On Skew McCoy Rings

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Abstract For a ring endomorphism $\alpha$, we introduce $\alpha$-skew McCoy rings which are generalizations of $\alpha$-rigid rings and McCoy rings, and investigate their properties. We show that if $\alpha^t = I_R$ for some positive integer $t$ and $R$ is an $\alpha$-skew McCoy ring, then the skew polynomial ring $R[x; \alpha]$ is $\alpha$-skew McCoy. We also prove that if $\alpha(1) = 1$ and $R$ is $\alpha$-rigid, then $R[x; \alpha]/\langle x^2 \rangle$ is $\bar{\alpha}$-skew McCoy.

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1. Introduction

All rings considered here are associative with identity. According to Nielsen [10], a ring $R$ is called a left McCoy ring if whenever $f(x), g(x) \in R\![x]\setminus\{0\}$ satisfy $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ with $rg(x) = 0$. Similarly, right McCoy rings can be defined. If a ring is both left and right McCoy, then we say that the ring is a McCoy ring. Some properties of McCoy rings have been studied in Camillo and Nielsen [2, 9], Yang et al. [11, 12].

According to Krempa [7], an endomorphism $\alpha$ of a ring $R$ is called rigid if $\alpha\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring $R$ $\alpha$-rigid if there exists a rigid endomorphism $\alpha$ of $R$. Note that any rigid endomorphism of a ring is a monomorphism and $\alpha$-rigid rings are reduced rings by Hong et al. [3, Proposition 5]. For an endomorphism $\alpha$ of a ring $R$, $R[x; \alpha]$ is reduced if and only if $R$ is $\alpha$-rigid by Hong et al. [4, Proposition 3]. Recall that for a ring $R$ with a ring endomorphism $\alpha : R \to R$, a skew polynomial ring (also called an Ore extension of endomorphism type) $R[x; \alpha]$ of $R$ is the ring obtained by giving the polynomial ring over $R$ with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. 

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Motivated by results in Hong et al. [3, 4], Nielsen [10] and so on, we investigate a generalization of α-rigid rings and McCoy rings which we call an α-skew McCoy ring.

2. Skew McCoy rings

**Definition 2.1** Let α be an endomorphism of a ring $R$. Assume that $f(x) = \sum_{i=0}^{n} a_{i}x^{i}$, $g(x) = \sum_{j=0}^{m} b_{j}x^{j} \in R[x; \alpha] \setminus \{0\}$ satisfy $f(x)g(x) = 0$. We say that $R$ is a left α-skew McCoy ring if there exists a nonzero element $r \in R$ with $rb_{j} = 0$ for all $0 \leq j \leq m$, and say that $R$ is a right α-skew McCoy ring if there exists a nonzero element $s \in R$ with $a_{i}\alpha^{i}(s) = 0$ for all $0 \leq i \leq n$. If a ring is both left α-skew McCoy and right α-skew McCoy, then we say that the ring is an α-skew McCoy ring.

It can be easily checked that if $R$ is a McCoy ring, then it is an $I_{R}$-skew McCoy ring, where $I_{R}$ is an identity endomorphism of $R$, and thus every reversible ring (or reduced ring) $R$ is $I_{R}$-skew McCoy since reversible rings are McCoy by Nielsen [10, Theorem 2]. However, the following example shows that there exists an $I_{R}$-skew McCoy ring $R$ which is not reversible.

**Example 2.2** Suppose that $R$ is a McCoy ring. Let $aUT_{3}(R) = \left\{ \begin{pmatrix} a & b & d \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$.

Then $aUT_{3}(R)$ is an $I_{R}$-skew McCoy ring since $aUT_{3}(R)$ is McCoy by Yang and Song [11, Proposition 2.5 and Corollary 2.8]. Let $A = E_{23}$, $B = E_{12}$, where $E_{ij}$, a $3 \times 3$ matrix, is the matrix unit with 1 in the $(i, j)$th position and 0 elsewhere. Then $AB = 0$. But $BA = E_{13} \neq 0$. Thus $aUT_{3}(R)$ is not reversible.

