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The Equivalence between Property (ω) and Weyl's Theorem

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Abstract We call $T \in B(H)$ consistent in Fredholm and index (briefly a CFI operator) if for each $B \in B(H)$, TB and BT are Fredholm together and the same index of B, or not Fredholm together. Using a new spectrum defined in view of the CFI operator, we give the equivalence of Weyl's theorem and property (ω) for T and its conjugate operator T^*. In addition, the property (ω) for operator matrices is considered.

Keywords Weyl's theorem; property (ω); spectrum.

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1. Introduction

Throughout this note let $B(H)$ ($K(H)$) denote the algebra of bounded linear operators (compact operators) acting on a complex, infinite dimensional Hilbert space H. If $T \in B(H)$, write $N(T)$ and $R(T)$ for the null space and the range of T; $\sigma(T)$ for the spectrum of T; $\pi_{00}(T) = \pi_0(T) \cap \operatorname{iso}\sigma(T)$, where $\pi_0(T) = \{\lambda \in \mathbb{C}, 0 < \dim N(T - \lambda I) < \infty\}$ are the eigenvalues of finite multiplicity. An operator $T \in B(H)$ is called upper semi-Fredholm if it has closed range with finite dimensional null space and if $R(T)$ has finite co-dimension, $T \in B(H)$ is called a lower semi-Fredholm operator. We call $T \in B(H)$ Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. For a semi-Fredholm operator, let $n(T) = \dim N(T)$ and $d(T) = \dim H/R(T)$. The index of a semi-Fredholm operator $T \in B(H)$ is given by $\operatorname{ind}(T) = \dim N(T) - \dim H/R(T) = n(T) - d(T)$. The ascent of T, $\operatorname{asc}(T)$, is the least non-negative integer n such that $N(T^n) = N(T^{n+1})$ and the descent, $\operatorname{des}(T)$, is the least non-negative integer n such that $R(T^n) = R(T^{n+1})$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. And $T \in B(H)$ is called Browder if it is Fredholm "of finite ascent and descent": equivalently [6, Theorem 7.9.3] if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, the Browder spectrum $\sigma_b(T)$, the upper (lower) semi-Fredholm spectrum $\sigma_{SF_+}(T)$ ($\sigma_{SF_-}(T)$) of $T \in B(H)$ are defined

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by ([6, 7]): $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Fredholm$\}$, $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Weyl$\}$, $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder$\}$, $\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not upper semi-Fredholm$\}$, $\sigma_{SF_-}(T) = \{\lambda \in \mathbb{C} : T-\lambda I$ is not lower semi-Fredholm$\}$. The property of consistency in Fredholm and index has been studied in [3]. Using a new spectrum defined in view of the property of consistency in Fredholm and index, the main purpose of this paper is to give the relation between the CFI property (see definition in Section 2) and Weyl type theorem. Also the equivalence of Weyl's theorem and property (ω) is studied.

2. CFI operator and Weyl type theorem

We begin with a definition [3]: we say $T \in B(H)$ is consistent in Fredholm and index (abbrev. a CFI operator) or T has CFI property, if for each $B \in B(H)$, TB and BT are Fredholm together and $\mathrm{ind}(TB) = \mathrm{ind}(BT) = \mathrm{ind}(B)$ or not Fredholm together.

Let

$$\rho_1(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is a CFI operator}\},$$

and let $\sigma_1(T) = \mathbb{C}\backslash\rho_1(T)$. Clearly, $\lambda_0 \in \sigma_1(T)$ if and only if $T - \lambda_0 I$ is semi-Fredholm but $T - \lambda_0 I$ is not Weyl. By perturbation theorem of semi-Fredholm operator, $\sigma_1(T)$ is an open set in the spectrum $\sigma(T)$ of operator T. Let $H(T)$ be the class of complex-valued functions which are analytic in a neighborhood of $\sigma(T)$ and are not constant on any neighbourhood of any component of $\sigma(T)$.

