgroup $Q\Phi(P)/\Phi(P)$ of $F(H)/\Phi(P) = F(H/\Phi(P))$, $q \neq p$, $Q$ is a Sylow $q$-subgroup of $F(H)$ and $Q_1 \in \mathcal{M}_q(Q)$. By the hypothesis, $Q_1$ is $s$-conditionally permutable in $G$. Hence by Lemma 2.1, $Q\Phi(P)/\Phi(P) = Q_1\Phi(P)/\Phi(P)$ is $s$-conditionally permutable in $G/\Phi(P)$. The minimal choice of $Q$ implies that $G/\Phi(P) \in \mathfrak{F}$. Then, since $\mathfrak{F}$ is a saturated formation, we obtain that $G \in \mathfrak{F}$, a contradiction.

(2) Every minimal normal subgroup of $G$ contained in $P$ is of order $p$.

Let $N$ be an arbitrary minimal normal subgroup of $G$ contained in $P$. Since $\Phi(P) = 1$, $N \not\subseteq \Phi(P)$. Without loss of generality, we may assume that $N \not\subseteq P_1$, where $P_1 \in \mathcal{M}_d(P)$. Let $N_1 = N \cap P_1$. Since $|N : N_1| = |N : N \cap P_1|$ is a maximal subgroup of $N$ and so $N_1 \trianglelefteq N$. Since $P_1$ is $s$-conditionally permutable in $G$, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow $q$-subgroup $Q$ such that $P_1Q \trianglelefteq G$. Hence $N_1 = N \cap P_1 \leq N \cap P_1Q \trianglelefteq P_1Q$. It follows that $P_1 \trianglelefteq N_G(N_1)$ and $Q \trianglelefteq N_G(N_1)$. Consequently, $P = NP_1 \leq N_G(N_1)$. Since $(|G : F(H)|, p) = 1$, $P$ is also a Sylow $p$-subgroup of $G$. This shows that $N_1 \trianglelefteq G$. Since $N$ is a minimal normal subgroup of $G$, $N_1 = 1$ and thereby $N$ is a cyclic subgroup of order $p$.

(3) The final contradiction.

By (2), we know that $F(H) = R_1 \times \cdots \times R_s$, where $R_i$ ($i = 1, \ldots, s$) is a minimal normal subgroup of order $p$ of $G$. Since $G/C_G(R_i) \cong \text{Aut}(R_i)$, $G/C_G(R_i)$ is cyclic. Thus, $G/(\cap_{i=1}^s C_G(R_i)) \in \mathfrak{F}$. Because $\cap_{i=1}^s C_G(R_i) = C_G(F(H))$, we have $G/C_G(F(H)) \in \mathfrak{F}$. Therefore, $G/C_H(F(H)) = G/(H \cap C_G(F(H))) \in \mathfrak{F}$. Since $F(H)$ is an abelian group, we know that $F(H) \leq C_H(F(H))$. On the other hand, we have $C_H(F(H)) \leq F(H)$ for $H$ is soluble. Hence, $F(H) = C_H(F(H))$. So $G/F(H) = G/C_H(F(H)) \in \mathfrak{F}$. Thus by Theorem 3.3, $G \in \mathfrak{F}$. The final contradiction completes the proof. $\square$
The influence of $s$-conditional permutability of subgroups on the structure of finite groups

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