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A Graph Associated with $|\mathrm{cd}(G)|-1$ Degrees of a Solvable Group

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Abstract Let G be a group. We consider the set $\mathrm{cd}(G)\backslash\{m\}$, where $m\in \mathrm{cd}(G)$. We define the graph $\Delta(G-m)$ whose vertex set is $\rho(G-m)$, the set of primes dividing degrees in $\mathrm{cd}(G)\backslash\{m\}$. There is an edge between p and q in $\rho(G-m)$ if pq divides a degree $a\in\mathrm{cd}(G)\backslash\{m\}$. We show that if G is solvable, then $\Delta(G-m)$ has at most two connected components.

Keywords solvable groups; irreducible character degrees.

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1. Introduction

Throughout the following, all groups are assumed to be finite. For a group G, $\mathrm{cd}(G)$ is the set of irreducible character degrees, $\Delta(G)$ denotes the prime degree graph of G, whose vertex set is $\rho(G)$, the set of primes that divide degrees in $\mathrm{cd}(G)$. There is an edge between p and q in $\rho(G)$ if pq divides some degree $a\in\mathrm{cd}(G)$, and $n(\Delta(G))$ denotes the number of connected components of $\Delta(G)$. The basic results on the relationship between $\mathrm{cd}(G)$ and the structure of G can be found in [1–3]. Many recent papers have studied the influence of $\mathrm{cd}(G)$ on the structure of G. Several papers have studied other graphs, such as [4–9]. It is well known that if G is solvable, then $\Delta(G)$ has at most two connected components. In this paper, we are particulary interested in the question of the set $\mathrm{cd}(G)\backslash\{m\}$, where $m\in\mathrm{cd}(G)$. We define the graph $\Delta(G-m)$ whose vertex set is $\rho(G-m)$, the set of primes dividing degrees in $\mathrm{cd}(G)\backslash\{m\}$. There is an edge between p and q in $\rho(G-m)$ if pq divides a degree $a\in\mathrm{cd}(G)\backslash\{m\}$. $\pi(m)$ denotes the set of primes which divide m, $n(\Delta(G-m))$ denotes the number of connected components of $\Delta(G-m)$. If G is abelian or $\mathrm{cd}(G)=\{1,a\}$ and $m=a$, then we have that $\mathrm{cd}(G)\backslash\{m\}=\emptyset$ or $\mathrm{cd}(G)\backslash\{m\}=\{1\}$, and we define $n(\Delta(G-m))=0$. It is obvious that ordinary $\Delta(G-m)$ has less edges or vertices than $\Delta(G)$ (such as, let $G=S_4\times D_4$. Then $\mathrm{cd}(G)=\{1,2,3,6\}$, where if $m=6$, then $\mathrm{cd}(G-m)=\{1,2,3\}$,

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so $\Delta(G-m)$ has less edges than $\Delta(G)$). But we can also show that if G is a solvable group, then $n(\Delta(G-m))\le 2$.

The following is our conclusion.

Theorem *Let G be a solvable group. Then $\Delta(G-m)$ has at most 2 connected components, that is, $n(\Delta(G-m))\le 2$.*

2. Proof of Theorem

At first we introduce some definitions [7] of the following Lemma 2.1. Fix a set of prime π. The set $\mathrm{cd}^{\pi}(G)$ denotes the set of character degrees that are divisible only by primes in π. The graph $\Delta^{\pi}(G)$, whose vertex set is $\rho^{\pi}(G)$, denotes the set of primes dividing degrees in $\mathrm{cd}^{\pi}(G)$. There is an edge between p and q if pq divides a degree $a\in\mathrm{cd}^{\pi}(G)$.

Lemma 2.1 ([7]) *Let π be a set of primes, and let G be a π-solvable group. Then $\Delta^{\pi}(G)$ has at most 2 connected components.*

Lemma 2.2 ([3]) *Let G be solvable and let π be a set of primes contained in $\Delta(G)$. Assume that $|\pi|\ge 3$. Then there exist distinct $u,v\in\pi$ such that $uv|\chi(1)$ for some $\chi\in\mathrm{Irr}(G)$.*

Lemma 2.3 ([2]) *Let G be solvable and assume that G' is the unique minimal normal subgroup of G. Then all nonlinear irreducible characters of G have equal degree f and one of the following situations holds:*

(a) G is a p-group, $Z(G)$ is cyclic and $G/Z(G)$ is elementary abelian of order f^2.

(b) G is a Frobenius group with an abelian Frobenius complement of order f. Also, G' is the Frobenius kernel and is an elementary abelian p-group.

Lemma 2.4 ([2]) *Let $N\lhd G$ and let $\chi\in\mathrm{Irr}(G)$ be such that $\chi_N=\theta\in\mathrm{Irr}(N)$. Then the characters $\beta\chi$ for $\beta\in\mathrm{Irr}(G/N)$ are irreducible, distinct for distinct β and are all of the irreducible constituents of θ^G.*

Lemma 2.5 ([2]) *Let $N\lhd G$ and $\chi\in\mathrm{Irr}(G)$. Let $\theta\in\mathrm{Irr}(N)$ be a constituent of χ_N. Then $\chi(1)/\theta(1)$ divides $|G:N|$.*

Proof of Theorem If G is abelian, then $n(\Delta(G-m))=0$. If $m=1$, then $\Delta(G-m)=\Delta(G)$, and thus $n(\Delta(G-m))=n(\Delta(G))\le 2$ by Lemma 2.2. So we can assume that G is not abelian and $m>1$. We prove the Theorem by the following two cases.