Weyl [13] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This "Weyl's theorem" has been extended to hyponormal and to Toeplitz operators [5], to seminormal and other operators [2, 4] and to Banach spaces operators [9, 10]. Variants have been discussed by Harte and Lee [8] and Rakočevič [11].

We say that the Weyl's theorem holds for $T \in B(H)$ if there is equality

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T).$$

Harte and Lee [8] have discussed a variant of the Weyl's theorem: "the Browder's theorem" holds for T if

$$\sigma(T) = \sigma_w(T) \cup \pi_{00}(T).$$

What is missing is the disjointness between the Weyl spectrum and the isolated eigenvalues of finite multiplicity: equivalently

$$\sigma_w(T) = \sigma_b(T).$$

Rakočevič [12] has looked at variants of "Weyl's theorem" and "Browder's theorem" in which the spectrum is replaced by the approximate point spectrum: "the a-Weyl's theorem" holds for T if

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \pi^a_{00}(T),$$

where we write $\sigma_a(T)$ for the approximate point spectrum of T, $\pi_{00}^a(T) = \pi_0(T) \cap iso\, \sigma_a(T)$ and $\sigma_{ea}(T) = \bigcap\{\sigma_a(T+K) : K \in K(H)\}$. It is well known that $\sigma_{ea}(T)$ coincides with $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(H)\}$, where $SF_+^-(H) = \{T \in B(H), T$ is upper semi-Fredholm of $\mathrm{ind}(T) \leq 0\}$. "The a-Browder's theorem" holds for T if

$$\sigma_{ea}(T) = \sigma_{ab}(T),$$

where $\sigma_{ab}(T) = \bigcap\{\sigma_a(T+K) : K \in K(H) \cap \mathrm{comm}(T)\}$. We know that $\lambda \notin \sigma_{ab}(T)$ if and only if $T - \lambda I$ is upper semi-Fredholm and $T - \lambda I$ has finite ascent. We call $\sigma_{ea}(T)$ and $\sigma_{ab}(T)$ the essential approximate point spectrum and the Browder essential approximate point spectrum, respectively. $T \in B(H)$ is said [1] to satisfy property (ω) if

$$\sigma_a(T) \backslash \sigma_{ea}(T) = \pi_{00}(T).$$

Property (ω) implies Weyl's theorem, a-Browder's theorem, Browder's theorem [1].

Let $\rho_2(T) = \{\lambda \in \mathbb{C}$: there exists $\epsilon > 0$ such that $T - \mu I$ is semi-Fredholm and

$$N(T - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \mu I)^n] \ \text{ if } \ 0 < |\mu - \lambda| < \epsilon\}$$

and let $\sigma_2(T) = \mathbb{C} \backslash \rho_2(T)$, $\sigma_G(T) = \{\lambda \in \mathbb{C} : R(T - \lambda I)$ is not closed$\}$.

Theorem 2.1 *Weyl's theorem holds for $T \in B(H)$,*

$\iff \sigma_b(T) = \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$

$\iff \sigma_b(T) = \sigma_1(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [acc\sigma(T) \bigcap \sigma_G(T)]$.

Proof Suppose Weyl's theorem holds for T. Clearly $\overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \subseteq \sigma_b(T)$, we only need prove $\overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \supseteq \sigma_b(T)$. Let $\lambda_0 \notin \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$. Then $0 < n(T - \lambda_0 I) < \infty$, $\lambda_0 \notin \overline{\sigma_1(T)}$, and there exists $\epsilon > 0$ such that $T - \lambda I$ is semi-Fredholm, and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda_0 - \lambda| < \epsilon$ since $\lambda_0 \notin \sigma_2(T)$. Since $\lambda_0 \notin \overline{\sigma_1(T)}$, it follows that $T - \lambda I$ is CFI if ϵ is small enough. The fact that $T - \lambda I$ is semi-Fredholm tells us that $T - \lambda I$ is Weyl [3, Theorem 3.2]. Then $T - \lambda I$ is Browder since Weyl's theorem holds for T. Thus $N(T - \lambda I) = N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$ (see [12, Theorem 3.4]), which means that $T - \lambda I$ is invertible if $0 < |\lambda - \lambda_0|$ is small enough. That is $\lambda_0 \in \mathrm{iso}\sigma(T)$. This shows that $\lambda_0 \in \pi_{00}(T)$ since $0 < n(T - \lambda I) < \infty$. Since Weyl's theorem holds for T, it follows that $T - \lambda_0 I$ is Browder. Now we prove $\lambda_0 \notin \sigma_b(T)$.

