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Moore-Smith Convergence in L-Fuzzifying Topological Spaces

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Abstract This paper presents a definition of L-fuzzifying nets and the related L-fuzzifying generalized convergence spaces. The Moore-Smith convergence is established in L-fuzzifying topology. It is shown that the category of L-fuzzifying generalized convergence spaces is a cartesian-closed topological category which embeds the category of L-fuzzifying topological spaces as a reflective subcategory.

Keywords L-fuzzifying topology; L-fuzzifying filter; L-fuzzifying net; L-fuzzifying generalized convergence space; topological category; cartesian-closed.

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1. Introduction and preliminaries

Convergence of filters and nets, called the Moore-Smith convergence, is an important topic in general topology. For convenience, sometimes we use filters and sometimes use nets to define and study convergence in topology since there is a close relation between them.

In L-topology theory, the Moore-Smith convergence theory had been completely established by Pu and Liu in [1] by means of L-fuzzy nets and L-fuzzy filters (of crisp degree). Analogously, in L-fuzzifying topology [2], in order to study convergence structures, L-fuzzifying filters or L-fuzzifying nets should be used. While there is no proper definition of L-fuzzifying nets corresponding to L-fuzzifying filters in fuzzy set theory.

The aim of this paper is to give a definition of L-fuzzifying nets corresponding to L-fuzzifying filters and then to establish the Moore-Smith convergence in L-fuzzifying topology. This paper is arranged as follows. In the rest of this section, we recall some materials which will be used

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throughout this paper. In Section 2, we give definitions of L-fuzzifying filters and L-fuzzifying nets and then study the Moore-Smith convergence in L-fuzzifying topology. In Section 3, we define an L-fuzzifying generalized convergence spaces and show that the resulting category L-FYGConv embeds the category of L-fuzzifying topological spaces as a reflective category. In Section 4, we show that L-FYGConv is a cartesian-closed topological category.

In the following, we will list some preliminaries which are used in this paper.

An element a of a lattice is called $\wedge$-irreducible if $a = b \wedge c$ always implies $a = b$ or $a = c$ for any elements b, c. A lattice with a $\wedge$-irreducible button 0 is called 0-$\wedge$-inaccessible. For example, the unit interval $[0, 1]$ is such a lattice. A DeMorgan algebra is a complete lattice equipped with an order-reversing involution.

A complete lattice L is a frame or a complete Heyting algebra if the binary meets are distributive over arbitrary joins, i.e., $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$ holds for all $a, b_i (i \in I) \in L$. For a frame L, an implicative operator $\rightarrow: L \times L \longrightarrow L$ can be defined as $a \rightarrow b = \bigvee\{c \in L |\ a \wedge c \leq b\}$ $(\forall a, b \in L)$. Then for any $a, b, c \in L,\ a \wedge c \leq b \Longleftrightarrow c \leq a \rightarrow b$. A frame is called spatial if it is generated by all $\wedge$-irreducible elements, that is, any element is the meets of all $\wedge$-irreducible elements less than or equal to it. Properties of frames can be found in many literatures, e.g. [3].

In this paper, L is always assumed to be a 0-$\wedge$ irreducible frame. We put $L_0 = L - \{0\}$.

2. L-fuzzifying filters, L-fuzzifying nets and their Moore-Smith convergence

Definition 1 ([4]) *We call a map $\mathcal{F} : 2^X \longrightarrow L$ an L-fuzzifying filter on X if*

(LF1) $\mathcal{F}(\emptyset) = 0, \mathcal{F}(X) = 1$*;*

(LF2) $\mathcal{F}(A \cap B) = \mathcal{F}(A) \wedge \mathcal{F}(B)$.

An L-fuzzyfying topology on a set X is a map $\tau : 2^X \longrightarrow L$ satisfying that (1) $\tau(\emptyset) = \tau(X) = 1$; (2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$; (3) $\tau(\bigcup_i A_i) \geq \bigwedge_i \tau(A_i)$. The pair (X, τ) is called an L-fuzzifying topological space [2]. A map $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ between two L-fuzzifying topological spaces is called continuous if $\tau_X(f^{-1}(B)) \geq \tau_Y(B)$ for all $B \subseteq Y$. Let L-FYS denote the category of all L-fuzzifying topological spaces with continuous maps as morphisms.

Let $\tau : L^X \longrightarrow L$ be an L-fuzzifying topology and $x \in X$. Define $\mathcal{N}_\tau^x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$. Then $\mathcal{N}_\tau^x$ is an L-fuzzifying filter, which is the neighborhood filter in [5].

Definition 2 *For a directed set Δ, we call a map $\xi = (p, v) : \Delta \longrightarrow X \times L_0$ an L-fuzzifying net on X if*

(LN1) $\bigvee_{d \in \Delta} v(d) = 1$*;*

(LN2) For any $d_1, d_2 \in \Delta$, there exists an upper bound d of d_1, d_2 such that $v(d_1) \wedge v(d_2) \leq v(d)$.

For $L = 2$, an L-fuzzifying net is just an ordinary net. Let $\xi = (p, v) : \Delta \longrightarrow X \times L_0$ be an L-fuzzifying net. We call ξ an L-fuzzifying net of crisp degree if $v \equiv 1$. We call ξ a constant net if $p : \Delta \longrightarrow X$ is a constant map with a value x and $v \equiv 1$, and in this case, ξ is also denoted by $\overline{x}$.

We denote the set of all L-fuzzifying filters (resp., nets) on a set X by $\mathbb{F}(X)$ (resp., $\mathbb{N}(X)$).

Let $\xi = (p_1, v_1) : D \longrightarrow X \times L_0, \eta = (p_2, v_2) : E \longrightarrow X \times L_0$ be two L-fuzzifying nets. We call η a subnet of ξ if there is a map $j : E \longrightarrow D$ satisfying that $p_2 = p_1 \circ j,\ v_2 = v_1 \circ j$ and for each $d \in D$, there exists an $e_0 \in E$ such that $v_2(e_0) \geq v_1(d)$ and $j(e) \geq d$ for all $e \geq e_0$.