Recall that a ring is called an Armendariz ring if $a_{i}b_{j} = 0$ for all $i, j$ whenever polynomials $f(x) = \sum_{i=0}^{n} a_{i}x^{i}$, $g(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x]$ satisfy $f(x)g(x) = 0$. For a monoid $M$, a ring $R$ is called an $M$-Armendariz ring if whenever elements $a = a_{1}g_{1} + a_{2}g_{2} + \cdots + a_{n}g_{n}$, $\beta = b_{1}h_{1} + b_{2}h_{2} + \cdots + b_{m}h_{m} \in R[M]$ satisfy $a\beta = 0$, then $a_{i}b_{j} = 0$ for each $i, j$. A ring $R$ is called a left $M$-McCoy ring if whenever elements $a = a_{1}g_{1} + a_{2}g_{2} + \cdots + a_{n}g_{n}$, $\beta = b_{1}h_{1} + b_{2}h_{2} + \cdots + b_{m}h_{m} \in R[M] \setminus \{0\}$ satisfy $a\beta = 0$, then there exists a nonzero element $r \in R$ with $r\beta = 0$, the right $M$-McCoy rings can be defined similarly. If a ring is both left and right $M$-McCoy, then we say that the ring is an $M$-McCoy ring. Armendariz rings are clearly McCoy. $M$-Armendariz rings are $M$-McCoy for any monoid $M$ by Yang and Song [11, Theorem 2.2]. Power-serieswise Armendariz rings are power-serieswise McCoy by Yang et al. [12, Theorem 2.2]. Some properties of these rings were studied in Anderson and Camillo [1], Hong et al. [4], Huh et al. [5], Kim et al. [6], Liu [8], Yang and Song [11], and Yang et al. [12]. Now let $\alpha$ be an endomorphism of a ring $R$, one may conjecture that if $R$ is $\alpha$-skew Armendariz, then $R$ is $\alpha$-skew McCoy. However, the following example eliminates the possibility.

**Example 2.3** ([4, Example 5]) Let $R = \mathbb{Z}_{2}[x]$, $\alpha : R \to R$ be an endomorphism defined by $\alpha(f(x)) = f(0)$. Then $R$ is $\alpha$-skew Armendariz by Hong et al. [4, Example 5]. However,
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Theorem 2.5 Let \( \alpha \) be an endomorphism of a ring \( R \). If \( R \) is \( \alpha \)-skew Armendariz, then \( R \) is right \( \alpha \)-skew McCoy.

Proof Let \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \alpha] \setminus \{0\} \) satisfy \( f(x)g(x) = 0 \). Then \( a_i \alpha^i(b_j) = 0 \) for all \( i, j \) since \( R \) is \( \alpha \)-skew Armendariz. Since \( g(x) \neq 0 \), there exists \( j_0 \) such that \( b_{j_0} \in R \setminus \{0\} \). Hence \( a_i \alpha^i(b_{j_0}) = 0 \) for all \( i \). Therefore \( R \) is right \( \alpha \)-skew McCoy. \( \Box \)

Theorem 2.5 Let \( \alpha \) be an endomorphism of a ring \( R \). If \( R \) is \( \alpha \)-rigid, then \( R \) is \( \alpha \)-skew McCoy.

Proof Let \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \alpha] \setminus \{0\} \) with \( f(x)g(x) = 0 \). Then \( R \) is \( \alpha \)-skew Armendariz by Hong et al. [4, Corollary 4]. Thus \( a_i \alpha^i(b_j) = 0 \) for all \( 0 \leq i \leq n, 0 \leq j \leq m \). Since \( f(x) \neq 0 \), there exists \( i_0 \) such that \( a_{i_0} \neq 0 \). Hence \( a_{i_0} \alpha^{i_0}(b_j) = 0 \) implies \( a_{i_0} b_j = 0 \) for all \( 0 \leq j \leq m \) by Hong et al. [3, Lemma 4(iii)]. Therefore \( R \) is left \( \alpha \)-skew McCoy. Moreover, \( R \) is right \( \alpha \)-skew McCoy by Proposition 2.4. The proof is completed. \( \Box \)

The following example shows that the converse of Theorem 2.5 is not true.

