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Existence of Positive Solutions of Generalized Sturm-Liouville Boundary Value Problems for a Singular Differential Equation

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Abstract By employing the fixed point theorem of cone expansion and compression of norm type, we investigate the existence of positive solutions of generalized Sturm-Liouville boundary value problems for a nonlinear singular differential equation with a parameter. Some sufficient conditions for the existence of positive solutions are established. In the last section, an example is presented to illustrate the applications of our main results.

Keywords generalized Sturm-Liouville boundary value problems; second-order differential equations; positive solutions.

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1. Introduction

In this paper, we consider the following nonlinear singular boundary value problem (BVP for short)

$$\begin{cases} u''(t)+\lambda[f(t,u(t))+q(t)]=0 \quad \text{for} \quad \text{a.e.} \quad t\in(0,1),\\ au(0)-bu'(0)=\sum\limits_{i=1}^{m-2}a_iu(\xi_i), \quad cu(1)+du'(1)=\sum\limits_{i=1}^{m-2}b_iu(\xi_i), \end{cases} \tag{1.1}$$

where λ is a positive parameter, $a\ge 0$, $b\ge 0$, $c\ge 0$, $d\ge 0$ such that $ac+bc+ad>0$, $\xi_i\in(0,1)$, a_i, $b_i\in(0,+\infty)$, $i=1,2,\ldots,m-2$ ($m\in\mathbb{N}$ and $m\ge 3$) are all constants, $f:(0,1)\times[0,\infty)\to[0,\infty)$ is continuous and may be singular at $t=0,1$, $q:(0,1)\to(-\infty,+\infty)$ is Lebesgue integrable and may have finitely many singularities in $[0,1]$. The precise meaning of singularity is given at the end of this section.

Il'in and Mosiseev [1] studied the existence of solutions for a linear multi-point boundary value problem. Motivated by the study of Il'in and Mosiseev [1], Gupta [2] studied certain

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three-point boundary value problems for nonlinear ordinary differential equations. Since then more general nonlinear multi-point boundary value problems have been widely studied by many authors (see [3–10] and some references therein) because multi-point boundary value problems describe many phenomena of applied mathematics and physics.

In recent years, many authors have studied nonlinear differential equations with Sturm-Liouville boundary value conditions or generalized Sturm-Liouville ones [7–11]. Especially Zhang [10] studied the following generalized Sturm–Liouville boundary value problem

$$\begin{cases} u''(t)+h(t)f(t,u(t),u'(t))=0, \quad t\in(0,1),\\ au(0)-bu'(0)=\sum\limits_{i=1}^{m-2}a_iu(\xi_i), \quad cu(1)+du'(1)=\sum\limits_{i=1}^{m-2}b_iu(\xi_i) \end{cases}$$

by applying the fixed point theorem due to Avery and Peterson. However the nonlinear term f is nonsingular in [10]. As far as we know, the BVP (1.1) is seldom investigated. Inspired by [8–10], our aim in the present paper is to establish the range of λ, for which there exists at least one positive solution for the BVP (1.1). In particular, we shall use the fixed point theorem of cone expansion and compression of norm type to prove our main result. In the last section, an example is presented to illustrate the applications of our main results.

By singularity we mean that the functions $f(t,u)$ and $q(t)$ in (1.1) are allowed to be unbounded at some points. In this paper, the function $q(t)$ is allowed to have finitely many singularities in [0, 1] and to change sign and tend to negative infinity. We call $u(t)\in C^1[0,1]\cap C^2(0,1)$ for a.e. $t\in[0,1]$ if $u(t)\in C^1[0,1]$ and $u''(t)\in C(0,1)$ for a.e. $t\in(0,1)$, where $u(t)\in C^1[0,1]$ means that $u(t)$ is first-order continuously differentiable on $[0,1]$, and $u''(t)\in C(0,1)$ for a.e. $t\in(0,1)$ means that there is a subset $Z(\subset(0,1))$ of Lebesgue measure 0 such that $u(t)$ is twice continuously differentiable on $(0,1)\backslash Z$. A function $u(t)\in C^1[0,1]\cap C^2(0,1)$ for a.e. $t\in[0,1]$ is called a positive solution of the BVP (1.1) if it satisfies the BVP (1.1) and $u(t)\geq 0$ for any $t\in[0,1]$.