For the converse, let $\lambda_0 \in \sigma(T) \backslash \sigma_w(T)$. Then $\lambda_0 \notin \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$, that is $\lambda_0 \notin \sigma_b(T)$. This means that $\sigma(T) \backslash \sigma_w(T) \subseteq \pi_{00}(T)$.

Let $\lambda_0 \in \pi_{00}(T)$. It is easy to see that $\lambda_0 \notin \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \bigcup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$. Then $\lambda_0 \notin \sigma_b(T)$, which means that $\lambda_0 \in \sigma(T) \backslash \sigma_w(T)$. That is $\pi_{00}(T) \subseteq \sigma(T) \backslash \sigma_w(T)$.

Hence, $\pi_{00}(T) = \sigma(T) \backslash \sigma_w(T)$.

In the same way, we can prove that Weyl's theorem holds for T if and only if $\sigma_b(T) = \sigma_1(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [acc\sigma(T) \bigcap \sigma_G(T)]$. $\square$

Remark 2.1 "T (T^*) satisfies Weyl's theorem" cannot imply "Weyl's theorem holds for T^* (T)".

For example, let $T \in B(\ell^2)$ be defined by: $T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$. Then $\sigma(T) = \sigma_w(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$, that is, $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, which means that Wely's theorem holds for T. But since $\sigma(T^*) = \sigma_w(T^*) = \{0\}$, while $\pi_{00}(T^*) = \{0\}$, that is, $\sigma(T) \setminus \sigma_w(T) \neq \pi_{00}(T)$, which shows that Wely's theorem fails for T^*.

It is well known that T is called isoloid if iso$\sigma(T) \subseteq \{\lambda \in \mathbb{C} : n(T - \lambda I) > 0\}$.

Theorem 2.2 *Suppose that both T and T^* are isoloid and satisfy Weyl's theorem,*

$\iff \sigma_b(T) = \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\}$.

$\iff \sigma_b(T) = \sigma_1(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup [acc\sigma(T) \bigcap \sigma_G(T)]$.

Proof Suppose that T and T^* are isoloid and both of them satisfy Weyl's theorem. It is clear that $\overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \subseteq \sigma_b(T)$. Let $\lambda_0 \notin \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\}$. Then $n(T - \lambda_0 I) < \infty$ or $n(T^* - \lambda_0 I) < \infty$, also $\lambda_0 \notin \overline{\sigma_1(T)}$, and there exists $\epsilon > 0$ such that $T - \lambda I$ is semi-Fredholm, and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda_0 - \lambda| < \epsilon$. Let ϵ be sufficiently small. Then $T - \lambda I$ is Weyl [3, Theorem 3.2]. Since Weyl's theorem holds for T, we have that $T - \lambda I$ is Browder. Then $N(T - \lambda I) = N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$ (see [12, Theorem 3.4]), which means that $T - \lambda I$ is invertible. That is $\lambda_0 \in iso\sigma(T)$. It is easy to see that $\lambda_0 \in \pi_{00}(T)$ or $\lambda_0 \in \pi_{00}(T^*)$ since both T and T^* are isoloid. Using the fact that Weyl's theorem holds for T and T^*, we have that $\lambda_0 \notin \sigma_b(T)$.