Example 2.6 Let \( R = \{ \begin{pmatrix} r & a & b \\ 0 & r & a \\ 0 & 0 & r \end{pmatrix} | r \in \mathbb{Z}, a, b \in \mathbb{Q} \} \), where \( \mathbb{Z} \) and \( \mathbb{Q} \) are the sets of all integers and all rational numbers, respectively. Let \( \alpha : R \to R \) be an automorphism defined by \( \alpha \left( \begin{pmatrix} r & a & b \\ 0 & r & a \\ 0 & 0 & r \end{pmatrix} \right) = \begin{pmatrix} r & a/2 & b/4 \\ 0 & r & a/2 \\ 0 & 0 & r \end{pmatrix} \). Then

1. \( R \) is not \( \alpha \)-rigid since \( \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \alpha \left( \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = 0 \), but \( \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0 \) if \( b \neq 0 \).

2. \( R \) is \( \alpha \)-skew McCoy.

Let \( f(x) = A_0 + A_1 x + \cdots + A_n x^n \), \( g(x) = B_0 + B_1 x + \cdots + B_m x^m \in R[x; \alpha] \setminus \{0\} \) with \( f(x)g(x) = 0 \), where \( A_i = \begin{pmatrix} r_i & a_i & b_i \\ 0 & r_i & a_i \\ 0 & 0 & r_i \end{pmatrix} \) and \( B_j = \begin{pmatrix} s_j & c_j & d_j \\ 0 & s_j & c_j \\ 0 & 0 & s_j \end{pmatrix} \) for \( 0 \leq i \leq n, 0 \leq j \leq m \).

Since \( f(x), g(x) \neq 0 \), by a similar proof to Hong et al. [4, Example 1] we have that \( A_i = \begin{pmatrix} 0 & a_i & b_i \\ 0 & 0 & a_i \\ 0 & 0 & 0 \end{pmatrix} \) and \( B_j = \begin{pmatrix} 0 & c_j & d_j \\ 0 & 0 & c_j \\ 0 & 0 & 0 \end{pmatrix} \) for \( 0 \leq i \leq n, 0 \leq j \leq m \). Take \( C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). We have \( CB_j = 0 \) for all \( j \), and \( A_i \alpha^i(C) = 0 \) for all \( i \). Thus \( R \) is \( \alpha \)-skew McCoy.

The following example shows that there exists an endomorphism \( \alpha \) of a McCoy ring \( R \) such that \( R \) is not \( \alpha \)-skew McCoy.

Example 2.7 ([4, Example 2]) Let \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Then \( R \) is a commutative reduced ring. Thus it is McCoy. Let \( \alpha : R \to R \) be an endomorphism defined by \( \alpha((a, b)) = (b, a) \). Then for \( f(x) = (1, 0) + (1, 0)x, g(x) = (0, 1) + (1, 0)x \in R[x; \alpha] \setminus \{0\} \), \( f(x)g(x) = 0 \). But for \( (a, b) \in R \), if \( (a, b)g(x) = 0 \), then \( a = b = 0 \). Thus \( R \) is not left McCoy. Similarly, if \( f(x)(a, b) = 0 \), then...
Recall that if \( \alpha \) is an endomorphism of a ring \( R \), then the map \( R[x] \to R[x] \) defined by 
\[
\sum_{i=0}^{n} a_i x^i \mapsto \sum_{i=0}^{n} \alpha(a_i) x^i
\]
is an endomorphism of the polynomial ring \( R[x] \) and clearly this map extends \( \alpha \). We shall also denote the extended map \( R[x] \to R[x] \) by \( \alpha \) and the image of \( f \in R[x] \) by \( \alpha(f) \).