Proposition 1 *(1) Let $f : X \longrightarrow Y$ be a map and $\xi = (p, v) \in \mathbb{N}(X),\ \mathcal{F} \in \mathbb{F}(X)$. Define $f(\xi) = (f \circ p, v) : \Delta \longrightarrow Y \times L_0$ and for all $B \subseteq Y,\ f(\mathcal{F})(B) = \mathcal{F}(f^{-1}(B))$, then $f(\xi) \in \mathbb{N}(Y),\ f(\mathcal{F}) \in \mathbb{F}(Y)$.*

(2) Let $f : X \longrightarrow Y$ be a map and ξ, η be two L-fuzzifying nets of X. If η is a subnet of ξ, then $f(\eta)$ is a subnet of $f(\xi)$.

Proof Straightforward. □

Definition 3 *Suppose that $\xi = (p, v) : D \longrightarrow X \times L_0$ is an L-fuzzifying net on X. Define $\mathcal{F}_\xi(A) = \bigvee\{v(d) |\ \forall e \geq d,\ p(e) \in A\}$, which can be considered as the degree for ξ eventually belonging to A.*

Proposition 2 *$\mathcal{F}_\xi$ is an L-fuzzifying filter.*

Proof (a) $\mathcal{F}_\xi(0_X) = \bigvee \emptyset = 0$, $\mathcal{F}_\xi(X) = \bigvee d \in \Delta v(d) = 1$. (b) Obviously, $\mathcal{F}_\xi$ is order-preserving. For any $A, B \in L^X$,

$$\begin{aligned}\mathcal{F}_\xi(A) \wedge \mathcal{F}_\xi(B) &= \bigvee\{v(d_1) |\ \forall e \geq d_1,\ p(e) \in A\} \wedge \bigvee\{v(d_2) |\ \forall e \geq d_2,\ p(e) \in B\}\\ &= \bigvee\{v(d_1) \wedge v(d_2) |\ \forall e_1 \geq d_1, \forall e_2 \geq d_2,\ p(e_1) \in A, p(e_2) \in B\}\\ &\leq \bigvee\{v(d) |\ \forall e \geq d,\ p(e) \in A \cap B\} = \mathcal{F}_\xi(A \cap B).\end{aligned}$$ □

Proposition 3 *If $\eta = (p_2, v_2) : E \longrightarrow X \times L_0$ is a subnet of $\xi = (p_1, v_1) : D \longrightarrow X \times L_0$, then $\mathcal{F}_\xi \leq \mathcal{F}_\eta$.*

Proof For $A \subseteq X$, suppose that $d \in D$ satisfying that $p_1(d_1) \in A$ for all $d_1 \geq d$. Since η is a subnet of ξ, there is a map $j : E \longrightarrow D$ satisfying that $p_2 = p_1 \circ j,\ v_2 = v_1 \circ j$ and for this d, there exists an $e_0 \in E$ such that $v_2(e_0) \geq v_1(d)$, $j(e) \geq d$ and $p_2(e) = p_1(j(e)) \in A$ for all $e \geq e_0$. By $v_2(e_0) \geq v_1(d)$, we have $\mathcal{F}_\xi \leq \mathcal{F}_\eta$. □

Proposition 4 Let $f : X \longrightarrow Y$ be a map. For every L-fuzzifying net ξ on a set X, $\mathcal{F}_{f(\xi)} = f(\mathcal{F}_\xi)$.

Proof For every $A \subseteq X$, we have

$$\begin{aligned}\mathcal{F}_{f(\xi)}(A) &= \bigvee\{v(d) |\ \forall e \geq d,\ f(p(e)) \in A\} = \bigvee\{v(d) |\ \forall e \geq d,\ p(e) \in f^{-1}(A)\}\\ &= \mathcal{F}_\xi(f^{-1}(A)) = f(\mathcal{F}_\xi)(A).\end{aligned}$$

For $\mathcal{F}$ an L-fuzzifying filter on X, put $\mathcal{F}^+ = \{A \subseteq X |\ \mathcal{F}(A) \in L_0\}$ and $\Delta_\mathcal{F} = \{(x, A) |\ x \in A \in \mathcal{F}^+\}$. Define a relation $\prec$ on $\Delta_\mathcal{F}$ as $(x, A) \prec (y, B)$ iff $B \subseteq A$. Define $\xi_\mathcal{F} = (p_\mathcal{F}, v_\mathcal{F}) : \Delta_\mathcal{F} \longrightarrow X \times L_0$ by $(x, A) \mapsto (x, \mathcal{F}(A))$. □

Proposition 5 *$\xi_{\mathcal{F}}$ is an L-fuzzifying net.*

Proof (1) $\Delta_{\mathcal{F}}$ is a direct set. For $(x,A),(y,B)\in\Delta_{\mathcal{F}}$, we have $\mathcal{F}(A),\mathcal{F}(B)\neq 0$ and then $\mathcal{F}(A\cap B)=\mathcal{F}(A)\wedge\mathcal{F}(B)\neq 0$. Then $A\cap B\neq 0$ and there exists $z\in A\cap B$. Thus $(x,A),(y,B)\prec(z,A\cap B)\in\Delta_{\mathcal{F}}$.

(2) $\xi_{\mathcal{F}}$ is an L-fuzzifying net. (LN1) $\bigvee_{(x,A)\in\Delta_{\mathcal{F}}}v_{\mathcal{F}}(x,A)\geq\mathcal{F}(X)=1$. (LN2) For $(x,A),(y,B)\in\Delta_{\mathcal{F}}$, then $(z,A\cap B)$ is an upper bound of $(x,A),(y,B)$. $v_{\mathcal{F}}(x,A)\wedge v_{\mathcal{F}}(y,B)=\mathcal{F}(A)\wedge\mathcal{F}(B)=\mathcal{F}(A\cap B)=v_{\mathcal{F}}(z,A\cap B)$. □

Proposition 6 For every L-fuzzifying filter $\mathcal{F}$ on a set X, $\mathcal{F}_{\xi_{\mathcal{F}}}=\mathcal{F}$.