2. Preliminaries and several important lemmas

Let $E=C[0,1]$ be equipped with norm $\|u\|=\max_{t\in[0,1]}|u(t)|$. Then $(E,\|\cdot\|)$ is a real Banach space. For convenience of readers, we provide some background materials in a real Banach space E.

Definition 2.1 (see Definition 1.1.1 in [12]) *Let E be a real Banach space. A nonempty convex closed set $P\subset E$ is called a cone if it satisfies the following two conditions:*

(i) $x\in P$, $\alpha\geq 0$ implies $\alpha x\in P$;

(ii) $x\in P$, $-x\in P$ implies $x=0$, where 0 denotes the zero element of E.

Definition 2.2 (see Definition 2.1.1 in [12]) *An operator is said to be completely continuous if it is continuous and compact.*

In this paper, we make the following assumptions:

(H_1) $f:(0,1)\times[0,+\infty)\to[0,+\infty)$ is continuous and there exist constants γ, μ, $0<\gamma<$

$\mu < +\infty$ *such that*

$$\delta^\mu f(t,u) \le f(t,\delta u) \le \delta^\gamma f(t,u) \quad \text{for any} \quad (t,u) \in (0,1)\times[0,\infty) \quad \text{and} \quad \delta \in [0,1]; \tag{2.1}$$

(H_2) λ *is a positive parameter,* $q:(0,1)\to(-\infty,+\infty)$ *is Lebesgue integrable such that*

$$0 < \int_0^1 q_-(s)\mathrm{d}s = r_1 < +\infty \quad \text{and} \quad 0 < \int_0^1 G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s = r_2 < +\infty,$$

where $q_+(s) = \max\{q(s),0\}$, $q_-(s) = \max\{-q(s),0\}$;

(H_3) $a \ge 0$, $b \ge 0$, $c \ge 0$, $d \ge 0$, $\rho = ac+bc+ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0,+\infty)$, $i = 1,2,\dots,m-2$ ($m \in \mathbb{N}$ *and* $m \ge 3$), $\rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0$, $\Delta < 0$, *where*

$$\Delta = \begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i\psi(\xi_i) & \rho - \sum\limits_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \rho - \sum\limits_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum\limits_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix}$$

and

$$\psi(t) = b + at, \quad \varphi(t) = d + c(1-t), \quad t \in [0,1].$$

Obviously ψ *is non-decreasing on* $[0,1]$ *and* φ *is non-increasing on* $[0,1]$.

Remark 2.1 The inequality (2.1) is equivalent to the following one

$$\delta^\gamma f(t,u) \le f(t,\delta u) \le \delta^\mu f(t,u) \quad \text{for any} \quad (t,u) \in (0,1)\times[0,\infty) \quad \text{and} \quad \delta \in [1,+\infty). \tag{2.2}$$

Remark 2.2 Typical functions that satisfy the above hypothesis of (H_1) are those taking the form

$$f(t,u) = \sum_{i=1}^n p_i(t)u^{l_i},$$

where $p_i(t) \in C(0,1)$, $p_i(t) > 0$ for $t \in (0,1)$, $0 < l_i < +\infty$, $i = 1,2,\dots,n$, $n \in \mathbb{N}$.

Remark 2.3 It is clear that a function q satisfying the following conditions also satisfies (H_2). For given points $t_1, t_2, \dots, t_j$, $q(t) \to \infty$ ($t \to t_i$), $i = 1,2,\dots,j$. Thus q can have finitely many singularities.

Lemma 2.1 (see Lemma 2.1 in [10] or Lemma 5.5.1 in [13]) *If (H_3) holds, then for* $y \in C[0,1]$, *the BVP*

$$\begin{cases} u''(t) + y(t) = 0, \quad 0 < t < 1, \\ au(0) - bu'(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + du'(1) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i) \end{cases} \tag{$\divideontimes$}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s + A(y)\psi(t) + B(y)\varphi(t),$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} \psi(s)\varphi(t), \quad 0 \le s \le t \le 1, \\ \psi(t)\varphi(s), \quad 0 \le t \le s \le 1, \end{cases} \tag{2.3}$$

$$A(y)=\frac{1}{\Delta}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)y(s)\mathrm{d}s & \rho-\sum\limits_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum\limits_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)y(s)\mathrm{d}s & -\sum\limits_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix}, \tag{2.4}$$

$$B(y)=\frac{1}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)y(s)\mathrm{d}s \\ \rho-\sum\limits_{i=1}^{m-2} b_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)y(s)\mathrm{d}s \end{vmatrix}. \tag{2.5}$$

Remark 2.4 Obviously $A(y)$ and $B(y)$ are nonnegative and nondecreasing in y if $y(t)\ge 0$ for $t\in[0,1]$ and (H_3) holds. Thus the unique solution of the BVP ($\circledast$) is nonnegative if $y(t)\ge 0$ for $t\in[0,1]$ in Lemma 2.1.