**Theorem 2.8** Let \( \alpha \) be an endomorphism of a ring \( R \) and \( \alpha^t = I_R \) for some positive integer \( t \). If \( R \) is \( \alpha \)-skew McCoy, then \( R[x; \alpha] \) is \( \alpha \)-skew McCoy.

**Proof** Let \( p(y) = f_0 + f_1 y + \cdots + f_n y^n \), \( q(y) = g_0 + g_1 y + \cdots + g_m y^m \in R[x; \alpha][y; \alpha]\{0\} \) with \( p(y)q(y) = 0 \). Assume that \( f_i = a_{i0} + a_{i1} x + \cdots + a_{iu} x^u \), \( g_j = b_{j0} + b_{j1} x + \cdots + b_{jv} x^v \) for each \( 0 \leq i \leq n \), and \( 0 \leq j \leq m \), where \( a_{i0}, a_{i1}, \ldots, a_{iu}, b_{j0}, b_{j1}, \ldots, b_{jv} \in R \). Take a positive integer \( k \) such that \( k > \max\{\deg(f_i), \deg(g_j)\} \) for any \( 0 \leq i \leq n \), and \( 0 \leq j \leq m \), where the degree is as polynomial in \( R[x; \alpha] \) and the degree of zero polynomial is taken to be 0. Suppose that 
\[
p(x^k) = f_0 + f_1 x^{k+1} + \cdots + f_n x^{ntk+n}, \quad q(x^k) = g_0 + g_1 x^{tk+1} + \cdots + g_m x^{mtk+m}.\]
Then \( p(x^k) \), \( q(x^k) \in R[x; \alpha]\{0\} \), and the set of coefficients of \( f_i \)'s (resp., \( g_j \)'s) equals the set of coefficients of \( p(x^k) \) (resp., \( q(x^k) \)). It is easy to check that \( p(x^k)q(x^k) = 0 \in R[x; \alpha] \) since \( p(y)q(y) = 0 \) in \( R[x; \alpha][y; \alpha] \) and \( \alpha^k = I_R \). Since \( R \) is \( \alpha \)-skew McCoy, there exist \( r, s \in R \{0\} \) such that \( rq(x^k) = 0 \), and \( p(x^k)s = 0 \). \( rq(x^k) = 0 \) implies \( rb_{jk} = 0 \) for any \( 0 \leq j \leq m \), and \( 0 \leq k \leq v_j \). Hence \( rg_j = 0 \) for any \( 0 \leq j \leq m \). Therefore \( R[x; \alpha] \) is left \( \alpha \)-skew McCoy. \( p(x^k)s = 0 \) implies \( a_{il} \alpha^{{tk+1}+l}(s) = 0 \) for any \( 0 \leq i \leq n \), and \( 0 \leq l \leq u_i \). Thus \( a_{il} \alpha^{i+l}(s) = 0 \) for any \( 0 \leq i \leq n \), and \( 0 \leq l \leq u_i \) since \( \alpha^{{tk}+n} = I_R \). Hence we have 
\[
f_i \alpha^i(s) = (a_{i0} + a_{i1} x + \cdots + a_{iu} x^u) \alpha^i(s) \\
= a_{i0} \alpha^{i+0}(s) + a_{i1} \alpha^{i+1}(s) x + \cdots + a_{iu} \alpha^{i+u}(s) x^u = 0
\]
for any \( 0 \leq i \leq n \). Therefore \( R[x; \alpha] \) is right \( \alpha \)-skew McCoy. The proof is completed. \( \square \)

Recall that an element \( a \) in \( R \) is called regular if \( r_r(a) = 0 = l_r(a) \), i.e., \( a \) is not a zero divisor. For subrings of an \( \alpha \)-skew McCoy ring, we have the following.