Proof Let $A\subseteq X$. If $\mathcal{F}(A)\neq 0$, then $A\neq\emptyset$. Choose $x\in A$, we have $(x,\mathcal{F}(A))\in\Delta_{\mathcal{F}}$. For any $(y,B)\in\Delta_{\mathcal{F}}$ with $(y,B)\succ(x,A)$, we have $\xi_{\mathcal{F}}(y,B)=y\in B\subseteq A$. Thus $v_{\mathcal{F}}(x,\mathcal{F}(A))=\mathcal{F}(A)\in P(\mathcal{F},A)$ and $\mathcal{F}_{\xi_{\mathcal{F}}}(A)\geq\mathcal{F}(A)$. Conversely, for any $(x,B)\in\Delta_{\mathcal{F}}$, $p_{\mathcal{F}}(y,C)=y\in A$ for any $(y,C)\succ(x,B)$. For $z\in B$, we have $(z,B)\in\Delta_{\mathcal{F}}$ and $(z,B)\succ(x,B)$, then $z\in A$. Then $B\subseteq A$. Hence $v_{\mathcal{F}}(x,B)=\mathcal{F}(B)\leq\mathcal{F}(A)$. Consequently we have $\mathcal{F}_{\xi_{\mathcal{F}}}(A)\leq\mathcal{F}(A)$. □

In the rest of this paper, we assume that L has an order-reversing involution *. For any L-fuzzifying topological space (X,τ), define $\mathrm{cl}:2^X\longrightarrow L^X$ by $\mathrm{cl}(A)(x)=(\mathcal{N}_\tau^x(A'))^*$. This is a special case of the closure operator cl in [6] (see Theorem 5.3).

Define $L_f:\mathbb{F}(X)\times X\longrightarrow L$ by $L_f(\mathcal{F},x)=\bigwedge_{A\in L^X}\mathrm{cl}(A)(x)\vee\mathcal{F}(A')$ and $L_n:\mathbb{N}(X)\times X\longrightarrow L$ by $L_n(\xi,x)=L_f(\mathcal{F}_\xi,x)$. The value $L_f(\mathcal{F},x)$ (resp., $L_n(\xi,x)$) can be considered as the degree of x to be a limit point of $\mathcal{F}$ (resp., ξ). By Proposition 6, we have $L_n(\xi_{\mathcal{F}},x)=L_f(\mathcal{F},x)$.

Remark 1 In a crisp topological space (X,T), a filter $\mathcal{F}$ is convergent to a point x iff $\mathcal{U}(x)\subseteq\mathcal{F}$ [7]. In fact, we can show that $\mathcal{U}(x)\subseteq\mathcal{F}$ iff for any $A\subseteq X$, $x\in A^-$ or $A'\in\mathcal{F}$. Thus L_f,L_n are generalizations of classical convergence in crisp topology.

Proof Suppose that $\mathcal{U}(x)\subseteq\mathcal{F}$. For $A\subseteq X$, if $A'\notin\mathcal{F}$, then $A'\notin\mathcal{U}(x)$. For all $U\in\mathcal{U}(x)$, if $U\cap A=\emptyset$, then $U\subseteq A'$, which implies that $A'\in\mathcal{U}(x)$, which is a contradiction to $A'\notin\mathcal{U}(x)$. Hence $x\in A^-$. Conversely, for any open neighborhood U of x, if $U=(U')'\notin\mathcal{F}$, then $x\in U'^-=U'$ (notice that U is open and U' is closed), which is a contradiction to $U\in\mathcal{U}(x)$.

We define $C_f:\mathbb{F}(X)\times X\longrightarrow L$ by $C_f(\mathcal{F},x)=\bigwedge_{A\in X}\mathrm{cl}(A)(x)\vee(\mathcal{F}(A))^*$ and $C_n:\mathbb{N}(X)\times X\longrightarrow L$ by $C_n(\xi,x)=C_f(\mathcal{F}_\xi,x)$. The value $C_f(\mathcal{F},x)$ (resp., $C_n(\xi,x)$) can be considered as the degree of x to be a cluster point of $\mathcal{F}$ (resp., ξ). By Proposition 6, $C_n(\xi_{\mathcal{F}},x)=C_f(\mathcal{F},x)$. □

Remark 2 In a crisp topological space (X,T), $x\in X$ is a cluster point of a filter $\mathcal{F}$ iff for all $A\in\mathcal{F}$, $x\in A^-$ (see [7]). Thus C_f,C_n are generalizations of cluster in crisp topology.

Proposition 7 For every $\mathcal{F}\in\mathbb{F}(\mathbb{X}),\xi\in\mathbb{N}(\mathbb{X})$ and $x\in X$, we have $L_f(\mathcal{F},\S)\leq\mathcal{C}_\{(\mathcal{F},\S)$, $L_n(\xi,x)\leq C_n(\xi,x)$.

Proof We only need to show $L_f\leq C_f$ or just $\mathcal{F}_\xi(A')\leq(\mathcal{F}_\xi(A))^*$. If $(\mathcal{F}_\xi(A))^*\neq 1$, then $\mathcal{F}_\xi(A)\neq 0$. While $\mathcal{F}_\xi(A')\wedge\mathcal{F}_\xi(A)=\mathcal{F}_\xi(A'\cap A)=0$, then $\mathcal{F}_\xi(A')=0$ since L is 0-$\wedge$-irreducible. □

For any L-fuzzifying topology τ on X and p a $\wedge$-irreducible element of L, it is easy to check that the family $\tau_{(p)} = \{A \subseteq X|\ \tau(A) \not\leq p\}$ is a crisp topology on X. In a topological space (X,T), for $x \in X,\ A \subseteq X$, we have $x \in A^-$ if and only if $U \cap A \neq \emptyset$ for any $U \in \mathcal{U}(x)$ (see [7]). Let $\mathcal{U}(x)$ be the neighborhood system of x. If $x \in A^-$, then there exists a net $\xi : \mathcal{U}(x) \longrightarrow X$ such that for any $U \in \mathcal{U}(x)$, $\xi(U) \in A \cap U$. We denote such a net by $\xi_{\mathcal{U}(x)}$.

