Convergence of Composition Operators on Hardy-Smirnov Space

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Abstract We consider the convergence of composition operators on Hardy-Smirnov space over a simply connected domain properly contained in the complex plane. The convergence of the power of a composition operator is also considered. Our approach is a method from Joel H. Shapiro and Wayne Smith’s celebrated work (Journal of Functional Analysis 205 (2003) 62-89). The resulting space is usually not the one obtained from the classical Hardy space of the unit disc by conformal mapping.

Keywords composition operators; convergence; Hardy-Smirnov space.

1. Introduction

For a simply connected domain $G$ that is properly contained in the complex plane, let $\tau$ be a Riemann map that takes the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ univalently onto $G$. Let $\Gamma_r$ denote the $\tau$-image of the circle $|z| = r$. For $0 < p < \infty$, let $\Lambda^p(G)$ be the collection of functions $F$ holomorphic on $G$ such that

$$
\|F\|_{\Lambda^p(G)} = \left( \sup_{0 < r < 1} \int_{\Gamma_r} |F(w)|^p |dw| \right)^{1/p} < \infty.
$$

Following [4] and [14], we call these the Hardy-Smirnov spaces of $G$.

The classical Hardy space $H^p$ $(0 < p < \infty)$ over the unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ is the collection of functions analytic on the unit disc $U$, satisfying

$$
\|f\|_{H^p} = \left( \sup_{0 < r < 1} \int_{\mathbb{T}} |f(re^{i\xi})|^p dm(\xi) \right)^{1/p} < \infty,
$$

where $\mathbb{T}$ is the unit circle $|z| = 1$, $m$ is the normalized arc-length measure on $\mathbb{T}$.

If $\Phi$ is a function holomorphic on $G$ with $\Phi(G) \subset G$, then $\Phi$ induces a linear composition operator (see [14]) $C_\Phi$ on the space $\text{Hol}(G)$ of all functions holomorphic on $G$ as follows

$$
C_\Phi F = F \circ \Phi, \quad F \in \text{Hol}(G).
$$
If $G$ is the unit disc, various prospects of such operators have been extensively studied [3, 10–14].

In [14], inspired by an earlier result of Matache [6], Shapiro and Smith investigated Hardy-Smirnov spaces that support compact operators. Their celebrated work takes place on the space $\Lambda^p(G)$ which is defined by (1). From [4] and [14], we know that the map $C_\tau : F \rightarrow F \circ \tau$ is an isomorphism of $\Lambda^p(G)$ onto $H^p$ if and only if both $\tau'$ and its reciprocal are bounded on $\mathbb{U}$. Either $\tau'$ or its reciprocal is unbounded on $\mathbb{U}$, our Hardy-Smirnov spaces are different from the conformally invariant ones. That is the point why we want to generalize the results on convergence of composition operators on the Hilbert Hardy space $H^2$ over the unit disc in [7].

In this paper, we will use the approach in [14] to consider the convergence of composition operators $C_\Phi$ on the Hardy-Smirnov spaces defined in (1).

Let us introduce some terminology from [14] that is proven to be quite useful and will be heavily used. For a simply connected domain $G$ that is properly contained in the complex plane, let $\tau$ be a Riemann map that takes the open unit disc $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$ univalently onto $G$. To every self-map $\Phi$ of $G$, the self-map $\varphi$ of $\mathbb{U}$ is defined as follows

$$\varphi = \tau^{-1} \circ \Phi \circ \tau. \quad (3)$$

For each index $0 < p < \infty$, the operator $V_p$ is defined as follows

$$(V_p F)(z) = \tau'(z)^{1/p} F(\tau(z)), \quad F \in \text{Hol}(G), \quad z \in \mathbb{U}. \quad (4)$$

The weighted operator

$$A_{\varphi,p} = V_p C_\Phi V_p^{-1}$$

maps $H^p$ boundedly into itself if and only if $C_\Phi$ is bounded on $\Lambda^p(G)$. Since $V_p$ establishes an isometric similarity between the two operators (for details see [14, p.66–67]), for a sequence of composition operators $\{C_{\Phi_n}\}$ induced by self-maps $\{\Phi_n\}$ of $G$ on $\Lambda^p(G)$, its norm convergence is equivalent to norm convergence of the corresponding sequence of composition operators $\{A_{\varphi_n,p}\}$ induced by $\{\varphi_n = \tau^{-1} \circ \Phi_n \circ \tau\}$ on $H^p$. Thus for $f \in H^p$,

$$(A_{\varphi,p} f)(z) = Q_{\varphi}(z)^{1/p} f(\varphi(z)), \quad \text{where} \quad Q_{\varphi}(z) = \frac{\tau'(z)}{\tau'(\varphi(z))} \quad (5)$$

Using the operator $A_{\varphi,p}$, we can generalize the results on convergence of composition operators on the Hilbert Hardy space $H^2$ over the unit disc in [7].

