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The Crossing Number of Two-Maps on Orientable Surfaces

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Abstract In this paper, we discuss the crossing numbers of two one-vertex maps on orientable surfaces. By using a reductive method, we give the crossing number of two one-vertex maps with one face on an orientable surface and the crossing number of a one-vertex map with one face and a one-vertex map with two faces on an orientable surface. This provides a lower bound for the crossing number of two general maps on an orientable surface.

Keywords crossing number; embedding; orientable surface.

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1. Introduction

In this paper, we consider the connected multi-graphs. All the concepts and terms are standard and may be found in [1–3]. A surface, always denoted by S, is a compact 2-manifold without boundary. An embedding of a graph G in a surface S is a continuous topological mapping (or drawing as some scholars named) $\Pi : G \mapsto S$ such that edges of $\Pi(G)$ have no crossing and each component of $S - \Pi(G)$ is an open disc called face (or region). In this case G is called an embedded graph or a map. A θ-map is a 2-connected embedded graph with exactly one face. Suppose that Π_i is an embedding of graph G_i $(i = 1, 2)$ on a surface S and D is a drawing of G_1, G_2 on S such that

(1) $D|_{G_i} = \Pi_i$) (i.e., Π_i is the restriction of D on G_i $(i = 1, 2)$);
(2) $D(V(G_1)) \cap D(V(G_2)) = \emptyset$;
(3) $\forall e_i \in E(G_i(i = 1, 2) \Rightarrow |D(e_1) \cap D(e_2)| \leq 1$.

Then D is called a good drawing of G_1, G_2 on S. The number $C_{rD}(G_1, G_2)$ is used to denote the number of edge-crossings resulting from those of $D(E(G_1)) \cup D(E(G_2))$. If D is a good drawing of two graphs G_1, G_2 on a surface S such that the crossing number $C_{rD}(G_1, G_2)$ is of minimum among all the possible good drawings of G_1, G_2 on S, then D is defined as an optimal drawing of

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G_1, G_2 on S and the corresponding value of $C_{rD}(G_1, G_2)$ is the crossing number of two graphs (or maps) G_1, G_2 on S.

In order to handle the problems of crossing numbers on orientable surfaces easily, one has to introduce some results and concepts on Polato presentation of the surface topology. By the theory, we have known that deciding the crossing number problems on a series of nontrivial graphs is very difficult. In fact, it has been proved NP-hard for one to find the crossing numbers of a graph in planar drawings [4]. So it is also very difficult for us to find the crossing numbers of a pair of graphs on a certain surface. In this field, Negami [5] and Archdeacon [6] did some works on lower surfaces. But for general orientable surface, little is known. In this paper, we try to investigate the crossing number problem(s) of two θ-maps.

Now we begin to introduce the polygonal presentation (i.e., the planar presentation) of a surface. In fact, our discussions follow from Liu's monograph [2]. By surface topology theory, a surface may also be obtained by identifying pairs of sides (always denoted by letters or words) of a polygon with even number of sides. Therefore, we may view a surface S as a set E of letters (or a string of letters) in cyclic order such that

(1) There are $n(\geq 1)$ distinct letters on S;

(2) Each letter appears exactly twice on S;

(3) Each occurrence of a letter with a power which is 1 (always omitted) or -1 distinguishes the two directions on S.

Let $\mathbf{S}$ be the set of all surfaces. If the two occurrences of each letter in a surface are with different powers, the surface is called orientable; otherwise, non-orientable. In a non-orientable surface, there is at least one letter whose two occurrences are with the same power. Let $\mathbf{P}$ and $\mathbf{Q}$ be the sets of all orientable and non-orientable surfaces, respectively. Then $\mathbf{S} = \mathbf{P} + \mathbf{Q}$. Two surfaces are treated as the same if one can be obtained from another by reversing the cyclic order, permuting some letters and/or replacing a letter by its inverse. Let $\mathbf{A}, \mathbf{B}, \ldots$ be sections of successive letters in linear order on $S \in \mathbf{S}$, or write $\mathbf{A}, \mathbf{B}, \ldots \subseteq S$. Of course, whenever $\mathbf{A} = \mathbf{S}, \mathbf{A}$ becomes with cyclic order in its own right. They are also allowed to be the empty or that contains only an occurrence of a letter. Sometimes, S may be a subset of E without confusion. This implies

$$\begin{cases} S = ABC \Rightarrow S = BCA = C^{-1}B^{-1}A^{-1}, \\ S = Aa^{\alpha}Ba^{\beta}C, b \in S \Rightarrow S = Ab^{\alpha}Bb^{\beta}C = Aa^{-\alpha}Ba^{-\beta}C. \end{cases}$$