Conversely, suppose that $\sigma_b(T) = \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\}$. Let $\lambda_0 \in \sigma(T)\setminus\sigma_w(T)$. Then $\lambda_0 \notin \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\}$, thus $\lambda_0 \notin \sigma_b(T)$. It implies that $\lambda_0 \in \pi_{00}(T)$, that is, $\sigma(T)\setminus\sigma_w(T) \subseteq \pi_{00}(T)$. For the converse, let $\lambda_0 \in \pi_{00}(T)$. Then $\lambda_0 \notin \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\}$, thus $\lambda_0 \notin \sigma_b(T)$. This shows that $\lambda_0 \in \sigma(T)\setminus\sigma_w(T)$, and we get $\pi_{00}(T) \subseteq \sigma(T)\setminus\sigma_w(T)$. Hence, Weyl's theorem holds for T.

We will prove that T is isoloid. Suppose $\lambda_0 \in iso\sigma(T)$, but $n(T - \lambda_0 I) = 0$. It implies $\lambda_0 \notin \overline{\sigma_1(T)} \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\}$. Then $\lambda_0 \notin \sigma_b(T)$. Since $n(T - \lambda_0 I) = 0$, we know $T - \lambda_0 I$ is invertible. It is in contradiction to the fact $\lambda_0 \in iso\sigma(T)$.

Similarly to the above proof, it can be proved that Weyl's theorem holds for T^* and T^* is isoloid.

In the same way, we can prove that T and T^* are isoloid and satisfy Weyl's theorem if and only if $\sigma_b(T) = \sigma_1(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup [acc\sigma(T) \bigcap \sigma_G(T)]$. □

Remark 2.2 "Weyl's theorem holds for T (T^*)" does not imply "Property (ω) holds for T (T^*)". For example, let $A, B \in B(\ell^2)$ be defined by: $A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$; $B(x_1, x_2, x_3, \ldots) = (x_1, 0, x_3, x_4, \ldots)$, and let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Then $\sigma(T) = \sigma_w(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, $\pi_{00}(T) = \emptyset$, $\sigma_a(T) = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Clearly, Weyl's theorem holds for T but Property (ω) fails for T.

In the following we explore when Property (ω) holds for T.

Theorem 2.3 *Property (ω) holds for T and T^* if and only if Weyl's theorem holds for T and T^* and $\sigma_b(T) = \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \sigma(T) : n(T^* - \lambda I) = 0\}$.*

Proof Suppose that Property (ω) holds for T and T^*. We only need to prove $\sigma_b(T) = \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \sigma(T) : n(T^* - \lambda I) = 0\}$. Clearly we have $\partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \sigma(T) : n(T^* - \lambda I) = 0\} \subseteq \sigma_b(T)$. Let $\lambda_0 \notin \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \sigma(T) : n(T^* - \lambda I) = 0\}$. We claim that $\lambda_0 \notin \overline{\sigma_1(T)}$. Otherwise, we have $\lambda_0 \in \sigma_1(T)$, then $\mathrm{ind}(T - \lambda_0 I) > 0$ or $\mathrm{ind}(T - \lambda_0 I) < 0$. Thus $\lambda_0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$ or $\lambda_0 \in \sigma_a(T^*) \setminus \sigma_{ea}(T^*)$. Since Property (ω) holds for T and T^*, we know that $T - \lambda_0 I$ is Browder. It is in contradiction to the fact that $\mathrm{ind}(T - \lambda_0 I) > 0$ or $\mathrm{ind}(T - \lambda_0 I) < 0$. By $\lambda_0 \notin \sigma_2(T)$, there exists $\epsilon > 0$ such that $T - \lambda I$ is semi-Fredholm, and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda_0 - \lambda| < \epsilon$. Let $\epsilon > 0$ be small enough. Since $\lambda \notin \overline{\sigma_1(T)}$, it follows that $T - \lambda I$ is Browder. Then $N(T - \lambda I) = N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$, which means that $T - \lambda I$ is invertible. This shows that $\lambda_0 \in iso\,\sigma(T)$. Then $\lambda_0 \in \pi_{00}(T)$ or $\lambda_0 \in \pi_{00}(T^*)$ since $n(T - \lambda_0 I) < \infty$ or $n(T^* - \lambda_0 I) < \infty$. Using the condition that Property (ω) holds for T and T^*, we know $T - \lambda_0 I$ is Browder. Thus $\lambda_0 \notin \sigma_b(T)$.