Motivated by the work in [9], in [7], Matache considered convergent sequence of composition operators $\{C_{\varphi_n}\}$ induced by $\{\varphi_n\}$ that converges in some sense to $\varphi$. Matache supposed $\{\varphi_n\}$ and $\varphi$ are bounded analytic self-maps of $\mathbb{U}$ endowed with the norm

$$\| f \|_\infty = \sup_{0 < r < 1} |f(z)|, \quad f \in \text{Hol}(\mathbb{U}) \quad (6)$$

He pointed out that only the case

$$|\varphi(\xi)| < 1 \quad \text{m-a.e.} \quad (7)$$

on $T$ is valuable for consideration.

Under the assumptions above, Matache proved that if $C_\varphi$ is a Hilbert-Schmidt operator, then $\|C_{\varphi_n} - C_\varphi\|_{\text{HS}} \rightarrow 0$ if and only if $\|C_{\varphi_n}\|_{\text{HS}} \rightarrow \|C_\varphi\|_{\text{HS}}$ and $\|\varphi_n - \varphi\|_{H^2} \rightarrow 0$. Let $\{\Phi_n\}$ be
a sequence of self-maps of \( G \) whose counterpart \( \{ \varphi_n \} \) is defined in (3), where \( \{ \varphi_n \} \) and \( \varphi \) are bounded in the norm (6) and satisfy (7). Let \( \{ Q_{\varphi_n}(z) \} \) and \( \{ A_{\varphi_n,p} \} \) be defined in (5). In Section 2, we show that if

\[
\int_T \frac{|Q_{\varphi}|}{1 - |\varphi|^2} dm < \infty,
\]

then with some restrictions on \( \{ Q_{\varphi_n} \} \), \( \| A_{\varphi_n,2} - A_{\varphi,2} \|_{\text{HS}} \to 0 \) if and only if \( \| A_{\varphi_n,2} \|_{\text{HS}} \to \| A_{\varphi,2} \|_{\text{HS}} \) and \( \| Q_{\varphi_n}^2 - Q_{\varphi}^2 \|_{H^2} \to 0 \).

Matache also considered the sequence \( \{ C^n_\varphi \} \) for non-inner \( \varphi \). Matache proved that if \( \varphi \) has a fixed point \( b \) in \( U \), then \( \| C^n_\varphi - C_b \|_{H^2} \to 0 \). We will investigate the same problem in \( \Lambda^p(G) \). We prove if \( \tau' \) and its reciprocal are in \( H^2 \) and \( \varphi \) defined by (3) is non-inner in \( U \), \( \Phi \) has a fixed point \( a \) in \( G \); if \( \| Q_{\varphi,n} \|_{H^2} \) is uniformly bounded, for the proper subclass \( F \in \Lambda^2(G) \subset \Lambda^1(G) \), \( \{ \| C^n_\varphi F \|_{A^2(G)} \} \) converges.

2. Norm convergence of a sequence of composition operators

Recall that on any Hilbert space, the Hilbert-Schmidt norm \( \| T \|_{\text{HS}} \) of an operator \( T \) is defined as

\[
\| T \|_{\text{HS}}^2 = \sum_{n=0}^{\infty} \| T e_n \|^2,
\]

where \( \{ e_n \} \) is an orthonormal basis [7, 11]. From the discussion in [7, p. 662], we know that \( \| T \|_{\text{HS}} \) is larger than or equals the operator norm \( \| T \| \). Since \( V_p \) establishes an isometric similarity between the two operators (for details see [14, pp. 66–67]), we just need to investigate the case where \( \| A_{\varphi,n,2} - A_{\varphi,2} \|_{\text{HS}} \to 0 \), which is equivalent to \( \| C_{\varphi,n} - C_{\varphi} \|_{\text{HS}(G)} \to 0 \).