**Proposition 2.9** Let \( \alpha \) be an endomorphism of a ring \( R \) and \( I \) be an ideal of \( R \) satisfying that every nonzero element in \( I \) is regular. If \( R \) is \( \alpha \)-skew McCoy, then \( I \) is \( \alpha \)-skew McCoy (without identity).

**Proof** Let \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( g(x) = \sum_{j=0}^{m} b_j x^j \in I[x; \alpha]\{0\} \) with \( f(x)g(x) = 0 \). Since \( I \) is an ideal of \( R \) and \( R \) is \( \alpha \)-skew McCoy, there exist nonzero elements \( r, s \in R \) satisfying \( rb_j = 0 \) for any \( 0 \leq j \leq m \), and \( a_i \alpha^i(s) = 0 \) for any \( 0 \leq i \leq n \). Therefore \( tr, st \in I \{0\} \) for any nonzero element \( t \in I \) (Otherwise, if \( tr = 0 \) (resp., \( st = 0 \)) for a element \( t \in I \{0\} \), then \( r \in r_r(t) \) (resp., \( s \in l_r(t) \)). Hence \( r = 0 \) (resp., \( s = 0 \)) since every nonzero element in \( I \) is regular. This is a contradiction). Consequently, we have 
\[
0 = t(rb_j) = (tr)b_j, \quad 0 = (a_i \alpha^i(s)) \alpha^i(t) = a_i \alpha^i(st)
\]
for any \( 0 \leq i \leq n \), and \( 0 \leq j \leq m \). Thus \( I \) is \( \alpha \)-skew McCoy. \( \square \)
Let $R_i$ be a ring and $\alpha_i$ an endomorphism of $R_i$ for each $i \in I$. For the product $\prod_{i \in I} R_i$ of $R_i$, the endomorphism $\bar{\alpha} : \prod_{i \in I} R_i \to \prod_{i \in I} R_i$ defined by $\bar{\alpha}((a_i)) = (\alpha(a_i))$. Yang et al. [12, Theorem 2.12] have shown that $\prod_{i \in I} R_i$ is a power-serieswise McCoy ring if and only if each $R_i$ is.

**Proposition 2.10** Let $\alpha_i$ be an endomorphism of $R_i$, $i \in I$. Then $\prod_{i \in I} R_i$ is $\bar{\alpha}$-skew McCoy if and only if each $R_i$ is $\bar{\alpha}_i$-skew McCoy.

**Proof** The proof is similar to Yang et al. [12, Theorem 2.12]. $\square$

**Corollary 2.11** Let $R$ be an abelian ring, $\alpha$ an endomorphism of $R$, and $e^2 = e \in R$. If $eR$ and $(1 - e)R$ are $\alpha$-skew McCoy, then $R$ is an $\alpha$-skew McCoy.

**Proof** Since $R$ is an abelian ring and $e^2 = e \in R$, $R = eR \times (1 - e)R$. Hence the conclusion follows from Proposition 2.10. $\square$

**Lemma 2.12** Let $\alpha$ be an endomorphism and $R$ an $\alpha$-rigid ring. If $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j \in R[x; \alpha]$ satisfy $f(x)g(x) = 0$, then $f(x)\alpha(g(x)) = 0$ and $\alpha(f(x))g(x) = 0$.

**Proof** Since $R$ in $\alpha$-skew Armendariz by Hong et al. [4, Corollary 4], $a_i \alpha^i(b_j) = 0$ for each $i, j$. Thus $a_i \alpha^i+1(b_j) = 0$ for each $i, j$ by Hong et al. [3, Lemma 4(i)]. Hence $f(x)\alpha(g(x)) = 0$. Since $R[x; \alpha]$ is reduced by Hong et al. [4, Proposition 3], $\alpha(f(x))g(x) = 0$. $\square$

For an ideal $I$ of a ring $R$, if $\alpha(I) \subseteq I$, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of a factor ring $R/I$.

**Theorem 2.13** Let $\alpha$ be an endomorphism of $R$ and $\alpha(1) = 1$. If $R$ is $\alpha$-rigid, then $R[x; \alpha]/\langle x^2 \rangle$ is $\bar{\alpha}$-skew McCoy.