Let $\xi = (p, v) : D \longrightarrow X \times L_0$ be an L-fuzzifying net and $A \subseteq X$. For a notation $\xi \subseteq A$, we mean $p(d) \in A$ for all $d \in D$. We also denote by $\mathbb{N}^c(X)$ the set of all L-fuzzifying nets of crisp degree on X.

We now give the main results of this section.

Theorem 1 *Suppose that L is a spatial frame. For $A \in L^X, x \in X$, the following five values are equal to each other:*

(1) $\mathrm{cl}(A)(x)$*; (2)* $\bigvee_{\xi\subseteq A} L_n(\xi, x)$*; (3)* $\bigvee_{\xi\subseteq A} C_n(\xi, x)$*;*

(4) $\bigvee_{\mathbb{N}^c(X)\ni\xi\subseteq A} L_n(\xi, x)$*; (5)* $\bigvee_{\mathbb{N}^c(X)\ni\xi\subseteq A} C_n(\xi, x)$.

Proof Obviously (4)$\leq$(5,2)$\leq$(3).

(3)$\leq$(1). First, it is easy to see that if $\xi \subseteq A$, then $\mathcal{F}_\xi(A) = 1$ and $(\mathcal{F}_\xi(A))^* = 0$. $\bigvee_{\xi\subseteq A} C_n(\xi, x) = \bigvee_{\xi\subseteq A} \bigwedge_{B\subseteq X} \mathrm{cl}(B)(x)\vee(\mathcal{F}_\xi(B))^* \leq \bigwedge_{\xi\subseteq A} \mathrm{cl}(A)(x)\vee(\mathcal{F}_\xi(A))^* = \bigwedge_{\xi\subseteq A} \mathrm{cl}(A)(x)\vee 0 = \mathrm{cl}(A)(x)$.

(1)$\leq$(4). We only need to show that

$$\mathcal{N}^x_\tau(A') \geq \bigwedge_{\mathbb{N}^c(X)\ni\xi\subseteq A} \bigvee_{B\in L^X} \mathcal{N}^x_\tau(B) \wedge (\mathcal{F}_\xi(B))^*.$$

In fact, for all prime element $p \geq \mathcal{N}^x_\tau(A')$, then $x \leq \overline{A}|\tau_{(p)}$ (otherwise, there exists an open neighborhood U of x in $\tau_{(p)}$ such that $U \subseteq A'$, then $x \in U \subseteq A$ and then $\tau(U) \leq p$, which is a contradiction to $U \in \tau_{(p)}$). Let $\mathcal{U}(x)$ be the neighborhood system of x in $\tau_{(p)}$. Clearly, $\xi_{\mathcal{U}(x)} \subseteq A$. Consider ξ as an L-fuzzifying net of crisp degree, then the value $(\mathcal{F}_\xi(B))^*$ is 0 or 1. If $(\mathcal{F}_{\xi_{\mathcal{U}(x)}}(B))^* = 1$, then $\mathcal{F}_{\xi_{\mathcal{U}(x)}}(B) = 0$. Then for all $U \in \mathcal{U}(x)$, there exists $V \in \mathcal{U}(x)$ such that $V \subseteq U$ and $\xi_{\mathcal{U}(x)}(V) \notin B$. In order to complete the proof, we only need to show that for all $x \in C \subseteq B$, $\tau(C) \leq p$. If not, then $C \in \tau_{(p)}$, and $C \in \mathcal{U}(x)$. For this C, there exists $D \subseteq C$ such that $\xi_{\mathcal{U}(x)}(D) \notin B$, while $\xi_{\mathcal{U}(x)}(D) \in D \subseteq C \subseteq B$, leading to a contradiction. □

3. Embed L-FYS in the category of L-fuzzifying generalized convergence spaces

A pair of functors (F, G) is called an adjunction [8] between two categories $\mathcal{A}$ and $\mathcal{B}$ if for any $A \in ob(\mathcal{A}), B \in ob(\mathcal{B})$, there is a bijection between $\hom_{\mathcal{A}}(A, G(B))$ and $\hom_{\mathcal{B}}(F(A), B)$. The functor F is called the left adjoint of G and G the right adjoint of F. If $\mathcal{A}$ is a subcategory of $\mathcal{B}$ and the inclusion functor $i : \mathcal{A} \longrightarrow \mathcal{B}$ has a left (resp., right) adjoint, then $\mathcal{A}$ is called a reflective (resp., coreflective) subcategory of $\mathcal{B}$.

In this section, we will show that the category of L-fuzzifying topological spaces can be embedded in the category of net-theoretical L-fuzzifying generalized convergence spaces as a reflective subcategory.

Definition 4 *We call a map $S : \mathbb{N}(X) \times X \longrightarrow L$ an L-fuzzifying generalized convergence structure on X if it satisfies*

(LC1) For all $x \in X$, $S(\overline{x}, x) = 1$;

(LC2) If η is a subnet of ξ, then for any $x \in X$, $S(\xi, x) \leq S(\eta, x)$.

The pair (X, S) is called an L-fuzzifying generalized convergence space.

For two L-fuzzifying generalized convergence spaces (X, S_1) and (Y, S_2), a map $f : X \longrightarrow Y$ is called continuous if for any $(\xi, x) \in \mathbb{N}(X) \times X$, $S_1(\xi, x) \leq S_2(f(\xi), f(x))$. Denote by L-FYGConv the category of L-fuzzifying generalized convergence spaces with continuous maps as morphisms.