The main result of this section is as follows.

**Theorem 1** Let \( \{ \Phi_n \} \) be a sequence of self-maps of \( G \) and \( \{ \varphi_n \} \) be defined in (3), replacing \( \{ \Phi \} \) by \( \{ \Phi_n \} \). Let \( \{ Q_{\varphi,n}(z) \} \) and \( \{ A_{\varphi,n,2} \} \) be defined in (5). If \( \{ \varphi_n \} \) and \( \varphi \) are bounded in the norm (6), satisfying (7), \( \{ \varphi_n \} \to \varphi \) a.e. on \( T \), \( Q_{\varphi}(z) \) satisfies (8), moreover, there is some function \( \chi(\varphi) \) defined on \( T \) such that

\[
\frac{|Q_{\varphi,n}|}{1 - |\varphi_n|^2} \leq \chi(\varphi), \quad m\text{-a.e.}
\]

and

\[
\int_T \chi(\varphi) dm < \infty,
\]

then \( \| A_{\varphi,n,2} - A_{\varphi,2} \|_{\text{HS}} \to 0 \) if and only if \( \| A_{\varphi,n,2} \|_{\text{HS}} \to \| A_{\varphi,2} \|_{\text{HS}} \) and \( \| Q_{\varphi,n}^2 - Q_{\varphi}^2 \|_{H^2} \to 0 \).

**Proof** As \( \{ 1, z, z^2, z^3, \ldots \} \) is the standard basis of \( H^2 \), the direct calculation of Hilbert-Schmidt norm (see also [14, 2.2 Example]) shows that if \( Q_{\varphi}(z) \) satisfies (8), then \( A_{\varphi,2} \) is a Hilbert-Schmidt composition operator.

If \( \| A_{\varphi,n,2} - A_{\varphi,2} \|_{\text{HS}} \to 0 \), then it is obvious that \( \| A_{\varphi,n,2} \|_{\text{HS}} \to \| A_{\varphi,2} \|_{\text{HS}} \), and we have

\[
\| Q_{\varphi,n}^2 - Q_{\varphi}^2 \|_{H^2} = \| A_{\varphi,n,2}(1) - A_{\varphi,2}(1) \|_{\text{HS}} \leq \| A_{\varphi,n,2} - A_{\varphi,2} \|_{\text{HS}} \to 0.
\]
Conversely, if $\|A_{\varphi,2}\|_{\text{HS}} \to A_{\varphi,2}\|_{\text{HS}}$ and $\|Q_{\varphi}^{1/2} - Q_{\varphi}^{1/2}\|_{H^2} \to 0$, but $\|A_{\varphi,2} - A_{\varphi,2}\|_{\text{HS}} \to 0$ does not hold, we will get a contradiction. In this case, for some $\varepsilon_0 > 0$, there is a subsequence $\{A_{\varphi_{n_k},2}\}$ of $\{A_{\varphi,2}\}$ satisfying

$$\|A_{\varphi_{n_k},2} - A_{\varphi,2}\|_{\text{HS}} \geq \varepsilon_0. \quad (11)$$

Since $\|Q_{\varphi}^{1/2} - Q_{\varphi}^{1/2}\|_{H^2} \to 0$, we can select a subsequence $\{Q_{\varphi_{m_k}}^{1/2}\}$ of $\{Q_{\varphi_{n_k}}^{1/2}\}$ such that $\{Q_{\varphi_{m_k}}^{1/2}\}$ converges a.e. to $Q_{\varphi}^{1/2}$. We have

$$\|A_{\varphi_{m_k},2} - A_{\varphi,2}\|_{\text{HS}} = \int_{\mathcal{T}} \frac{|Q_{\varphi_{m_k}}^{1/2}|}{1 - |\varphi_{m_k}|^2} \, dm + \int_{\mathcal{T}} \frac{|Q_{\varphi}^{1/2}|}{1 - |\varphi|^2} \, dm - 2R \int_{\mathcal{T}} \frac{Q_{\varphi_{m_k}}^{1/2} \bar{Q}_{\varphi}^{1/2}}{1 - \varphi \varphi_{m_k}} \, dm. \quad (12)$$

