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Application of the Residue Theorem to Trigonometric Sum Identities

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Abstract By evaluating a contour integral with the Cauchy residue theorem, we prove a general summation formula on trigonometric sum, which contains several interesting trigonometric identities as special cases.

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1. Introduction

Trigonometric sums have important applications in classical analysis, such as integer valued problems by Byrne and Smith [2], Dedekind sums by Gessel [6] and Zagier [8], the matrix spectrum by Calogero [3, §2.4.5.3] as well as trigonometric approximation and interpolation in Kress [7, §8.2]. Berndt and Yeap [1] have employed the Cauchy residue theorem to treat the trigonometric reciprocity; Chu [5] and Wang [9] have established many closed formulae of trigonometric sums. The purpose of this paper is to investigate some parametric trigonometric sums. The main theorem will be shown in the second section, where the Cauchy residue theorem will be employed to evaluate a contour integral with the integrand and contour being properly devised. As applications, several interesting examples will be illustrated in the last section, including those due to Chu [4] and Wang [9].

2. Contour integration

Theorem 1 *Let $P(\theta)$ be a polynomial of degree $< 2n$ in $\cos\theta$. Then for a real parameter y, there holds the following trigonometric sum identity:*

$$\sum_{k=0}^{2n-1}\frac{\sin(y+\frac{k\pi}{n})P(y+\frac{k\pi}{n})}{\cos(y+\frac{k\pi}{n})-\cos\theta}=\frac{2n\sin 2nyP(\theta)}{\cos 2ny-\cos 2n\theta}.$$

Proof First, we suppose $0<y<\pi/n$ and $0<\theta<2\pi$. Let $C=C_R$ denote the positively oriented indented rectangle with vertices at $(\pm iR)$ and $(2\pi\pm iR)$ where R is a real number. For the complex function defined by

$$f(\alpha)=\frac{2n\sin 2ny\sin\alpha P(\alpha)}{\big(\cos 2n\alpha-\cos 2ny\big)\big(\cos\alpha-\cos\theta\big)},$$

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consider the following contour integral

$$\frac{1}{2\pi i}\oint_C f(\alpha)\mathrm{d}\alpha. \tag{1}$$

It is not hard to see that $f(\alpha)$ has $4n+2$ simple poles inside C which can be explicitly displayed as $\{\theta, 2\pi-\theta\}$ and $\{y+\frac{k\pi}{n}, 2\pi-y-\frac{k\pi}{n}\}$ with $k=0,1,\ldots,2n-1$.

For $\{\alpha=\theta\}$ and $\{\alpha=2\pi-\theta\}$, it is routine to compute the corresponding residues

$$\begin{aligned}
\operatorname*{Res}_{\alpha=\theta} f(\alpha) &= \lim_{\alpha\to\theta}\frac{\alpha-\theta}{\cos\alpha-\cos\theta}\frac{2n\sin 2ny\sin\alpha P(\alpha)}{\cos 2n\alpha-\cos 2ny} = \frac{2n\sin 2nyP(\theta)}{\cos 2ny-\cos 2n\theta};\\
\operatorname*{Res}_{\alpha=2\pi-\theta} f(\alpha) &= \lim_{\alpha\to 2\pi-\theta}\frac{\alpha-2\pi+\theta}{\cos\alpha-\cos\theta}\frac{2n\sin 2ny\sin\alpha P(\alpha)}{\cos 2n\alpha-\cos 2ny}\\
&= \frac{2n\sin 2nyP(2\pi-\theta)}{\cos 2ny-\cos 2n(2\pi-\theta)} = \frac{2n\sin 2nyP(\theta)}{\cos 2ny-\cos 2n\theta}.
\end{aligned}$$

When $\alpha=y+\frac{k\pi}{n}$ and $\alpha=2\pi-y-\frac{k\pi}{n}$ with $k=0,1,\ldots,2n-1$, we can show that

$$\begin{aligned}
\operatorname*{Res}_{\alpha=y+\frac{k\pi}{n}} f(\alpha) &= \lim_{\alpha\to y+\frac{k\pi}{n}}\frac{\alpha-y-\frac{k\pi}{n}}{\big(\cos 2n\alpha-\cos 2ny\big)}\frac{2n\sin 2ny\sin\alpha P(\alpha)}{\big(\cos\alpha-\cos\theta\big)}\\
&= -\frac{\sin(y+\frac{k\pi}{n})P(y+\frac{k\pi}{n})}{\cos(y+\frac{k\pi}{n})-\cos\theta};\\
\operatorname*{Res}_{\alpha=2\pi-y-\frac{k\pi}{n}} f(\alpha) &= \lim_{\alpha\to 2\pi-y-\frac{k\pi}{n}}\frac{\alpha-2\pi+y+\frac{k\pi}{n}}{\cos 2n\alpha-\cos 2ny}\frac{2n\sin 2ny\sin\alpha P(\alpha)}{\cos\alpha-\cos\theta}\\
&= -\frac{\sin 2ny\sin\left(2\pi-y-\frac{k\pi}{n}\right)P(2\pi-y-\frac{k\pi}{n})}{\sin 2n(2\pi-y-\frac{k\pi}{n})\big\{\cos(2\pi-y-\frac{k\pi}{n})-\cos\theta\big\}}\\
&= -\frac{\sin(y+\frac{k\pi}{n})P(y+\frac{k\pi}{n})}{\cos(y+\frac{k\pi}{n})-\cos\theta}.
\end{aligned}$$