Let (X, S) be an L-fuzzifying generalized convergence space. Define $\mathcal{U}_S^x : L^X \longrightarrow L$ by

$$\mathcal{U}_S^x(A) = \begin{cases} \bigwedge\limits_{\xi \in \mathbb{N}(X)} S(\xi, x) \to \mathcal{F}_\xi(A), & x \in A; \\ 0, & x \notin A. \end{cases}$$

Lemma 1 *(1) For all $x \in X$, $\mathcal{U}_S^x$ is an L-fuzzifying filter.*

(2) Let $f : (X, S_1) \longrightarrow (Y, S_2)$ be a continuous map. Then $f(\mathcal{U}_{S_1}^x) \geq \mathcal{U}_{S_2}^{(f(x))}$.

Proof (1) $\mathcal{U}_S^x(\emptyset) = 0$, $\mathcal{U}_S^x(X) = 1$ are obvious. For all $A, B \subseteq X$, $x \in A \cap B$ iff $x \in A, x \in B$. Then

$$\begin{aligned} \mathcal{U}_S^x(A) \wedge \mathcal{U}_S^x(B) \leq & \bigwedge_{\xi \in \mathbb{N}(X)} (S(\xi, x) \to \mathcal{F}_\xi(A)) \wedge (S(\xi, x) \to \mathcal{F}_\xi(B)) \\ = & \bigwedge_{\xi \in \mathbb{N}(X)} S(\xi, x) \to (\mathcal{F}_\xi(A) \wedge \mathcal{F}_\xi(B)) \\ = & \bigwedge_{\xi \in \mathbb{N}(X)} S(\xi, x) \to \mathcal{F}_\xi(A \cap B) \\ = & \, \mathcal{U}_S^x(A \cap B). \end{aligned}$$

Clearly, $\mathcal{U}_S^x$ is order-preserving. Hence $\mathcal{U}_S^x(A \cap B) = \mathcal{U}_S^x(A) \wedge \mathcal{U}_S^x(B)$.

(2) For all $B \subseteq Y$,

$$\begin{aligned} f(\mathcal{U}_{S_1}^x)(B) &= \mathcal{U}_{S_1}^x(f^{-1}(B)) = \bigwedge_{\xi \in \mathbb{N}(X)} S_1(\xi, x) \to \mathcal{F}_\xi(f^{-1}(B)) \\ &\geq \bigwedge_{\xi \in \mathbb{N}(X)} S_2(f(\xi), f(x)) \to f(\mathcal{F}_\xi)(B) = \bigwedge_{\xi \in \mathbb{N}(X)} S_2(f(\xi), f(x)) \to \mathcal{F}_{f(\xi)}(B) \\ &\geq \bigwedge_{\eta \in \mathbb{N}(Y)} S_2(\eta, f(x)) \to \mathcal{F}_\eta(B) \\ &= \mathcal{U}_{S_2}^{f(x)}(B). \ \Box \end{aligned}$$

Define $\tau_S : 2^X \longrightarrow L$ by

$$\tau_S(A) = \bigwedge_{x \in A} \mathcal{U}_S^x(A), \quad \forall A \subseteq X.$$

Proposition 8 *The map τ_S is an L-fuzzifying topology on X.*

Proof (O1) Obviously, $\tau_S(\emptyset) = \tau_S(X) = 1$.

(O2) For any $A, B \subseteq X$, if $A \cap B = \emptyset$, then $\tau_S(A \cap B) = 1 \geq \tau_S(A) \wedge \tau_S(B)$. Otherwise,

$$\begin{aligned}\tau_S(A) \wedge \tau_S(B) &= \bigwedge_{x\in A} \mathcal{U}_S^x(A) \wedge \bigwedge_{y\in B} \mathcal{U}_S^y(B) \leq \bigwedge_{z\in A\cap B} \mathcal{U}_S^z(A) \wedge \mathcal{U}_S^z(B) \\ &= \bigwedge_{z\in A\cap B} \mathcal{U}_S^z(A\cap B) = \tau_S(A \cap B).\end{aligned}$$

(O3)

$$\tau_S(\bigcup_i A_i) = \bigwedge_{x\in\bigcup_i A_i} \mathcal{U}_S^x(\bigcup_i A_i) \geq \bigwedge_{\exists i,\ x\in A_i} \mathcal{U}_S^x(A_i) = \bigwedge_i \bigwedge_{x\in A_i} \mathcal{U}_S^x(A_i) = \bigwedge_i \tau_S(A_i). \quad \square$$

Proposition 9 *If $f : (X, S_1) \longrightarrow (Y, S_2)$ is continuous, then $f : (X, \tau_{S_1}) \longrightarrow (Y, \tau_{S_2})$ is continuous.*

Proof For all $B \subseteq Y$,

$$\tau_{S_1}(f^{-1}(B)) = \bigwedge_{x\in f^{-1}(B)} \mathcal{U}_{S_1}^x(f^{-1}(B)) = \bigwedge_{f(x)\in B} f^{\rightarrow}(\mathcal{U}_{S_1}^x)(B) \geq \bigwedge_{f(x)\in B} \mathcal{U}_{S_2}^{f(x)}(B) \geq \tau_{S_2}(B). \quad \square$$

By Propositions 8 and 9, we obtain a concrete functor T_S from L-FYS to L-FYGConv transferring an L-fuzzifying generalized convergence structure S to τ_S.

Let (X, τ) be an L-fuzzifying topological space. Define $S_\tau : \mathbb{N}(X) \times X \longrightarrow L$ by

$$S_\tau(\xi, x) = \bigwedge_{x\in A} \tau(A) \rightarrow \mathcal{F}_\xi(A), \quad \forall(\xi, x) \in \mathbb{N}(X) \times X.$$

Proposition 10 *For an L-fuzzifying topology τ on X, we have*

(1) S_τ is an L-fuzzifying generalized convergence structure on X.