Conversely, suppose Weyl's theorem holds for T and T^* and $\sigma_b(T) = \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \sigma(T) : n(T^* - \lambda I) = 0\}$. We have that $\pi_{00}(T) = \sigma(T) \setminus \sigma_w(T) \subseteq \sigma_a(T) \setminus \sigma_{ea}(T)$ and $\pi_{00}(T^*) = \sigma(T^*) \setminus \sigma_w(T^*) \subseteq \sigma_a(T^*) \setminus \sigma_{ea}(T^*)$. We only need to prove $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \pi_{00}(T)$ and $\sigma_a(T^*) \setminus \sigma_{ea}(T^*) \subseteq \pi_{00}(T^*)$. Let $\lambda_0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$. It is easy to see that $n(T^* - \lambda I) \neq 0$, then $\lambda_0 \notin \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = n(T^* - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \sigma(T) : n(T^* - \lambda I) = 0\}$. This shows that $\lambda_0 \notin \sigma_b(T)$, then $\lambda_0 \in \pi_{00}(T)$, that is $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \pi_{00}(T)$. Also we can prove that $\sigma_a(T^*) \setminus \sigma_{ea}(T^*) \subseteq \pi_{00}(T^*)$. $\square$

Theorem 2.4 *Property (ω) holds for T if and only if $\sigma_b(T) = \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\}$.*

Proof Suppose Property (ω) holds for T. The inclusion $\partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \subseteq \sigma_b(T)$ is clear. For the converse inclusion, let $\lambda_0 \notin \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\}$. Then $0 < n(T - \lambda_0 I) < \infty$, $n(T - \lambda_0 I) \leq d(T - \lambda_0 I)$, and $\lambda_0 \notin \partial\sigma_1(T)$. We assert that $\lambda_0 \notin \overline{\sigma_1(T)}$. Otherwise, we get $\lambda_0 \in \sigma_1(T)$, then $\lambda_0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Thus $T - \lambda_0 I$ is Browder since Property (ω) holds for T, it is in contradiction to $\mathrm{ind}(T - \lambda_0 I) < 0$. Then $\lambda_0 \in \mathrm{int}\,\rho_1(T)$. Using the fact that $\lambda_0 \notin \sigma_2(T)$, we can prove that $\lambda_0 \in \mathrm{iso}\,\sigma(T)$. Then $\lambda_0 \in \pi_{00}(T)$ since $0 < n(T - \lambda_0 I) < \infty$. Since property (ω) holds for T and T^*, $T - \lambda_0 I$ is Browder, that is, $\lambda_0 \notin \sigma_b(T)$.

Conversely, suppose $\sigma_b(T) = \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)$. Let $\lambda_0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $\lambda_0 \notin \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)$, which implies that $\lambda_0 \notin \sigma_b(T)$. We get $\lambda_0 \in \pi_{00}(T)$, that is, $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \pi_{00}(T)$. For the converse inclusion, let $\lambda_0 \in \pi_{00}(T)$. Then $n(T-\lambda_0 I) \leq d(T-\lambda_0 I)$. Thus $\lambda_0 \notin \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) > d(T-\lambda I)\}$. It follows that $\lambda_0 \notin \sigma_b(T)$, then $\lambda_0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$, that is, $\pi_{00}(T) \subseteq \sigma_a(T) \setminus \sigma_{ea}(T)$. Therefore, property (ω) holds for T. $\Box$

Remark 2.3 "Property (ω) holds for T (T^*)" does not imply "Property (ω) holds for T^* (T)". For example let $T \in B(\ell^2)$ be defined as in Remark 2.2. We know that $\sigma_a(T) = \sigma_{ea}(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$, which means that Property (ω) holds for T. But since $\pi_{00}(T^*) = \{0\}$ and $\sigma_a(T^*) = \sigma_{ea}(T^*) = \{0\}$, we know that that Property (ω) fails for T^*.