**Proof** Suppose that $p(y) = \sum_{i=0}^n \bar{f}_i y^i$, $q(y) = \sum_{j=0}^m \bar{g}_j y^j \in (R[x]/\langle x^2 \rangle)[y; \bar{\alpha}] \setminus \{0\}$ with $p(y)q(y) = 0$. Let $\bar{f}_i = a_{i0} + a_{i1} \bar{x}$, $\bar{g}_j = b_{j0} + b_{j1} \bar{x}$, where $a_{i0}, a_{i1}, b_{j0}, b_{j1} \in R$, $\bar{x} = x + \langle x^2 \rangle$. Note that $\bar{x}y = y\bar{x}$ since $\alpha(1) = 1$. Thus $p(y) = h_0 + h_1 \bar{x}$ and $q(y) = k_0 + k_1 \bar{x}$, where $h_0 = \sum_{i=0}^n a_{i0} y^i$, $h_1 = \sum_{i=0}^n a_{i1} y^i$, $k_0 = \sum_{j=0}^m b_{j0} y^j$ and $k_1 = \sum_{j=0}^m b_{j1} y^j$. Since $\bar{x}^2 = 0$ and $\bar{a} = \alpha(a)\bar{x}$ for any $a \in R$, we have

$$0 = p(y)q(y) = (h_0 + h_1 \bar{x})(k_0 + k_1 \bar{x}) = h_0 k_0 + (h_0 k_1 + h_1 \alpha(k_0))\bar{x}.$$  

Hence in $R[y; \bar{\alpha}]$ we have $h_0 k_0 = 0$ and $h_0 k_1 + h_1 \alpha(k_0) = 0$. Thus $h_0 k_0 = 0$ implies $h_0 \alpha(k_0) = 0$ by Lemma 2.12. Since $R[y; \alpha](\cong R[x; \alpha])$ is reduced, $\alpha(k_0) h_0 = 0$, and so $0 = \alpha(k_0)(h_0 k_1 + h_1 \alpha(k_0)) = \alpha(k_0) h_1 \alpha(k_0) = (h_1 \alpha(k_0))^2$. Thus $h_1 \alpha(k_0) = 0$, and hence $h_0 k_1 = 0$.

If $h_0 \neq 0$, then the equation $0 = h_0 (k_0 + k_1)$ implies that

$$0 = h_0 (k_0 + k_1) = h_0 (\sum_{j=0}^m b_{j0} y^j + \sum_{j=0}^m b_{j1} y^{j+m+1}).$$

Since $R$ is (left) $\alpha$-skew McCoy by Theorem 2.5, there exists $r \in R \setminus \{0\}$ such that $rb_{j0} = 0$ and $rb_{j1} = 0$ for any $j$. Hence $r\bar{g}_j = 0$ for any $0 \leq j \leq m$.

Otherwise, if $h_0 = 0$, then $h_1 \neq 0$, and $0 = p(y)q(y) = (h_1 \bar{x})(k_0 + k_1 \bar{x}) = (h_1 \alpha(k_0))\bar{x}$. Thus
$h_1\alpha(k_0) = 0$. If $\alpha(k_0) \neq 0$, then there exists $s \in R\setminus\{0\}$ such that $\alpha(s) = 0$ for any $j$ since $R$ is (left) $\alpha$-skew McCoy. Let $r = s\bar{x}$. Then $r \in (R[x; \alpha]/(x^2))\setminus\{0\}$ and $r\bar{g}_j = s\bar{x}(b_{j0} + b_{j1} \bar{x}) = s\alpha(b_{j0})\bar{x} = 0$ for any $0 \leq j \leq m$. If $\alpha(k_0) = 0$, then let $r = \bar{x}$, and $r\bar{g}_j = 0$ for any $0 \leq j \leq m$.

So $R[x; \alpha]/(x^2)$ is left $\bar{\alpha}$-skew McCoy.