(2) For each $x \in X$, $\mathcal{U}_{S_\tau}^x \geq \mathcal{U}_\tau^x$.

Proof (1) (LC1) is straightforward and (LC2) can be implied by Proposition 3.

(2) For all $x \in A \subseteq X$ and all $x \in B \subseteq A$,

$$\begin{aligned}\mathcal{U}_{S_\tau}^x(A) \geq \mathcal{U}_{S_\tau}^x(B) &= \bigwedge_{\xi\in\mathbb{N}(X)} S_\tau(\xi, x) \rightarrow \mathcal{F}_\xi(B) \\ &= \bigwedge_{\xi\in\mathbb{N}(X)} (\bigwedge_{x\in C} (\tau(C) \rightarrow \mathcal{F}_\xi(C)) \rightarrow \mathcal{F}_\xi(B) \\ &\geq \bigwedge_{\xi\in\mathbb{N}(X)} (\tau(B) \rightarrow \mathcal{F}_\xi(B)) \rightarrow \mathcal{F}_\xi(B) \geq \tau(B).\end{aligned}$$

Hence $\mathcal{U}_{S_\tau}^x \geq \mathcal{U}_\tau^x$. $\square$

Proposition 11 *If $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ is continuous, then so is $f : (X, S_{\tau_1}) \longrightarrow (Y, S_{\tau_2})$.*

Proof For all $(\xi, x) \in \mathbb{N}(X) \times X$,

$$\begin{aligned}S_{\tau_2}(f(\xi), f(x)) &= \bigwedge_{B\ni f(x)} \tau_2(B) \rightarrow \mathcal{F}_{f(\xi)}(B) \\ &\geq \bigwedge_{B\ni f(x)} \tau_1(f^{-1}(B)) \rightarrow \mathcal{F}_\xi(f^{-1}(B))\end{aligned}$$

$$\geq \bigwedge_{A \ni x} \tau_1(A) \to \mathcal{F}_\xi(A) = S_{\tau_1}(\xi, x). \quad \square$$

By Propositions 10 and 11, we obtain a concrete functor S_T from L-FYS to L-FYGConv transferring an L-fuzzifying topology τ to S_τ.

Proposition 12 *Let τ be an L-fuzzifying topology and S be an L-fuzzifying generalized convergence space. Then $S_{\tau_S} \geq S$, $\tau_{S_\tau} \geq \tau$.*

Proof (1) For all $(\xi, x) \in \mathbb{N}(X) \times X$,

$$S_{\tau_S}(\xi, x) = \bigwedge_{x \in A} \tau_S(A) \to \mathcal{F}_\xi(A) \geq \bigwedge_{x \in A} (S(\xi, x) \to \mathcal{F}_\xi(A)) \to \mathcal{F}_\xi(A) \geq S(\xi, x).$$

(2) For all $A \subseteq X$,

$$\tau_{S_\tau}(A) = \bigwedge_{x \in A} \mathcal{U}^x_{S_\tau}(A) \geq \bigwedge_{x \in A} \mathcal{U}^x_\tau(A) \geq \tau(A). \quad \square$$

Theorem 2 *The category L-FYS can be embedded in L-FYGConv as a reflective subcategory.*

4. L-FYGConv is a cartesian-closed topological category

A construct $\mathcal{C}$ over Set (U is the forgetful functor) is called topological [8] if for any U-source $(f_i \longrightarrow (X_i, \xi_i))_{i \in I}$, there exists a unique initial U-lift (X, ξ), that is for any $\mathcal{C}$-object (Y, η), a map $g : (Y, \eta) \longrightarrow (X, \xi)$ is a $\mathcal{C}$-morphism if and only if for any $i \in I$, $f_i \circ g : (Y, \eta) \longrightarrow (X_i, \xi_i)$ is a $\mathcal{C}$-morphism.

A category with finite products is called cartesian-closed [8] if for each pair (A, B) of objects there exists an object $[A \to B]$ and an evaluation morphism $ev : [A \to B] \times A \longrightarrow B$ with the following universal property: for each morphism $f : C \times A \longrightarrow B$ there exists a unique morphism $\hat{f} : C \longrightarrow [A \to B]$ such that $ev \circ (\hat{f} \times \mathrm{id}) = f$.

Theorem 3 *L-FYGConv is a topological category.*

Proof Let U : L-FYGConv$\longrightarrow$ Set be the forgetful functor and $(X, f_i, (X_i, S_i))_{i \in I}$ be a U-source. Define $S : \mathbb{N}(X) \times X \longrightarrow L$ by $S(\xi, x) = \bigwedge_i S_i(f_i(\xi), f_i(x))$. It is routine to show that (X, S) is an L-fuzzifying generalized convergence space.

Suppose that (Y, S_Y) is an L-fuzzifying generalized convergence space. A map $g : (Y, S_Y) \longrightarrow (X, S)$ is an L-FYGconv-morphism if and only if $\forall (\xi, y) \in \mathbb{N}(Y) \times Y$, $S_Y(\xi, y) \leq S(g(\xi), (g(y)))$ if and only if $\forall (\xi, y) \in \mathbb{N}(Y) \times Y$, $\forall i \in I$, $S_Y(\xi, y) \leq S_i((f_i g)(\xi), ((f_i g)(y)))$ if and only if $\forall i \in I$, $f_i g : (Y, S_Y) \longrightarrow (X_i, S_i)$ is an L-FYGConv-morphism. Hence (X, S) is a unique initial U-lift for the given U-source (the uniqueness is obvious). $\square$

Since L-FYGConv is topological, there exist products in L-FYGConv [8]. By Theorem 3, let $\{(X_i, S_i) |\ i \in I\}$ be a nonempty family of L-fuzzifying generalized convergence spaces and $X = \prod_{i \in X} X_i$. For all $(\xi, x) \in \mathbb{N}(X) \times X$, let $S(\xi, x) = \bigwedge_{i \in I} S_i(p_i(\xi), (p_i(x)))$. Then (X, S) is the product of $\{(X_i, S_i) |\ i \in I\}$ in L-FYGConv. For an L-fuzzifying net $\xi = (p, v) : \Delta \longrightarrow X = \prod_i X_i$, we have $p_i \circ \xi = (p_i \circ p, v) : \Delta \longrightarrow X_i$ is an L-fuzzifying net for any $i \in I$.