Let $\rho_a(T) = \mathbb{C}\backslash\sigma_a(T)(\rho_s(T) = \mathbb{C}\backslash\sigma_s(T))$. Similarly to the proof of Theorem 2.3, we can prove the following

Theorem 2.5 *T and T^* are isoloid and satisfy property (ω) if and only if $\sigma_b(T) = \partial\sigma_1(T) \cup \sigma_2(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = n(T^*-\lambda I) = \infty\} \cup \{[\rho_a(T) \cup \rho_s(T)] \cap \sigma(T)\}$.*

3. Weyl type theorem for operator matrices

The study of upper triangular operator matrices arises naturally from the following fact: if A is a Hilbert space operator and M is an invariant subspace for A, then A has the following 2×2 upper triangular operator matrix representation:

$$A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : M \oplus M^{\perp} \longrightarrow M \oplus M^{\perp},$$

and one way to study operator is to see them as entries of simpler operators. The upper triangular operator matrices have been studied by many authors [14, 15]. When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C an operator acting on $H \oplus K$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where $C \in B(K,H)$. If $C = 0$, let $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Wely's theorem (property (ω)) may not hold for a direct sum of operators for which Weyl's theorem (property (ω)) holds. In this section, using the new spectrum set $\sigma_1(T)$ and $\sigma_2(T)$, we explore the Weyl's type theorem (property (ω)) for 2×2 operator matrices. We begin with [16, Lemma 3.1].

Lemma 3.1 *For a given pair (A,B) of operators, if both A and B have finite ascent, then for every $C \in B(K,H)$, M_C has finite ascent.*

Theorem 3.1 *Let $A \in B(H)$ be such that $\partial\sigma_1(A) \cup \sigma_2(A) = \sigma_w(A)$ and let $B \in B(K)$.*

(1) If Weyl's theorem holds for M_{C_0} for some $C_0 \in B(K,H)$, then it holds for M_C for every $C \in B(K,H)$;

(2) If property (ω) holds for M_{C_0} for some $C_0 \in B(K,H)$, then it holds for M_C for every $C \in B(K,H)$.

Proof (1) For any $C \in B(K,H)$, suppose $M_C - \lambda_0 I$ is Weyl but not invertible. From

$$M_C - \lambda_0 I = \begin{pmatrix} I & 0 \\ 0 & B-\lambda_0 I \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A-\lambda_0 I & C \\ 0 & I \end{pmatrix}$$

we know $A - \lambda_0 I$ is upper semi-Fredholm, $B - \lambda_0 I$ is lower semi-Fredholm and $A - \lambda_0 I$ is Fredholm if and only if $B - \lambda_0 I$ is Fredholm. Since $A - \lambda_0 I$ is upper semi-Fredholm, there is $\lambda_0 \notin \partial\sigma_1(A) \cup \sigma_2(A)$. It follows that $\lambda_0 \notin \sigma_w(A)$, which means that $A - \lambda_0 I$ is Weyl. Then $B - \lambda_0 I$ is Fredholm and hence $\mathrm{ind}(M_{C_0} - \lambda_0 I) = \mathrm{ind}(A - \lambda_0 I) + \mathrm{ind}(B - \lambda_0 I) = \mathrm{ind}(M_C - \lambda_0 I) = 0$. This shows that $M_{C_0} - \lambda_0 I$ is Weyl. But since Weyl's theorem holds for M_{C_0}, it follows that $M_{C_0} - \lambda_0 I$ is Browder, that is $asc(A - \lambda_0 I) < \infty$ and $\mathrm{des}(B - \lambda_0 I) < \infty$. Thus $A - \lambda_0 I$ is Browder since $A - \lambda_0 I$ is Weyl. The fact that both $M_{C_0} - \lambda_0 I$ and $A - \lambda_0 I$ are Browder tells us that $B - \lambda_0 I$ is Browder. This proves that $M_C - \lambda_0 I$ is Browder. Now we get that $\sigma(M_C) \backslash \sigma_w(M_C) \subseteq \pi_{00}(M_C)$.