For convenience, throughout this paper, we set

$$\beta=\min\left\{\frac{\psi(\theta)}{\psi(1)},\ \frac{\varphi(\vartheta)}{\varphi(0)}\right\},\quad V=\max\{G(t,s)|\ 0\le t\le 1,\ 0\le s\le 1\},$$

$$I=\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i & \rho-\sum\limits_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum\limits_{i=1}^{m-2} b_i & -\sum\limits_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix},\quad J=\begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} a_i \\ \rho-\sum\limits_{i=1}^{m-2} b_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \end{vmatrix},$$

$$M=Vr_1+\frac{\psi(1)r_1VI}{\Delta}+\frac{\varphi(0)r_1VJ}{\Delta},$$

where $0<\theta<\vartheta<1$ are given constants, r_1 is defined in (H_2). It is obvious that $I<0$, $J<0$ and $M>0$ if (H_3) holds.

Let

$$P=\{u\in E\mid u(t)\ge 0,\ \min_{t\in[\theta,\vartheta]}u(t)\ge\beta\|u\|\}.$$

Then it is clear that P is a cone of E.

Proposition 2.1 *For $t,s\in[0,1]$, we have*

$$0\le G(t,s)\le G(s,s). \tag{2.6}$$

Proof By the monotonicity of φ and ψ, it is evident that (2.6) holds. □

Proposition 2.2 *For $t\in[\theta,\vartheta]$, we have*

$$G(t,s)\ge\beta G(s,s),\quad s\in[0,1]. \tag{2.7}$$

Proof For $t\in[\theta,\vartheta]$ and $s\in(0,1)$, by (2.3), we obtain

$$\frac{G(t,s)}{G(s,s)}\ge\min\left\{\frac{\psi(\theta)}{\psi(s)},\frac{\varphi(\vartheta)}{\varphi(s)}\right\}\ge\min\left\{\frac{\psi(\theta)}{\psi(1)},\frac{\varphi(\vartheta)}{\varphi(0)}\right\}=\beta.$$

If $s=0$ and $t\in[\theta,\vartheta]$, by (2.3), we have

$$G(t,0)=\frac{\psi(0)\varphi(t)}{\rho}\ge\frac{1}{\rho}\cdot\frac{\varphi(\vartheta)}{\varphi(0)}\cdot\psi(0)\varphi(0)=\frac{\varphi(\vartheta)}{\varphi(0)}\cdot G(0,0).$$

If $s=1$ and $t \in [\theta, \vartheta]$, by (2.3), we get

$$G(t,1) = \frac{\psi(t)\varphi(1)}{\rho} \geq \frac{1}{\rho} \cdot \frac{\psi(\theta)}{\psi(1)} \cdot \psi(1)\varphi(1) = \frac{\psi(\theta)}{\psi(1)} \cdot G(1,1).$$

Therefore, (2.7) holds. This completes the proof. □

Let

$$w(t) = \int_0^1 \lambda G(t,s) q_-(s) \mathrm{d}s + \lambda[\mathrm{A}(\mathrm{q}_-)\psi(\mathrm{t}) + \mathrm{B}(\mathrm{q}_-)\varphi(\mathrm{t})],$$

where $G(t,s), A(q_-)$ and $B(q_-)$ are defined by (2.3)–(2.5), respectively. Obviously $w(t)$ is continuous on $[0,1]$. According to (H_2), we obtain