We are now in position to evaluate the integral displayed in (1). Since $f(\alpha)$ has period 2π, the integral on the two opposite vertical sides of C vanishes. Therefore, we only consider the integral on the two horizontal sides of C. Recalling the Euler formulae

$$\sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}, \quad \cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$$

and keeping in mind that $P(\theta)$ is a polynomial of degree $< 2n$ in $\cos\theta$, we may consider $P(\alpha)\sin\alpha$ as a formal polynomial consisting of terms $e^{mi\alpha}$ with $|m|\le 2n$ and the coefficients of $e^{\pm 2ni\alpha}$ different from zero. For the same reason, we can also consider the trigonometric function $(\cos 2n\alpha-\cos 2ny)(\cos\alpha-\cos\theta)$ as a formal polynomial consisting of terms $e^{mi\alpha}$ with $|m|\le 2n+1$ and the coefficients of $e^{\pm(2n+1)i\alpha}$ different from zero.

Therefore, $f(\alpha)$ is a proper fraction in $e^{i\alpha}$. Writing $\alpha=\lambda+i\mu$ with λ and μ being real, we have no difficulty in verifying that

$$\lim_{\mu\to\pm\infty} f(\alpha) = 0,$$

which implies consequently

$$\frac{1}{2\pi i}\oint_C f(\alpha)\mathrm{d}\alpha = \frac{1}{2\pi i}\int_{y+2\pi-\varepsilon}^{y-\varepsilon} 0\mathrm{d}\lambda + \frac{1}{2\pi i}\int_{y-\varepsilon}^{y+2\pi-\varepsilon} 0\mathrm{d}\lambda = 0.$$

According to the residue theorem, this completes the proof of Theorem 1. The conditions $0 <$

$y < \pi/n$ and $0 < \theta < \pi$ can be removed in view of the periodicity of $f(\alpha)$ and the analytic continuation, even though they have been assumed at the beginning of the proof. □

Performing the replacement $y \to y + \pi/2n$ in Theorem 1, we get another trigonometric sum identity.

Proposition 2 *Let $P(\theta)$ be a polynomial of degree $< 2n$ in $\cos\theta$. Then for a real parameter y, there holds the following trigonometric sum identity:*

$$\sum_{k=0}^{2n-1} \frac{\sin(y + \frac{1+2k\pi}{2n})P(y + \frac{1+2k\pi}{2n})}{\cos(y + \frac{1+2k\pi}{2n}) - \cos\theta} = \frac{2n\sin 2nyP(\theta)}{\cos 2ny + \cos 2n\theta}.$$

Observe that $\sin n\theta/\sin\theta$ is a polynomial of degree $n-1$ in $\cos\theta$. Specifying $P(\theta) = \frac{Q(\theta)\sin n\theta}{\sin\theta\sin ny}$ in Theorem 1 and Proposition 2, we can deduce the following two trigonometric formulae.

Proposition 3 *Let $Q(\theta)$ be a polynomial of degree $\le n$ in $\cos\theta$. Then for a real parameter y, there holds the following trigonometric sum identity:*

$$\sum_{k=0}^{2n-1} \frac{(-1)^k Q(y + \frac{k\pi}{n})}{\cos(y + \frac{k\pi}{n}) - \cos\theta} = \frac{4n\,\sin n\theta\cos ny\,Q(\theta)}{\sin\theta\,(\cos 2ny - \cos 2n\theta)},$$

$$\sum_{k=0}^{2n-1} \frac{(-1)^k Q(y + \frac{(1+2k)\pi}{2n})}{\cos\,(y + \frac{(1+2k)\pi}{2n}) - \cos\theta} = \frac{4n\,\sin n\theta\,\sin ny\,Q(\theta)}{\sin\theta\,(\cos 2ny + \cos 2n\theta)}.$$

Observe also that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$. Letting $P(\theta) = Q(\theta)\cos n\theta$ in Proposition 2 leads us directly to the following trigonometric formula.