Proposition 13 (Product of two L-fuzzifying nets) *Let $\xi = (p_1, v_1) : D \longrightarrow X \times L_0$, $\eta = (p_2, v_2) : E \longrightarrow Y \times L_0$ be two L-fuzzifying nets. Define $\xi \times \eta : D \times E \longrightarrow (X \times Y) \times L_0$ by*

$$(\xi \times \eta)(d, e) = ((p_1(d), p_2(e)), v_1(d) \wedge v_2(e)), \quad \forall (d, e) \in D \times E.$$

Then $\xi \times \eta$ is an L-fuzzifying net of $X \times Y$.

Proof Since L is 0-$\wedge$-inaccessible, for any $(d, e) \in D \times E$, $h_1(d) \wedge h_2(e) \in L_0$, $v_1(d) \wedge v_2(e) \in L_0$.

(LN1) $\bigvee_{(d,e)\in D\times E} v_1(d) \wedge v_2(e) = \bigvee_{d\in D} v_1(d) \wedge \bigvee_{e\in E} v_2(e) = 1 \wedge 1 = 1$.

(LN2) For all $(d_1, e_1), (d_2, e_2) \in D \times E$, there exists an upper bound d of d_1, d_2 such that $v_1(d_1) \wedge v_1(d_2) \leq v_1(d)$ and an upper bound e of e_1, e_2 such that $v_2(e_1) \wedge v_2(e_2) \leq v_2(e)$. Then (d, e) is an upper bound of $(d_1, e_1), (d_2, e_2)$ such that $(v_1(d_1) \wedge v_2(e_1)) \wedge (v_1(d_2) \wedge v_2(e_2)) \leq v_1(d) \wedge v_2(e)$. $\square$

Let (X, S_X) and (Y, S_Y) be two L-fuzzifying generalized convergence spaces and $[X \to Y]$ the set of all continuous maps from (X, S_X) to (Y, S_Y). For any $(\xi, f) \in \mathbb{N}([X \to Y]) \times [X \to Y]$, define

$$S_{[X\to Y]}(\xi, f) = \bigwedge_{(\eta,x)\in\mathbb{N}(X)\times X} S_X(\eta, x) \to S_Y(ev(\xi \times \eta), f(x)).$$

Lemma 2 *For all $f \in [X \to Y], \eta \in \mathbb{N}(X), a \in L_0$, $ev(\overline{f} \times \eta)$ is a subnet of $f(\eta)$, where $\overline{f}$ is a constant net on $[X \to Y]$.*

Proof Let $\overline{f} : D \longrightarrow [X \to Y] \times L_0$, $\eta = (p, v) : E \longrightarrow X \times L_0$ be two L-fuzzifying nets. Then the net $ev(\overline{f} \times \eta) : D \times E \longrightarrow Y \times L_0$ is defined by $(d, e) \mapsto (f(\eta(e)), v(e))$ and $[f(\eta)] : E \longrightarrow Y$ by $e \mapsto (f(\eta(e)), v(e))$. Define $j : D \times E \longrightarrow E$ by $(d, e) \mapsto e$. Then $ev(\overline{f} \times \eta)$ is a subnet of $[f(\eta)]$. $\square$

Lemma 3 *If ξ_1, ξ_2 are two L-fuzzifying nets on X and ξ_1 is a subnet of ξ_2. Then for any L-fuzzifying net η on Y and any map $f : X \times Y \longrightarrow Z$, $\xi_1 \times \eta$ is a subnet of $\xi_2 \times \eta$ and consequently $f(\xi_1 \times \eta)$ is a subnet of $f(\xi_2 \times \eta)$.*

Proof Straightforward. $\square$

Proposition 14 *$S_{[X\to Y]}$ is an L-fuzzifying convergence structure on $[X \to Y]$.*

Proof (LC2) can be easily derived by Lemma 3.

(LC1) By Lemma 2,

$$\begin{aligned} S_{[X\to Y]}(\overline{f}, f) &= \bigwedge_{(\eta,x)\in\mathbb{N}(X)\times X} S_X(\eta, x) \to S_Y(ev(\overline{f} \times \eta), f(x)) \\ &\geq \bigwedge_{(\eta,x)\in\mathbb{N}(X)\times X} S_X(\eta, x) \to S_Y(f(\eta), f(x)) \\ &= \bigwedge_{(\eta,x)\in\mathbb{N}(X)\times X} S_X(\eta, x) \to S_Y(f(\eta), f(x)) = 1. \quad \square \end{aligned}$$

Proposition 15 *The evaluation $ev : ([X \to Y], S_{[X\to Y]}) \times (X, S_X) \longrightarrow (Y, S_Y)$ is a continuous map.*

Proof For any $(\xi, f) \in \mathbb{N}([X \to Y]) \times [X \to Y]$ and any $(\eta, x) \in \mathbb{N}(X) \times X$,

$$\begin{aligned} S_{[X\to Y]}(\xi, f) &= \bigwedge_{(\beta,x)\in\mathbb{N}(X)\times X} S_X(\beta, x) \to S_Y(ev(\xi \times \beta), f(x)) \\ &\le S_X(\eta, x) \to S_Y(ev(\xi \times \eta), f(x)). \end{aligned}$$

Then

$$S_{[X\to Y]\times X}((\xi,\eta),(f,x)) = S_{[X\to Y]}(\xi, f) \wedge S_X(\eta, x) \le S_Y(ev(\xi\times\eta), f(x)).$$

That is, ev is continuous. $\Box$

Now let us consider the following situation. Let $f : X \times Y \longrightarrow Z$ be a map. Define for $x \in X$ the map $f_x : Y \longrightarrow Z, y \longmapsto f(x, y)$ and with this the map $f^* : X \longrightarrow Z^Y, x \longmapsto f_x$. The map $\varphi : Z^{X\times Y} \longrightarrow (Z^Y)^X, f \longmapsto f^*$ is called the exponential map.