One may think of what a lower surface looks like. It is easily seen from the above operation that

$$\begin{cases} S_0 = aa^{-1}, \\ S_p = \prod\limits_{1 \leqslant i \leqslant p} a_i b_i a_i^{-1} b_i^{-1}, p \geqslant 1, \\ N_q = \prod\limits_{1 \leqslant i \leqslant p} c_i c_i^{-1}, q \geqslant 1, \end{cases}$$

are all the possible surfaces which will be seen to be the simplest in every case [4]. Of course, S_p $(p \geq 0)$ are all orientable surfaces and S_0 is the sphere, S_1 the torus, N_1 the projective plane, and N_2 the Klein bottle and so on. Instinctively, $S_p\,(N_k)$ is obtained by adding p handles (k crosscaps) on the sphere S_0.

A polyhedron is defined to be a set of polygons denoted by $\Sigma = (X_1, X_2, \ldots, X_n)$, on s sets of letters E such that each letter appears exactly twice without a proper subset of Σ with the same property. Moreover, another polyhedron denoted by $\Sigma^* = (X_1^*, X_2^*, \ldots, X_n^*)$ can be defined on the same set E from Σ in the following way:

$$\forall x, y \in E, x^{-1}y \subseteq X \in \Sigma \Rightarrow xy \subseteq X^* \in \Sigma^*$$

such that the product of the powers of the two occurrences of a letter in Σ^* is the same as that in Σ. It is easy to see $\Sigma^{**} = \Sigma$. Thus, Σ^* is called a dual of Σ. Polygons in Σ are called faces, the letters are edges and polygons in Σ^* are vertices of Σ. For a polyhedron Σ, the number

$$\chi(\Sigma) = |V(G)| - |E(G)| + |F|$$

is said to be the Euler characteristic of Σ, where $|V(G)|$, $|E(G)|$, $|F|$ are the numbers of vertices, edges and faces of a graph G on Σ, respectively.

2. The main result

In this section we only consider graphs on orientable surfaces. First, we have to do some preparations.

A cycle (curve) C on S_g is contractible if $S_g - C$ has one component that is homeomorphic to an open disc; otherwise C is essential or non-contractible.

Claim 1 *Suppose that G is a θ-map on S_g. Then G has a spanning tree T that every fundamental cycle of T is non-contractible on S_g.*

In fact, after a series of edges (in G) are contracted, the resulting subgraph of G is a one-vertex map with one face on S_g whose loop edges are essential cycles. If splitting the vertex inversely, we will get a spanning tree T of G, called an inner tree.

Reduction Lemma 1 *Suppose that θ_1, θ_2 are two θ-maps on $S = S_g$. Then there are two one-vertex maps θ_1', θ_2' with one face on S such that*

(1) θ_i' is obtained by contracting a series of clean edges of θ_i, $1 \le i \le 2$;

(2) $C_r(\theta_1, \theta_2) = C_r(\theta_1', \theta_2')$.

Here we define that an edge is clean if there is no crossing on it.

Proof It is easy to see that if two θ-maps θ_1', θ_2' (each of which has exactly one-vertex, one face) satisfy (1)–(2) above, then $C_r(\theta_1, \theta_2) \le C_r(\theta_1', \theta_2')$. On the other hand, suppose that D is an optimal drawing of θ_1, θ_2 on S such that $C_{r_D}(\theta_1, \theta_2) = C_r(\theta_1, \theta_2)$. □

Claim 2 *If $|V(\theta_1)| > 1$, then there is an edge of θ_1 which is clean in an optimal drawing D of θ_1, θ_2.*

In fact, we consider the dual map θ_1^* of θ_1. Then $|E(\theta_1)| = |E(\theta_1^*)| = |V(\theta_1)| + 2g - 1 \ge 2g + 1$, where g is genus of $S = S_g$. This shows that θ_1^* is a spanning subgraph of a one-vertex graph with at least two faces and has no contractible edges. After deleting some edges $e_1^*, e_2^*, \ldots, e_m^*$ of θ_1^*, we get a one-vertex map with one face on S. Consider the inner tree of θ_2 as a single vertex which is corresponding to the vertex of $\theta_1^* - e_1^* - e_2^* \cdots - e_m^*$ and let the other $2g$ edges be the copies of edges of $\theta_1^* - e_1^* - e_2^* \cdots - e_m^*$, we get a drawing D' of θ_1 and θ_2 such that $|C_{r_{D'}}(\theta_1, \theta_2)| < |E(\theta_1)|$. For D is an optimal drawing, we conclude (2). Claim 2 is proved.