Case 1 Suppose $|\pi(m)\cap\rho(G-m)|<|\pi(m)|$, that is to say there is $p\in\pi(m)$ and p is not in $\rho(G-m)$.

Let $\pi=\rho(G-m)$. By the previous paragraph we know that $\Delta(G-m)=\Delta^{\pi}(G)$, and by Lemma 2.1 this implies that $n(\Delta(G-m))\le 2$.

Case 2 Suppose $|\pi(m)\cap\rho(G-m)|=|\pi(m)|$, so $\rho(G)=\rho(G-m)$.

Let $K\trianglelefteq G$ such that K is maximal among those subgroups with G/K nonabelian. By the results of Isaacs in [2, Chapter 12], we know that $\mathrm{cd}(G/K)=\{1,f\}$. Using Lemma 2.3, we know that G/K is either a p-group for some prime p, or a Frobenius group. Let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = m$.

At first consider G/K is a p-group. Then, it follows that $f = p^e$ for some positive integer e. If $\gcd(p, m) = 1$, then χ restricts irreducibly to K. By using Gallagher's Theorem (Lemma 2.4), we have that $\chi(1)f \in \mathrm{cd}(G)$. So $n(\Delta(G)) = n(\Delta(G - m))$, and thus $n(\Delta(G - m)) \leq 2$. If, on the other hand, p divides m, then suppose $\pi(m) = \{p\}$, and it is seen that $n(\Delta(G - m)) \leq 2$. So we suppose that $\pi(m) \supset \{p\}$ and $\pi(m) \neq \{p\}$. Let $\chi_K(1) = et\theta(1)$, where $\theta \in \mathrm{Irr}(K)$. So $m = \chi_K(1) = et\theta(1)$, but by Lemma 2.5, we know that $et \mid |G/K|$, a power of prime p. And thus $\theta(1) \neq 1$. As the condition of Theorem is subgroup-closed, by induction of the orders of groups, we have $n(\Delta(K - \theta(1))) \leq 2$.

By Lemma 2.5 we know that $\pi(\psi_K(1)) = \pi(\varphi(1))$ or $\pi(\psi_K(1)) = \pi(\varphi(1)) \cup \{p\}$ for every $\psi \in \mathrm{Irr}(G)$ and φ is an irreducible constituent of ψ_K, and thus $\rho(G - m) = \rho(G) \subseteq \rho(K) \cup \{p\}$. If there is $\varphi_1(1) \in \mathrm{cd}(K)\backslash\{\theta(1)\}$ and $p|\varphi_1(1)$, then $n(\Delta(G - m)) = n(\Delta(K - \theta(1))) \leq 2$. So we know that p does not divide $\varphi(1)$ to every $\varphi(1) \in \mathrm{cd}(K)\backslash\{\theta(1)\}$. Let $\chi_i(1)$ be an irreducible constituent of φ^G for some $1 \neq \varphi(1) \in \mathrm{cd}(K)\backslash\{\theta(1)\}$. Then either p does not divide $\chi_i(1)$ or $p|\chi_i(1)$. If p does not divide $\chi_i(1)$, then $\gcd(\chi_i(1), |G/K|) = 1$. It follows that $\chi_i(1)f \in \mathrm{cd}(G)$ by Lemma 2.4, this implies that $n(\Delta(G - m)) = n(\Delta(K - \theta(1))) \leq 2$. Otherwise, if $p|\chi_i(1)$, and thus there is an edge between p and some vertex on $\Delta(G - m)$. We can conclude that $n(\Delta(G - m)) = n(\Delta(K - \theta(1))) \leq 2$. So we have $n(\Delta(G - m)) \leq 2$ in every case.

Suppose G/K is a Frobenius group. As stated in the previous paragraph, we know that the Frobenius Kernel N/K is an elementary abelian p-group and $|G : N| = f$. Consider $\psi \in \mathrm{Irr}(G)$ such that $\psi(1) > 1$. We will show that $\psi(1)$ either lies in the same connected component as f or is divisible by p. Suppose that $\psi(1)$ and f lie in different connected components of $\Delta(G)$. If ψ restricts irreducibly to K, then from Gallagher's Theorem (Lemma 2.4) we know that $\psi(1)f \in \mathrm{cd}(G)$. This implies that $\psi(1)$ lies in the same component as f, which contradicts the assumption. So we know that ψ does not restrict irreducibly to K. But since $\psi(1)$ is coprime to f, we know that ψ restricts irreducibly to N. Hence, we can conclude that p divides $\psi(1)$. And thus for every $\psi(1) \in \mathrm{cd}(G)\backslash\{m\}$, we have $\psi(1)$ either lies in the same connected components as f or is divisible by p. So $n(\Delta(G - m)) \leq 2$. We have proved the theorem. □

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