By (9), we have

$$\frac{|Q_{\varphi_{m_k}}^{1/2} \bar{Q}_{\varphi}^{1/2}|}{1 - |\varphi \varphi_{m_k}|} \leq \frac{|Q_{\varphi_{m_k}}^{1/2} Q_{\varphi}^{1/2}|}{(1 - |\varphi_{m_k}|^2)^{1/2}(1 - |\varphi|^2)^{1/2}} \leq \chi(\varphi) + \frac{|Q_{\varphi}^{1/2}|}{1 - |\varphi|^2}, \quad \text{m-a.e.}$$

By the a.e. convergence and the dominated convergence theorem, combining (8) and (10), we have

$$\int_{\mathcal{T}} \frac{Q_{\varphi_{m_k}}^{1/2} \bar{Q}_{\varphi}^{1/2}}{1 - |\varphi \varphi_{m_k}|} \, dm \to \int_{\mathcal{T}} \frac{|Q_{\varphi}^{1/2}|}{1 - |\varphi|^2} \, dm, \quad (13)$$

and

$$\int_{\mathcal{T}} \frac{|Q_{\varphi_{m_k}}|}{1 - |\varphi_{m_k}|^2} \, dm \to \int_{\mathcal{T}} \frac{|Q_{\varphi}^{1/2}|}{1 - |\varphi|^2} \, dm. \quad (14)$$

From (12), (13) and (14), we have

$$\|A_{\varphi_{m_k},2} - A_{\varphi,2}\|_{\text{HS}} \to 0$$

which contradicts (11).

### 3. Powers of composition operators

In this section we treat the convergence of the operator sequence $\{C^n_{\phi^n,\mathcal{F}}\}$ for $F \in \Lambda^p(G)$ where $C^n_{\phi^n,\mathcal{F}} = \mathcal{F}(\Phi^{[n]} \circ \cdots \circ \Phi)$ is the n-fold iteration of $\Phi$. From the reasoning in Section 2, we only need to investigate the corresponding sequence $\{A^n_{\phi,\mathcal{F}}\}$ in $H^p$, here

$$A^n_{\phi,\mathcal{F}}(z) = Q_{\phi,[n]}(z)^{1/p} f(\phi^{[n]}(z)), \quad Q_{\phi,[n]}(z) = \frac{\phi^{[n]}(z)}{\phi^{[n]}(\tau^{-1}(z))}, \quad z \in \mathbb{U}, \quad (15)$$

where $\phi^{[n]} = \phi \circ \cdots \circ \phi$ is the n-fold iteration of $\phi$. Suppose $\Phi$ is a self-map of $G$ satisfying $\Phi(G) \subset G$ and for some $a \in G$, $\Phi(a) = a$, then $\phi$ defined in (3) has a fixed point $b = \tau^{-1}(a)$ in $\mathbb{U}$ (see [14]).

Recall an inner function $\phi$ is an analytic self-map of $\mathbb{U}$ whose radial limit-function is unimodular m-a.e. on $\mathbb{T}$ (see [4, 5, 8]). From [8, p. 353, Exercise 6], we know that $H^2$ is a proper subclass of $H^1$.

The main results of this section are as follows.

**Theorem 2** Let $\Phi$ be a self-map of $G$ satisfying $\Phi(G) \subset G$ and for some $a \in G$, $\Phi(a) = a$. 

If there is a Riemann map \( \tau \) from \( G \) to \( U \) such that both \( \tau' \) and its reciprocal are in \( H^2 \), \( \{\|Q_{\varphi[n]}\|_{H^2}\} \) is uniformly bounded, \( \varphi \) defined in (3) is bounded in the norm (6) and non-inner, then \( \{\|C_{\varphi}^nF\|_{\Lambda^1(G)}\} \) converges for the proper subclass \( F \in \Lambda^2(G) \subset \Lambda^1(G) \).

**Theorem 3** Let \( \Phi \) be a self-map of \( G \) satisfying \( \Phi(G) \subset G \) and for some \( a \in G, \Phi(a) = a \). If there is a Riemann map \( \tau \) from \( G \) to \( U \) such that \( \tau' \) and its reciprocal are bounded on \( T \) in the norm (6) and non-inner, then for each \( F \in \Lambda^p(G) \) and \( 0 < p < \infty \), the sequence \( \{\|C_{\Phi}^nF\|_{\Lambda^p(G)}\} \) converges.