$$\begin{aligned} w(t) &= \int_0^1 \lambda G(t,s) q_-(s) \mathrm{d}s + \lambda[A(q_-)\psi(t) + B(q_-)\varphi(t)] \\ &\leq \lambda \int_0^1 V q_-(s) \mathrm{d}s + \frac{\lambda\psi(1)}{\Delta} \begin{vmatrix} Vr_1 \sum\limits_{i=1}^{m-2} a_i & \rho - \sum\limits_{i=1}^{m-2} a_i \varphi(\xi_i) \\ Vr_1 \sum\limits_{i=1}^{m-2} b_i & -\sum\limits_{i=1}^{m-2} b_i \varphi(\xi_i) \end{vmatrix} + \\ &\qquad \frac{\lambda\varphi(0)}{\Delta} \begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i \psi(\xi_i) & Vr_1 \sum\limits_{i=1}^{m-2} a_i \\ \rho - \sum\limits_{i=1}^{m-2} b_i \psi(\xi_i) & Vr_1 \sum\limits_{i=1}^{m-2} b_i \end{vmatrix} \\ &= \lambda V r_1 + \frac{\lambda\psi(1) r_1 V I}{\Delta} + \frac{\lambda\varphi(0) r_1 V J}{\Delta} = \lambda M < +\infty, \end{aligned} \tag{2.8}$$

so $w(t)$ is well defined in E. By direct computation, we have

$$\begin{cases} w''(t) + \lambda q_-(t) = 0 \quad \text{for} \quad \text{a.e.} \quad t \in (0,1), \\ aw(0) - bw'(0) = \sum\limits_{i=1}^{m-2} a_i w(\xi_i), \quad cw(1) + dw'(1) = \sum\limits_{i=1}^{m-2} b_i w(\xi_i), \end{cases}$$

which implies that $w(t)$ is a positive solution of the following boundary value problem

$$\begin{cases} u''(t) + \lambda q_-(t) = 0 \quad \text{for} \quad \text{a.e.} \quad t \in (0,1), \\ au(0) - bu'(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + du'(1) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i). \end{cases}$$

For any $u(t) \in C[0,1]$, let us define a function $[\cdot]^*$ by

$$[u(t)]^* = \begin{cases} u(t), & u(t) \geq 0, \\ 0, & u(t) < 0. \end{cases}$$

Now we consider the following BVP

$$\begin{cases} u''(t) + \lambda[f(t,[u(t) - w(t)]^*) + q_+(t)] = 0 \quad \text{for} \quad \text{a.e.} \quad t \in (0,1), \\ au(0) - bu'(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + du'(1) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i). \end{cases} \tag{2.9}$$

By Lemma 2.1, a function $u(t) \in C^1[0,1] \cap C^2(0,1)$ for a.e. $t \in [0,1]$ is a solution of the BVP (2.9) if and only if $u(t)$ is a solution of the following nonlinear integral equation

$$u(t) = \int_0^1 \lambda G(t,s)[f(s,[u(s)-w(s)]^*) + q_+(s)]\mathrm{d}s + \lambda[A(\widehat{f}+q_+)\psi(t) + B(\widehat{f}+q_+)\varphi(t)], \quad t \in [0,1],$$

where $\widehat{f}$ denotes $f(s,[u(s)-w(s)]^*)$, $G(t,s)$, $A(\widehat{f}+q_+)$ and $B(\widehat{f}+q_+)$ are defined by (2.3)–(2.5), respectively.

Let

$$(Tu)(t) = \int_0^1 \lambda G(t,s)[f(s,[u(s)-w(s)]^*) + q_+(s)]\mathrm{d}s + \lambda[A(\widehat{f}+q_+)\psi(t) + B(\widehat{f}+q_+)\varphi(t)], \quad t \in [0,1]. \tag{2.10}$$

Obviously the existence of solutions of the BVP (2.9) is equivalent to the existence of fixed points of the operator T in the real Banach space E.

Lemma 2.2 *Suppose that (H_1) holds, then $f(t,u)$ is nondecreasing on $u \in [0,+\infty)$ for any fixed $t \in (0,1)$.*

Proof For any fixed $t \in (0,1)$ and any $u_1, u_2 \in [0,+\infty)$, without loss of generality, let $0 \le u_1 \le u_2$. If $u_2 = 0$, obviously equations $f(t,u_1) = f(t,u_2) = f(t,0)$ hold. If $u_2 \neq 0$, let $\delta_0 = \frac{u_1}{u_2}$. Then we obtain $0 \le \delta_0 \le 1$. It follows from (2.1) that

$$f(t,u_1) = f(t,\delta_0 u_2) \le \delta_0^\gamma f(t,u_2) \le f(t,u_2),$$

i.e., $f(t,u)$ is nondecreasing on $u \in [0,+\infty)$ for any fixed $t \in (0,1)$. This proves Lemma 2.2. $\square$