Proposition 4 *Let $Q(\theta)$ be a polynomial of degree $< n$ in $\cos\theta$. Then for a real parameter y, there holds the following trigonometric sum identity:*

$$\sum_{k=0}^{2n-1} (-1)^k \frac{\sin\,(y + \frac{1+2k}{2n}\pi)Q(y + \frac{1+2k}{2n}\pi)}{\cos\theta - \cos\,(y + \frac{1+2k}{2n}\pi)} = \frac{4n\,\cos ny\,\cos n\theta\,Q(\theta)}{\cos 2ny + \cos 2n\theta}.$$

3. Examples of trigonometric identities

The general results displayed in the last section imply numerous identities on trigonometric sums, which will be exhibited in this section.

(i) Letting $P(\theta) = \frac{\sin n\theta}{\sin\theta}$ in Theorem 1, we obtain

$$\sum_{k=0}^{2n-1} \frac{\sin n(y + \frac{k\pi}{n})}{\cos(y + \frac{k\pi}{n}) - \cos\theta} = \frac{2n\sin 2ny\sin n\theta}{\sin\theta\,(\cos 2ny - \cos 2n\theta)}.$$

According to the parity of n, we can further deduce from the last identity the following two formulae

$$\sum_{k=0}^{n-1} \frac{(-1)^k\cos\,(y + k\pi/n)}{\cos^2\,(y + k\pi/n) - \cos^2\theta} = \frac{2n\sin n\theta\cos ny}{\sin\theta(\cos 2ny - \cos 2n\theta)}, \quad n\text{-odd}; \tag{2a}$$

$$\sum_{k=0}^{n-1} \frac{(-1)^k\cos\theta}{\cos^2\,(y + k\pi/n) - \cos^2\theta} = \frac{2n\sin n\theta\cos ny}{\sin\theta(\cos 2ny - \cos 2n\theta)}, \quad n\text{-even}. \tag{2b}$$

(ii) Letting $P(\theta) = \frac{\sin 2n\theta}{\sin\theta}$ in Theorem 1, we get

$$\sum_{k=0}^{2n-1} \frac{\sin 2n(y+\frac{k\pi}{n})}{\cos(y+\frac{k\pi}{n})-\cos\theta} = \frac{2n\sin 2ny\sin 2n\theta}{\sin\theta(\cos 2ny-\cos 2n\theta)}.$$

Splitting the last sum into two parts according to $0 \le k < n$ and $n \le k < 2n$ and then replacing k by $k+n$ for the second one, we find, after some simplification, the following identity:

$$\sum_{k=0}^{n-1} \frac{\cos\theta}{\cos^2(y+k\pi/n)-\cos^2\theta} = \frac{n\sin 2n\theta}{\sin\theta(\cos 2ny-\cos 2n\theta)}. \tag{3}$$

(iii) Let $P(\theta) = \cos n\theta$ in Proposition 2. Then for even n, there holds

$$\sum_{k=0}^{2n-1} \frac{\sin(y+\frac{1+2k}{2n}\pi)\cos n(y+\frac{1+2kX}{2n}\pi)}{\cos(y+\frac{1+2k}{2n}\pi)-\cos\theta} = \frac{2n\sin 2ny\cos n\theta}{\cos 2ny+\cos 2n\theta}$$

which is equivalent to

$$\frac{4n\cos ny\cos n\theta}{\cos 2ny+\cos 2n\theta} = \sum_{k=0}^{2n-1} \frac{(-1)^k\sin(y+\frac{1+2k}{2n}\pi)}{\cos\theta-\cos(y+\frac{1+2k}{2n}\pi)}.$$

Following the same process as described in the last paragraph, we deduce from the equation just displayed the following identity:

$$\sum_{k=0}^{n-1}(-1)^k \frac{\sin(y+\frac{1+2k}{2n}\pi)\cos(y+\frac{1+2k}{2n}\pi)}{\cos^2\theta-\cos^2(y+\frac{1+2k}{2n}\pi)} = \frac{2n\cos n\theta\cos ny}{\cos 2ny+\cos 2n\theta}, \quad n\text{-even}. \tag{4}$$

(iv) Letting $P(\theta) = \frac{\sin 2n\theta}{\sin\theta}$ in Proposition 2, we have

$$\sum_{k=0}^{n-1} \frac{\cos\theta}{\cos^2\theta-\cos^2(y+\frac{1+2k}{2n}\pi)} = \frac{n\sin 2n\theta}{\sin\theta(\cos 2ny+\cos 2n\theta)}. \tag{5}$$

When $y = 0$, the corresponding sums displayed from (2) to (5) have appeared in Chu and Marini [4]. There exist other formulae related to these four identities, which are not going to be reproduced.

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