In order to prove the theorems, we need the following lemmas.

**Lemma 1** ([8, p. 339]) Suppose \( 0 < p < \infty \), \( f \in H^p \) and \( f \) is not equivalently zero, \( B \) is the Blaschke product formed with the zeros of \( f \). Then there is a zero-free function \( h \in H^2 \) such that

\[
f = B \cdot h^{2/p}.
\]

**Lemma 2** ([14]) If a composition operator \( C_{\varphi} \) is bounded on \( \Lambda^p(G) \) for some \( 0 < p < \infty \), then it is bounded for all such \( p \). Actually, for arbitrary \( 0 < p < \infty \) and \( 0 < q < \infty \),

\[
\|A_{\varphi,p}\|^p = \|A_{\varphi,q}\|^q.
\]

The following lemma is the main result of Section 3 in [7], here we cite it as a lemma for later use. For \( f \in H^p \), let \( C_a f = f(a) \). From [7] we know it is a bounded linear operator in \( H^p \).

**Lemma 3** ([7]) Let \( \varphi \) be a self-map of \( U \) with a fixed point in \( U \) which is bounded in the norm defined in (6) and non-inner. Then for \( f \in H^2 \), \( C_{\varphi}^nf(z) = f(\varphi^n)(z) \) and \( C_b f(z) = f(b) \), \( \|C_{\varphi}^n - C_b\|_{H^2} \to 0 \) as \( n \to \infty \).

**Proof of Theorem 2** Since convergence of \( \{\|C_{\Phi}^nF\|_{\Lambda^p(G)}\} \) is equivalent to the convergence of \( \{\|A_{\varphi,p}^n f\|_{H^p}\} \), we just need to show \( \|A_{\varphi,p}^n f - A_{b,p} f\|_{H^p} \to 0 \) where \( A_{b,p} f(z) = \frac{\tau_{\varphi^n}(z)}{\tau_{\tau^n}(b)} f(b), b = \tau^{-1}(a) \). If

\[
\|A_{\varphi,p}^n f - A_{b,1} f\|_{H^1} \to 0,
\]

for the proper subclass \( f \in H^2 \subset H^1 \), then Theorem 2 follows. By the triangular inequality, for \( f \in H^2 \subset H^1 \) and \( \|f\|_{H^1} = 1 \)

\[
\|A_{\varphi,1}^n f - A_{b,1} f\|_{H^1} = \|Q_{\varphi[n]} f(\varphi[n]) - \tau'/\tau'(b) \cdot f(b)\|_{H^1} \\
\leq \|Q_{\varphi[n]} f(\varphi[n]) - Q_{\varphi[n]} f(b)\|_{H^1} + \|Q_{\varphi[n]} f(b) - \tau'/\tau'(b) \cdot f(b)\|_{H^1}.
\]

Then we just need to prove

\[
\|Q_{\varphi[n]} f(\varphi[n]) - Q_{\varphi[n]} f(b)\|_{H^1} \to 0
\]

and

\[
\|Q_{\varphi[n]} f(b) - \tau'/\tau'(b) \cdot f(b)\|_{H^1} \to 0.
\]

By triangular inequality and the decomposition \( f = B \cdot h \) for \( f \in H^2 \subset H^1 \) where \( h \in H^2 \) is the
decomposition in Lemma 1,

\[ \|Q_{\varphi[n]}f(\varphi[n]) - Q_{\varphi[n]}f(b)\|_{H^1} = \|Q_{\varphi[n]}B(\varphi[n])h(\varphi[n]) - Q_{\varphi[n]}B(b)h(b)\|_{H^1} \]

\[ \leq \|Q_{\varphi[n]}B(\varphi[n])h(\varphi[n]) - Q_{\varphi[n]}B(\varphi[n])h(b)\|_{H^1} + \|Q_{\varphi[n]}B(\varphi[n])h(b) - Q_{\varphi[n]}B(b)h(b)\|_{H^1}. \]

By the boundedness of \( B(z) \), we have

\[ \|Q_{\varphi[n]}f(\varphi[n]) - Q_{\varphi[n]}f(b)\|_{H^1} \leq \|Q_{\varphi[n]}(h(\varphi[n]) - h(b))\|_{H^1} + |h(b)| \cdot \|Q_{\varphi[n]}(B(\varphi[n]) - B(b))\|_{H^1}. \]

(21)

Since \( \{\|Q_{\varphi[n]}\|_{H^2}\} \) is uniformly bounded, there exists some \( M > 0 \), such that \( \|Q_{\varphi[n]}\|_{H^2} \leq M \).