Lemma 2.3 *Assume that (H_2) and (H_3) hold. If $x(t)$ with $x(t) \ge w(t)$ is a positive solution of the BVP (2.9), then $x(t) - w(t)$ is a positive solution of the BVP (1.1).*

Proof Suppose that $x(t)$ is a positive solution of the BVP (2.9) such that $x(t) \ge w(t)$, then from (2.9) and the definition of $[\cdot]^*$, we have

$$\begin{cases} x''(t) + \lambda\{f(t,[x(t)-w(t)]) + q_+(t)\} = 0 \quad \text{for} \quad \text{a.e.} \quad t \in (0,1), \\ ax(0) - bx'(0) = \sum\limits_{i=1}^{m-2} a_i x(\xi_i), \quad cx(1) + dx'(1) = \sum\limits_{i=1}^{m-2} b_i x(\xi_i). \end{cases} \tag{2.11}$$

Let $u(t) = x(t) - w(t)$. Then $u''(t) = x''(t) - w''(t)$ for a.e. $t \in (0,1)$, which implies that

$$x''(t) = u''(t) - \lambda q_-(t) \quad \text{for a.e. } t \in (0,1).$$

Thus (2.11) becomes

$$\begin{cases} u''(t) + \lambda[f(t,u(t)) + q_+(t) - q_-(t)] = 0 \quad \text{for} \quad \text{a.e.} \quad t \in (0,1), \\ au(0) - bu'(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + du'(1) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i). \end{cases} \tag{2.12}$$

Noticing $q(t) = q_+(t) - q_-(t)$ and (2.12), we know that $u(t)$ is a positive solution of the BVP (1.1), i.e., $x(t) - w(t)$ is a positive solution of the BVP (1.1). This proves Lemma 2.3. $\square$

Lemma 2.4 *Assume that (H_1)–(H_3) hold. Then the operator $T : P \to P$ is well defined and*

$T: P \to P$ is a completely continuous operator.

Proof For any fixed $u \in P$, choose $0 < \eta < 1$ such that $\eta\|u\| < 1$, then we obtain $\eta[u(t)-w(t)]^* \leq \eta u(t) \leq \eta\|u\| < 1$. Thus by (2.1)–(2.2) and Lemma 2.2, we have

$$f(t,[u(t)-w(t)]^*) \leq (\frac{1}{\eta})^{\mu} f(t,\eta[u(t)-w(t)]^*) \leq \eta^{-\mu} f(t,\eta\|u\|) \leq \eta^{\gamma-\mu}\|u\|^{\gamma} f(t,1). \tag{2.13}$$