For the reverse inclusion, first suppose $\lambda_0 \in \pi_{00}(M_C)$. Then $0 < n(M_C - \lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $M_C - \lambda_0 I$ is invertible if $0 < |\lambda - \lambda_0| < \epsilon$. It follows that $A - \lambda I$ is bounded from below and $B - \lambda I$ is surjective if $0 < |\lambda - \lambda_0| < \epsilon$. Then $\lambda \notin \partial\sigma_1(A) \cup \sigma_2(A)$. Thus $A - \lambda I$ is Weyl and hence $A - \lambda I$ is invertible. This implies $B - \lambda I$ is invertible too. Hence $\lambda_0 \in iso\sigma(M_{C_0})$. We will show that $0 < n(M_{C_0} - \lambda_0 I) < \infty$. First of all observe that there is a general inclusion

$$N(M_C - \lambda_0 I) \subseteq (A - \lambda_0 I)^{-1}[CN(B - \lambda_0 I)] \oplus N(B - \lambda_0 I),$$

which forces $N(A - \lambda_0 I) \oplus N(B - \lambda_0 I)$ to be nontrivial because otherwise $N(M_C - \lambda_0 I)$ would be trivial, a contradiction. Now we must show that $N(A - \lambda_0 I) \oplus N(B - \lambda_0 I)$ is finite-dimensional. But since $N(A - \lambda_0 I) \oplus \{0\} \subseteq N(M_C - \lambda_0 I)$, it follows that $n(A - \lambda_0 I) < \infty$. Thus we only need to prove that $n(B - \lambda_0 I) < \infty$. If $n(B - \lambda_0 I) = \infty$, without loss of generality, suppose $\lambda_0 \in \sigma(A)$, then $\lambda_0 \in iso\sigma(A)$, which implies that $\lambda_0 \notin \partial\sigma_1(A) \cup \sigma_2(A)$. Thus $A - \lambda_0 I$ is Browder. Now there are two cases to consider.

Suppose that $CN(B - \lambda_0 I)$ is finite-dimensional. Then $N(C)$ must contain an orthonormal sequence $\{y_i\}$ in $N(B - \lambda_0 I)$. But then $\binom{0}{y_i} \in N(M_C - \lambda_0 I)$, which means that $N(M_C - \lambda_0 I)$ is infinite-dimensional, a contradiction.

Suppose that $CN(B - \lambda_0 I)$ is infinite-dimensional. Since $A - \lambda_0 I$ is Browder, $R(A - \lambda_0 I)^\perp$ must be finite-dimensional. Therefore $CN(B - \lambda_0 I) \cap R(A - \lambda_0 I)$ is infinite-dimensional. Now we can find an orthonormal sequence $\{y_i\}$ in $N(B - \lambda_0 I)$ for which there exists a sequence$\{x_i\}$ in H such that $(A - \lambda_0 I)x_i = Cy_i$ for each $i = 1, 2, \ldots$. Then $\binom{x_i}{-y_i} \in N(M_C - \lambda_0 I)$, which implies that $N(M_C - \lambda_0 I)$ is infinite-dimensional, a contradiction again.