This procedure shows there are clean edges in D. Obviously, after contracting a clean edge of θ_1, we can get an optimal drawing of the corresponding map pair keeping the same crossing number. Repeating this procedure until we get a good drawing D' of θ_1' and θ_2' such that $C_{r_D}(\theta_1, \theta_2) = C_{r_{D'}}(\theta_1', \theta_2')$. Thus, Reduction Lemma 1 is proved.

Example 1 The crossing number of two θ maps on S_1 is 2.

Proof Let θ_1, θ_2 be a pair of θ-maps on S_1. Then by Reduction Lemma 1, we may further assume that both of θ_1, θ_2 are one-vertex maps with only one face in S_1. Cutting S_1 along the edges of θ_1, then we get a polygonal representation of S_1 (as depicted in Figure 1), where the four sides are copies of edges of θ_1. Now put θ_2 into the region bounded by the four sides of S_1. This shows that $C_r(\theta_1, \theta_2) = 2$.

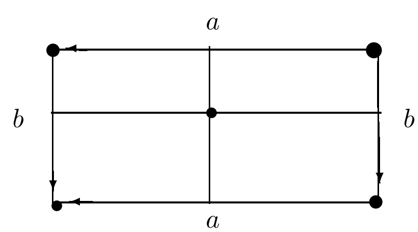

Figure 1 An optimal drawing of θ_1, θ_2 on S_1

(where the vertices are equal to the vertices of θ_1, parallel sides are copies of a and b.)

Theorem 1 *The crossing number of two θ-maps on S_n is $2n$.*

Proof Let θ_1, θ_2 be a pair of θ-maps on S_n. Then by Reduction Lemma 1 we may further suppose that both of them are one-vertex, one-face maps on S_n. We cut S_n along the $2n$ edges of θ_1, say $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. Then we obtain a polygonal representation of S_n with $4n$ sides where each pair a_i, a_i^{-1} (b_i, b_i^{-1}) of sides are copies of $a_i(b_i)$ of θ_1 $(1 \leq i \leq 2n)$. Now put θ_2 into the inner region of the polygon. Since θ_2 is also a one-vertex, one-face map on S_n, its loops are all noncontractible cycles and hence each of them must destroy a genus of S_n. Therefore, $C_r(\theta_1, \theta_2) \geq 2n$. Now the drawing D shown in Figure 2 attains this bound $2n$. D is an optimal drawing of θ_1, θ_2 on S_n.

Remark Since every map on an orientale surface S_g must contain a θ-map (as its submap) and the crossing number of a map is never less than that of its submaps, Theorem 1 thus provides a lower bound for the crossing number of two general maps on a surface (i.e., for any two maps G_1, G_2 on S_g, their crossing number is at least $2g$).

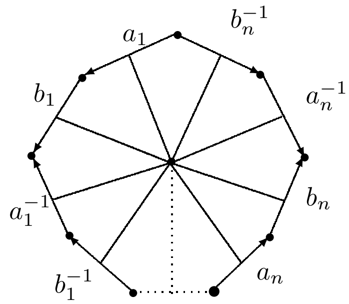

Figure 2 An optimal drawing of two one-vertex maps with one face on S_n

In the following, we shall discuss a more complicated situation which concerns the crossing number of a θ-map and a one-vertex map with two faces.

Lemma 2 *Suppose that θ is a one-vertex map with two faces on S_n. Then there are $2n+1$ types of such maps, all of which has two face boundaries with the lengthes k and $4n-k+2$ $(1 \leqslant k \leqslant 2n+1)$, respectively.*

Proof A one-vertex map θ with two faces is composed by a one-vertex map θ' with exactly one face W and an additional loop e_2. It is clear that e_2 connects two copies of the vertex of θ' on the boundary of the $4n$-gon (which is a plane representation of S_n) and thus divides the only face of θ' into two. Though there are $\binom{4n}{2}$ ways of doing so, there are $2n+1$ types of distinct pair of regions whose lengths are, respectively, k and $4n-k+2$ for $k = 1, 2, \ldots, 2n+1$.