Hence for any $t \in [0,1]$, by (2.6), (2.10), (2.13) and Lemma 2.2, we get

$$\begin{aligned}
(Tu)(t) &= \int_0^1 \lambda G(t,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s + \lambda[A(\widehat{f}+q_+)\psi(t)+B(\widehat{f}+q_+)\varphi(t)] \\
&\leq \lambda \int_0^1 G(s,s)[\eta^{\gamma-\mu}\|u\|^{\gamma} f(s,1)+q_+(s)]\mathrm{d}s + \lambda\psi(1)\cdot A(\widehat{f}+q_+) + \lambda\varphi(0)\cdot B(\widehat{f}+q_+) \\
&\leq \lambda K \int_0^1 G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s + \lambda\psi(1)\cdot A(\widehat{f}+q_+) + \lambda\varphi(0)\cdot B(\widehat{f}+q_+) \\
&\leq \lambda K r_2 + \frac{\lambda\psi(1)}{\Delta}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)[\eta^{\gamma-\mu}\|u\|^{\gamma} f(s,1)+q_+(s)]\mathrm{d}s & \rho-\sum\limits_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum\limits_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)[\eta^{\gamma-\mu}\|u\|^{\gamma} f(s,1)+q_+(s)]\mathrm{d}s & -\sum\limits_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix} + \\
&\qquad \frac{\lambda\varphi(0)}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)[\eta^{\gamma-\mu}\|u\|^{\gamma} f(s,1)+q_+(s)]\mathrm{d}s \\ \rho-\sum\limits_{i=1}^{m-2} b_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)[\eta^{\gamma-\mu}\|u\|^{\gamma} f(s,1)+q_+(s)]\mathrm{d}s \end{vmatrix} \\
&\leq \lambda K r_2 + \frac{\lambda\psi(1)}{\Delta}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i \int_0^1 KG(s,s)[f(s,1)+q_+(s)]\mathrm{d}s & \rho-\sum\limits_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum\limits_{i=1}^{m-2} b_i \int_0^1 KG(s,s)[f(s,1)+q_+(s)]\mathrm{d}s & -\sum\limits_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix} + \\
&\qquad \frac{\lambda\varphi(0)}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} a_i \int_0^1 KG(s,s)[f(s,1)+q_+(s)]\mathrm{d}s \\ \rho-\sum\limits_{i=1}^{m-2} b_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \int_0^1 KG(s,s)[f(s,1)+q_+(s)]\mathrm{d}s \end{vmatrix} \\
&= \lambda K r_2 + \frac{\lambda\psi(1)}{\Delta}\begin{vmatrix} Kr_2\sum\limits_{i=1}^{m-2} a_i & \rho-\sum\limits_{i=1}^{m-2} a_i\varphi(\xi_i) \\ Kr_2\sum\limits_{i=1}^{m-2} b_i & -\sum\limits_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix} + \\
&\qquad \frac{\lambda\varphi(0)}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i\psi(\xi_i) & Kr_2\sum\limits_{i=1}^{m-2} a_i \\ \rho-\sum\limits_{i=1}^{m-2} b_i\psi(\xi_i) & Kr_2\sum\limits_{i=1}^{m-2} b_i \end{vmatrix} \\
&= \lambda K r_2 + \frac{\lambda\psi(1)r_2KI}{\Delta} + \frac{\lambda\varphi(0)r_2KJ}{\Delta} < +\infty,
\end{aligned} \tag{2.14}$$

where $K=\eta^{\gamma-\mu}\|u\|^{\gamma}+1$. Thus $T:P\to E$ is well defined. Next for any $u\in P$ and $t\in[0,1]$, by (2.6) and (2.10), we obtain

$$\begin{aligned}(Tu)(t)&=\int_0^1\lambda G(t,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\lambda[A(\widehat{f}+q_+)\psi(t)+B(\widehat{f}+q_+)\varphi(t)]\\&\le\int_0^1\lambda G(s,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\lambda\psi(1)\cdot A(\widehat{f}+q_+)+\lambda\varphi(0)\cdot B(\widehat{f}+q_+).\end{aligned}$$

Then we have

$$\|Tu\|\le\int_0^1\lambda G(s,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\lambda\psi(1)\cdot A(\widehat{f}+q_+)+\lambda\varphi(0)\cdot B(\widehat{f}+q_+).\quad(2.15)$$

Thus for any $u\in P$ and $t\in[\theta,\vartheta]$, by (2.7), (2.10) and (2.15), we get

$$\begin{aligned}&(Tu)(t)\\&=\int_0^1\lambda G(t,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\lambda[A(\widehat{f}+q_+)\psi(t)+B(\widehat{f}+q_+)\varphi(t)]\\&\ge\int_0^1\beta\lambda G(s,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\lambda\psi(\theta)\cdot A(\widehat{f}+q_+)+\lambda\varphi(\vartheta)\cdot B(\widehat{f}+q_+)\\&=\int_0^1\beta\lambda G(s,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\frac{\psi(\theta)}{\psi(1)}\cdot\lambda\psi(1)A(\widehat{f}+q_+)+\\&\quad\frac{\varphi(\vartheta)}{\varphi(0)}\cdot\lambda\varphi(0)B(\widehat{f}+q_+)\\&\ge\beta\Big\{\int_0^1\lambda G(s,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\lambda\psi(1)\cdot A(\widehat{f}+q_+)+\lambda\varphi(0)\cdot B(\widehat{f}+q_+)\Big\}\\&\ge\beta\|Tu\|.\end{aligned}$$

This implies that $T:P\to P$ is well defined.