Lemma 4 *Let $f : X \times Y \longrightarrow Z$ be a map, $\xi = (p, v) : D \longrightarrow Y \times L_0$ an L-fuzzifying net and $\overline{x} : E \longrightarrow X \times L_0$ a constant L-fuzzifying net on X. Then $f_x(\xi)$ is a subnet of $f(\overline{x} \times \xi)$.*

Proof We suppose that E has a top element t. Otherwise, we first do a transformation for $\overline{x}$. Put $E_t = E \cup \{t\}$ such that t is the top of E, then E_t is also a direct set. Define $(\overline{x})^* : E_t \longrightarrow X \times L_0$ by $e \mapsto (x, 1)$ for all $e \in E_t$. Thus $(\overline{x})^*$ is also a constant net on X, which has hardly difference from $\overline{x}$. Now we consider $(\overline{x})^*$ and $\overline{x}$ are the same, that is, E has a top element t.

The net $f(\overline{x} \times \xi) : E \times D \longrightarrow Z \times L_0$ is defined by $f(\overline{x} \times \xi)(e, d) = (f(x, p(d)), v(d))$ and the net $f_x(\xi) : D \longrightarrow Z \times L_0$ is defined by $f_x(\xi)(d) = (f(x, p(d)), v(d))$. Define $j : D \longrightarrow E \times D$ by $j(d) = (t, d)$. Then we have $f_x(\xi) = f(\overline{x} \times \xi) \circ j$, and for any $(e, d) \in E \times D$, $v_{[f_x(\xi)]}(d) = v(d) = v_{f(\overline{x}\times\xi)}$ and $j(d_1) = (t, d_1) \ge (e, d)$ for any $d_1 \ge d$. Hence $f_x(\xi)$ is a subnet of $f(\overline{x} \times \xi)$. $\Box$

Lemma 5 *Let $f : (X, S_X) \times (Y, S_Y) \longrightarrow (Z, S_Z)$ be a continuous map. Then for each $x \in X$, $f_x : (Y, S_Y) \longrightarrow (Z, S_Z)$ is also continuous.*

Proof For any $(\xi, y) \in \mathbb{N}(Y) \times Y$,

$$\begin{aligned} S_Z(f_x(\xi), f_x(y)) &\ge S_Z(f(\overline{x} \times \xi), f(x, y)) \ge S_{X\times Y}(\overline{x} \times \xi, (x, y)) \\ &= S_X(\overline{x}, x) \wedge S_Y(\xi, y) = S_Y(\xi, y). \end{aligned}$$

Thus f_x is continuous. $\Box$

Lemma 6 *For all $\xi \in \mathbb{N}(X)$, $\eta \in \mathbb{N}(Y)$, $f : X \times Y \longrightarrow Z$, we have $ev(\varphi(f)(\xi) \times \eta) = f(\xi \times \eta)$.*

Proof Let $\xi = (p_1, v_1) : D \longrightarrow X \times L_0$ and $\eta = (p_2, v_2) : E \longrightarrow Y \times L_0$ be two L-fuzzifying nets. The net $\varphi(f)(\xi) : D \longrightarrow Z^Y \times L_0$ is defined by $d \mapsto (f_{p_1(d)}, v_1(d))$. And $ev(\varphi(f)(\xi) \times \eta) : D \times E \longrightarrow Z \times L_0$ is defined by $(d, e) \mapsto (ev(f_{p_1(d)}, p_2(e)), v_1(d) \wedge v_2(e))) = (f(p_1(d), p_2(e)), v_1(d) \wedge v_2(e))$. Therefore, $ev(\varphi(f)(\xi) \times \eta) = f(\xi \times \eta)$. $\Box$

Proposition 16 *If the map $f : X \times Y \longrightarrow Z$ is continuous, then so is $\varphi(f) : X \longrightarrow [Y \to Z]$.*

Proof If $f : X \times Y \longrightarrow Z$ is continuous, then by Lemma 5, for any $x \in X$, $\varphi(f)(x) = f_x :$

$Y \longrightarrow Z$ is continuous and then $\varphi(f)$ is a well-defined map. For any $(\xi, x) \in \mathbb{N}(X) \times X$,

$$
\begin{aligned}
& S_{[Y\to Z]}(\varphi(f)(\xi), \varphi(f)(x)) \\
& = \bigwedge_{(\eta,y)\in\mathbb{N}(Y)\times Y} S_Y(\eta, y) \to S_Z(ev(\varphi(f)(\xi)\times\eta), f_x(y)) \\
& = \bigwedge_{(\eta,y)\in\mathbb{N}(Y)\times Y} S_Y(\eta, y) \to S_Z(f(\xi\times\eta), f(x,y)) \\
& \geq \bigwedge_{(\eta,y)\in\mathbb{N}(Y)\times Y} S_Y(\eta, y) \to S_{X\times Y}(\xi\times\eta, (x,y)) \\
& \geq \bigwedge_{(\eta,y)\in\mathbb{N}(Y)\times Y} S_Y(\eta, y) \to (S_X(\xi, x)\wedge S_Y(\eta, y)) \\
& \geq S_X(\xi, x).
\end{aligned}
$$

Hence $S_{[Y\to Z]}(\varphi(f)(\xi), \varphi(f)(x)) \geq S_X(\xi, x)$. Therefore, $\varphi(f)$ is continuous. □

By Propositions 14, 15 and 16, we have

Theorem 4 *L-FYGConv is cartesian-closed.*

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