By the Hölder’s inequality, we have

\[ \|Q_{\varphi[n]}(h(\varphi[n]) - h(b))\|_{H^1} \leq \|Q_{\varphi[n]}||^{1/2} \cdot \|h(\varphi[n]) - h(b)\|^{|1/2}_{H^2} \]

\[ \leq M^{1/2} \cdot \|h(\varphi[n]) - h(b)\|^{|1/2}_{H^2}. \]

From Lemma 3 and \( h \in H^2 \), it follows

\[ \|h(\varphi[n]) - h(b)\|_{H^2} = \|C_{\varphi}^n h - C_b h\|_{H^2} \leq \|h\|_{H^2} \cdot \|C_{\varphi}^n - C_b\|_{H^2} \to 0. \]

From the boundedness of

\[ \|1/\tau'(\varphi[n])\|_{H^2} = \|C_{\varphi}^n (1/\tau')\|_{H^2}, \]

we have

\[ \|Q_{\varphi[n]}(h(\varphi[n]) - h(b))\|_{H^1} \to 0. \]

(22)

Repeating the above reasoning gives

\[ \|Q_{\varphi[n]}(B(\varphi[n]) - B(b))\|_{H^1} \leq \|Q_{\varphi[n]}||^{1/2} \cdot \|(B(\varphi[n]) - B(b))\|^{|1/2}_{H^2} \]

\[ \leq M^{1/2} \cdot \|(B(\varphi[n]) - B(b))\|^{|1/2}_{H^2}. \]

By the boundedness of \( B(z) \), from [1] and [11], we know that \( B(\varphi[n]) \) converges to \( B(b) \) a.e..

Thus applying the bounded convergence theorem yields

\[ \|Q_{\varphi[n]}(B(\varphi[n]) - B(b))\|_{H^1} \to 0. \]

(23)

From (21), (22) and (23), we can get (19). By the Hölder’s inequality, we have

\[ \|Q_{\varphi[n]}f(b) - \tau'/\tau'(b) \cdot f(b)\|_{H^1} \leq |f(b)| \cdot \|\tau'|^{1/2}_{H^2} \cdot \|1/\tau'(\varphi[n]) - 1/\tau'(b)|^{1/2}_{H^2}. \]

By Lemma 3, we get (20).

Remark In the proof of Theorem 2, the boundedness of Blaschke product is applied. We refer to [4], [5] and [8] for more details.

Proof of Theorem 3 Since convergence of \( \{\|C_{\varphi}^n F\|_{\mathcal{A}(G)}\} \) is equivalent to the convergence of \( \{\|A_{\varphi,p}^n f\|_{H^p}\} \), we just need to prove the convergence of the latter one. Denote \( A_{b,p} f(z) = \tau'/\tau'(b) \cdot f(b), b = \tau^{-1}(a) \), it is obvious that \( A_{b,p} \) is bounded by the supposition of Theorem 3, and we just need to prove \( \|A_{\varphi,p}^n - A_{b,p}\|_{H^p} \to 0 \). By Lemma 2, we only need to prove the case where
p = 2. By the boundedness of \( \tau' \) and its reciprocal we know that the operator \( T f = \tau' f \) and its inverse operator \( T^{-1} f = \frac{1}{\tau} f \) are both bounded in \( H^2 \). Taking \( f \in H^2 \) and \( \|f\|_{H^2} = 1 \) gives

\[
\|A_{\varphi,2}^n - A_{b,2}\|_{H^2} = \|A_{\varphi,2}^n f - A_{b,2} f\|_{H^2} = \|T C_{\varphi}^n T^{-1} f - T C_b T^{-1} f\|_{H^2} \\
\leq \|T\|_{H^2} \cdot \|T^{-1}\|_{H^2} \cdot \|C_{\varphi}^n - C_b\|_{H^2}.
\]

Thus by Lemma 3, we have \( \|A_{\varphi,2}^n - A_{b,2}\|_{H^2} \to 0 \).

References