Moreover, the equations $h_0k_0 = 0$ and $h_0k_1 + h_1\alpha(k_0) = 0$ yield that $h_0\alpha(k_0) = 0$ and $h_1\alpha(k_0) = 0$. Thus $h_1y^{n+1}\alpha(k_0) = h_1\alpha^{n+2}(k_0)y^{n+1} = 0$ by Lemma 2.12. Hence we have

$$0 = (h_0 + h_1y^{n+1})\alpha(k_0) = \sum_{i=0}^{n} a_{i0}y^i + \sum_{i=0}^{n} a_{i1}y^{i+n+1} + \sum_{j=0}^{m} a_{j0}\alpha(b_{j0})y^j.$$  

Then the right case can be proved similarly as above. $\square$

**Proposition 2.14** Let $\alpha$ be an endomorphism of a ring $R$ and $I$ an ideal of $R$ with $\alpha(I) \subseteq I$. If $a\alpha(a) \in I$ implies $a \in I$ for $a \in R$, then $R/I$ is $\bar{\alpha}$-skew McCoy.

**Proof** By the proof of Hong et al. [4, Proposition 9], $R/I$ is $\bar{\alpha}$-rigid. Thus $R/I$ is $\bar{\alpha}$-skew McCoy by Theorem 2.5. $\square$

**Lemma 2.15** Let $\alpha$ be a monomorphism of a ring $R$, $I$ an $\alpha$-rigid ideal (without identity) of $R$ with $\alpha(I) \subseteq I$, $r \in I$ and $s \in R$. Then we have the following:

1. If $rs = 0$, then $\alpha^k(s) = \alpha^k(r)s = 0$ for any positive integer $k$.
2. If $r\alpha^k(s) = 0$ (or $\alpha^k(r)s = 0$) for some positive integer $k$, then $rs = 0$.

**Proof** The proof is similar to Hong et al. [3, Lemma 4]. $\square$

**Proposition 2.16** Let $\alpha$ be a monomorphism of a ring $R$, $I$ an $\alpha$-rigid ideal (without identity) of $R$ with $\alpha(I) \subseteq I$ and every nonzero element in $I$ regular. Suppose $br \in I\setminus\{0\}$ implies $b\alpha(r) \in I\setminus\{0\}$ for $b, r \in R$. If $R/I$ is reversible and $\bar{\alpha}$-skew McCoy, then $R$ is $\alpha$-skew McCoy.

**Proof** Let $f(x) = \sum_{i=0}^{n} a_{i}x^i$, $g(x) = \sum_{j=0}^{m} b_{j}x^j \in R[x; \alpha]\setminus\{0\}$ with $f(x)g(x) = 0$. Consider the following three cases.

**Case 1** Both $f(x)$ and $g(x)$ are in $I[x; \alpha]$. By Theorem 2.5, $I$ is $\alpha$-skew McCoy. Thus there exist nonzero $r, s \in I \subseteq R$ such that $rb_j = 0$, $a_i\alpha^i(s) = 0$ for all $i$ and $j$.

**Case 2** One and only one of $f(x)$, $g(x)$ is in $I[x; \alpha]$. Without loss of generality, assume that $f(x) \in I[x; \alpha]$, but $g(x) \notin I[x; \alpha]$. Using Lemma 2.15 and $I$ is $\alpha$-rigid repeatedly, similar to the proof of Hong et al. [3, Proposition 6], we have that $a_i b_j = 0$ for all $i, j$. Since $f(x), g(x) \neq 0$, there are $i_0, j_0$ such that $a_{i_0}, b_{j_0} \in R\setminus\{0\}$. Take $r = a_{i_0}$, $s = b_{j_0}$, and hence $rb_j = 0$, and $a_i\alpha^i(s) = 0$ for all $i$ and $j$.