From the preceding proof, we know that $0 < \dim[N(A - \lambda_0 I) \oplus N(B - \lambda_0 I)] < \infty$. The fact that $N(M_C - \lambda_0 I) \subseteq (A - \lambda_0 I)^{-1}[CN(B - \lambda_0 I)] \oplus N(B - \lambda_0 I)$ implies $n(M_C - \lambda_0 I) < \infty$. If $N(M_C - \lambda_0 I) = \{0\}$, then $N(A - \lambda_0 I) = \{0\}$, which means that $A - \lambda_0 I$ is invertible. Thus $0 < n(B - \lambda_0 I) < \infty$. Let $y_0 \in N(B - \lambda_0 I)$ and $y_0 \neq 0$. There exists $x_0 \in H$ such that $(A - \lambda_0 I)x_0 = C_0 y_0$, because $R(A - \lambda_0 I)$ is surjective. Then $\binom{x_i}{-y_i} \in N(M_C - \lambda_0 I)$, a contradiction. Hence $\lambda_0 \in \pi_{00}(M_{C_0})$. Since Weyl's theorem holds for M_{C_0}, it follows that $M_{C_0} - \lambda_0 I$ is Browder. We can prove that $A - \lambda_0 I$ is Browder, so is $B - \lambda_0 I$. Hence $M_C - \lambda_0 I$

is Browder. Thus $\lambda_0 \in \sigma(M_C)\backslash\sigma_w(M_C)$. Now we prove that Weyl's theorem holds for M_C for each $C \in B(K,H)$.

(2) For any $C \in B(K,H)$, suppose $\lambda_0 \in \sigma_a(M_C)\backslash\sigma_{ea}(M_C)$. Then $A-\lambda_0 I$ is upper semi-Fredholm. We assert that $A-\lambda_0 I$ is Weyl and $B-\lambda_0 I$ is upper semi-Fredholm, this is an immediate consequence of $\partial\sigma_1(A)\cup\sigma_2(A)=\sigma_w(A)$. It follows that $\mathrm{ind}(M_{C_0}-\lambda_0 I)=\mathrm{ind}(A-\lambda_0 I)+\mathrm{ind}(B-\lambda_0 I)=\mathrm{ind}(M_C-\lambda_0 I)\leq 0$. From the general inclusion $N(M_C-\lambda_0 I)\subseteq (A-\lambda_0 I)^{-1}[CN(B-\lambda_0 I)]\oplus N(B-\lambda_0 I)$ and the condition that $n(M_C-\lambda_0 I)>0$, we get that $0<\dim[N(A-\lambda_0 I)\oplus N(B-\lambda_0 I)]<\infty$. We claim that $n(M_{C_0}-\lambda_0 I)>0$. Otherwise, let $N(M_{C_0}-\lambda_0 I)=\{0\}$. Then $A-\lambda_0 I$ is invertible. This shows that $0<\dim N(B-\lambda_0 I)<\infty$. Let $y_0\in N(B-\lambda_0 I)$ and $y_0\neq 0$. Using the fact that $R(A-\lambda_0 I)$ is surjective, we know that there is $x_0\in H$ such that $(A-\lambda_0 I)x_0=C_0x_0$. Then $\binom{x_0}{-y_0}\in N(M_{C_0}-\lambda_0 I)$, it is in contradiction to the assumption that $N(M_{C_0}-\lambda_0 I)=\{0\}$. Then $n(M_{C_0}-\lambda_0 I)>0$, which means that $\lambda_0\in \sigma_a(M_{C_0})\backslash\sigma_{ea}(M_{C_0})$. Since property (ω) holds for M_{C_0}, $M_{C_0}-\lambda_0 I$ is Browder. Similarly to the proof (1), we get that $M_C-\lambda_0 I$ is Browder. Now we have proved $\sigma_a(M_C)\backslash\sigma_{ea}(M_C)\subseteq\pi_{00}(M_C)$.

Conversely, similarly to the proof (1), we can prove that $\pi_{00}(M_C)\subseteq\sigma(M_C)\backslash\sigma_w(M_C)$.

From the above proof, property (ω) holds for M_C. $\square$

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