Jordan Curve Theorem ([2, 3]) *A simple closed curve C on the plane divides the plane into two inner-disjoint connected regions with C as their common boundary.*

Theorem 2 *The crossing number of a one-vertex map θ_1 with one face and a one-vertex map θ_2 with two faces on S_n is*

$$C_r(\theta_1, \theta_2) = 2n + k - 1, 1 \leqslant k \leqslant 2n + 1$$

provided that the two face boundaries of θ_2 are k and $4n-k+2$, respectively.

Proof It is easy to see that there is an edge e_2 of θ_2 on the common boundary of its two faces such that $\theta_2 - e_2$ is a one-vertex map with one face on S_n. Suppose the polygonal presentation of S_n is

$$\prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1}$$

where a_i, b_i represent $2n$ edges of $\theta_2 - e_2$ and a_i^{-1} and b_i^{-1} are two copies of a_i and b_i with anti-orientation for $1 \leq i \leq n$. Then the vertices of the $4n$-polygon are just the copies of the only one vertex of θ_2. Add e_2 back to the $4n$-polygon and suppose that the two facial boundaries are, respectively, $W_1 : C_1\, C_2\, \ldots\, C_k\, e_2$ and $W_2 : e_2\, C_{k+1}\, C_{k+2}\, \ldots\, C_{4n}$, where $k = 4m + r$, $0 \leqslant r \leq 3$ and $W_1 = (\prod_{i=1}^{m} a_i b_i a_i^{-1} b_i^{-1}) C_{k-r+1}, \ldots, C_{k-1} C_k e_2$. If $m = 0$, we define $\prod_{i=1}^{m} a_i b_i a_i^{-1} b_i^{-1} = \emptyset$.

Then

$$C_{k-r+1}C_{k-r+2}\ \ldots\ C_k = \begin{cases} \emptyset, & r=0,\\ a_{m+1}, & r=1,\\ a_{m+1}b_{m+1}, & r=2,\\ a_{m+1}b_{m+1}a_{m+1}^{-1}, & r=3 \end{cases}$$

and $W_2 = e_2, C_{k+1}, \ldots, C_{4m+4}(\prod_{i=1}^{m} a_i b_i a_i^{-1} b_i^{-1})$. Hence,

$$C_{k+1}C_{k+2}\ \ldots\ C_{4m+4} = \begin{cases} a_{m+1}b_{m+1}a_{m+1}^{-1}b_{m+1}^{-1}, & r=0,\\ b_{m+1}a_{m+1}^{-1}b_{m+1}^{-1}, & r=1,\\ a_{m+1}^{-1}b_{m+1}^{-1}, & r=2,\\ b_{m+1}^{-1}, & r=3 \end{cases}$$

We may suppose that W_1 is a shorter boundary (i.e., $1 \leqslant k \leqslant 2n$) without loss of generality. Then we have

Claim 3 *Suppose that D_i is a good drawing of θ_1, θ_2 on S_n (according to that the vertex of θ_1 is put into the inner region of W_i $(i = 1, 2)$, where W_1, W_2 are two faces of θ_2). Then $C_{r_{D_1}}(\theta_1, \theta_2) \geq C_{r_{D_2}}(\theta_1, \theta_2)$.*

In fact, since every edge of θ_1 is a noncontractible loop on S_n, θ_1 will cross the boundary of the $4n$-polygon $2n$ times. The Jordan Curve Theorem implies that it will also cross the edge e_2 exactly $k-1$ or $4n-k+1$ times according to that the vertex of θ_1 is put into W_1 or W_2. Now that $|W_1| \leq |W_2|$ implies that $C_{r_{D_1}}(\theta_1, \theta_2) \geq C_{r_{D_2}}(\theta_1, \theta_2)$

Next we will show that D_2 is an optimal drawing of graph pair θ_1, θ_2 on S_n.

Suppose that D_2' is a good drawing of θ_1, θ_2 on S_n such that the only one vertex of θ_1 is put into the region bounded by W_2 (of θ_2). Then $\theta_2 - e_2$ is a one-vertex map with one face on S_n. It is clear that $C_{rD_2'}(\theta_2 - e_2, \theta_1) = 2n$. If W_1 has length k, then edges of θ_1 will cross e_2 exactly $k-1$ times by Jordan Curve Theorem. Therefore, $C_{rD'}(\theta_1, \theta_2) = 2n + k - 1$. This completes the proof of Theorem 2. □

In fact, after a series of edges (in G) are contracted, the resulting subgraph of G is a one-vertex map with one face on S_g whose loop edges are essential cycles. If splitting the vertex inversely, we will get a spanning tree T of G, called an inner tree.

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