**Case 3** Neither $f(x)$ nor $g(x)$ is in $I[x; \alpha]$. Then $\sum_{i=0}^{n} a_{i}x^i, \sum_{j=0}^{m} b_{j}x^j \in (R/I)[x; \bar{\alpha}]\setminus\{0\}$. Since $R/I$ is $\bar{\alpha}$-skew McCoy, there exist $\overline{r}, \overline{s} \in R/I\setminus\{0\}$ such that $\overline{r}\overline{b}_j = \overline{0}$, $\overline{a}_i\alpha^i(\overline{s}) = \overline{0}$. Since $R/I$ is reversible, $\overline{r}\overline{s} = \overline{0}$, $\alpha^i(\overline{s})\overline{a}_i = \overline{0}$. So $r\bar{b}_j, b_j r, a_i\alpha^i(s), \alpha^i(s)a_i \in I$ for all $i, j$. We claim that $\alpha^i(s)a_i = 0$ for all $i$, and $b_j r = 0$ for all $j$.

Assume that there exists some $i$ such that $\alpha^i(s)a_i \neq 0$. Let $t$ be the smallest one relation
to the property. Then \(0 = \alpha^i(s)f(x)g(x) = (\sum_{i=0}^n \alpha^i(s)a_ix^i)(\sum_{j=0}^mb_jx^j)\) implies \(g(x) = 0\) since \(\alpha^i(s)a_i \in R\{0\}\) is regular and \(\alpha\) is monomorphism, a contradiction. Similarly, if there exists some \(j\) such that \(b_jr \neq 0\). Let \(n\) be the smallest one relation to the property. Since \(I\) is reduced, \(b_jr = 0\) yields \(b_j\alpha^i(r) = 0\) for \(0 \leq j \leq n - 1\). Thus \(0 = f(x)g(x)r = (\sum_{i=0}^n a_ix^i)(\sum_{j=1}^mb_j\alpha^i(r)x^j)\). Since \(b_jr \in I\{0\}\), we have \(b_i\alpha^i(r) \in I\{0\}\). Hence \(0 = f(x)g(x)r\) implies \(f(x) = 0\) since \(b_i\alpha^i(r) \in I\{0\}\) is regular and \(\alpha\) is monomorphism, this is a contradiction. Thus \(\alpha^i(s)a_i = 0\) for all \(i\), and \(b_jr = 0\) for all \(j\).

Hence \(R\) is \(\alpha\)-skew McCoy. \(\square\)

In the last part of this section, we consider the \(n \times n\) upper triangular matrix ring \(T_n(R)\) over a ring \(R\). Let \(aUTn(R)\) be the ring consisting of \(n \times n\) upper triangular matrices with equal diagonal entries over \(R\), where \(n \geq 2\) is a positive integer. Hong et al. [4, Proposition 17] proved that if \(R\) is an \(\alpha\)-rigid ring, then \(aUT_3(R)\) is \(\tilde{\alpha}\)-skew Armendariz, but \(aUTn(R)\) is not \(\tilde{\alpha}\)-skew Armendariz for \(n \geq 4\) (see [4, Example 18]), where \(\alpha\) is an endomorphism of a ring \(R\) and \(\tilde{\alpha}\) is the endomorphism of \(aUTn(R)\) defined by \(\tilde{\alpha}((a_{ij})) = (\alpha(a_{ij}))\).

By Camillo and Nielsen [2, Proposition 10.2], the full matrix ring \(M_n(R)\) and \(T_n(R)\) over a nonzero ring \(R\) need not to be \(I_R\)-skew McCoy.

**Proposition 2.17** Let \(n \geq 2\). Then a ring \(R\) is \(\alpha\)-skew McCoy if and only if \(aUTn(R)\) is \(\tilde{\alpha}\)-skew McCoy.

**Proof** The proof is similar to Yang and Song [11, Theorem 2.7] \(\square\)

**Corollary 2.18** A ring \(R\) is \(\alpha\)-skew McCoy if and only if the trivial extension

\[T(R, R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\}\]

of \(R\) is \(\tilde{\alpha}\)-skew McCoy.

**References**