Let $D\subset P$ be any bounded set. Then there exists a constant $L>0$ such that $\|x\|\le L$ for any $x\in D$. Thus for any $x\in D$ and $s\in[0,1]$, we have

$$[x(s)-w(s)]^*\le x(s)\le\|x\|\le L\le L+1.\quad(2.16)$$

By (2.2), (2.16) and Lemma 2.2, for any $x\in D$ and $s\in[0,1]$, we obtain that

$$f(s,[x(s)-w(s)]^*)\le f(s,L+1)\le(L+1)^{\mu}f(s,1).\quad(2.17)$$

From (2.6), (2.10), (2.17), (H_2) and Lemma 2.2, proceeding similarly to the above (2.14), we can have

$$\begin{aligned}(Tx)(t)&=\int_0^1\lambda G(t,s)[f(s,[x(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+\lambda[A(\widehat{f}+q_+)\psi(t)+B(\widehat{f}+q_+)\varphi(t)]\\&\le\lambda r_2[(L+1)^{\mu}+1]\Big(1+\frac{\psi(1)I}{\Delta}+\frac{\varphi(0)J}{\Delta}\Big)<+\infty\quad\text{for any}\quad x\in D.\end{aligned}$$

Therefore, $T(D)$ is uniformly bounded.

Next we shall show that $T(D)$ is equicontinuous on $[0,1]$. For any $x\in D$ and $t\in(0,1)$, by (2.10), (2.17) and Lemma 2.2, we obtain

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}(Tx)(t)\Big|$$

$$
\begin{aligned}
&= \lambda \Big| -\frac{c}{\rho}\int_0^t (b+as)[f(s,[x(s)-w(s)]^*)+q_+(s)]\mathrm{d}s+ \\
&\quad \frac{a}{\rho}\int_t^1 [d+c(1-s)][f(s,[x(s)-w(s)]^*)+q_+(s)]\mathrm{d}s + a\cdot A(\widehat{f}+q_+) - c\cdot B(\widehat{f}+q_+)\Big| \\
&\le \lambda[(L+1)^\mu+1]\Big\{\int_0^t \frac{c}{\rho}(b+as)[f(s,1)+q_+(s)]\mathrm{d}s+ \\
&\quad \int_t^1 \frac{a}{\rho}[d+c(1-s)][f(s,1)+q_+(s)]\mathrm{d}s\Big\} + a\cdot\lambda\cdot A(\widehat{f}+q_+) + c\cdot\lambda\cdot B(\widehat{f}+q_+). \qquad (2.18)
\end{aligned}
$$

Exchanging the integral order and combining with (H_2), we have

$$
\begin{aligned}
&\int_0^1 \Big\{\int_0^t \frac{c}{\rho}(b+as)[f(s,1)+q_+(s)]\mathrm{d}s + \int_t^1 \frac{a}{\rho}[d+c(1-s)][f(s,1)+q_+(s)]\mathrm{d}s\Big\}\mathrm{d}t \\
&= \int_0^1 \mathrm{d}s\int_s^1 \frac{c}{\rho}(b+as)[f(s,1)+q_+(s)]\mathrm{d}t + \int_0^1 \mathrm{d}s\int_0^s \frac{a}{\rho}[d+c(1-s)][f(s,1)+q_+(s)]\mathrm{d}t \\
&= \int_0^1 \frac{c(1-s)(b+as)}{\rho}\cdot[f(s,1)+q_+(s)]\mathrm{d}s + \int_0^1 \frac{as[d+c(1-s)]}{\rho}\cdot[f(s,1)+q_+(s)]\mathrm{d}s \\
&\le \int_0^1 \frac{[d+c(1-s)](b+as)}{\rho}\cdot[f(s,1)+q_+(s)]\mathrm{d}s+ \\
&\quad \int_0^1 \frac{(b+as)[d+c(1-s)]}{\rho}\cdot[f(s,1)+q_+(s)]\mathrm{d}s \\
&= 2\int_0^1 G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s = 2r_2 < +\infty. \qquad (2.19)
\end{aligned}
$$

Thus for any $x\in D$, by (2.18) and (2.19), we obtain that

$$
\begin{aligned}
&\int_0^1 \Big|\frac{\mathrm{d}}{\mathrm{d}t}(Tx)(t)\Big|\mathrm{d}t \\
&\quad \le 2\lambda[(L+1)^\mu+1]\int_0^1 G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s + a\cdot\lambda A(\widehat{f}+q_+) + c\cdot\lambda B(\widehat{f}+q_+) \\
&\quad \le 2\lambda r_2[(L+1)^\mu+1]\Big(1+\frac{aI}{2\Delta}+\frac{cJ}{2\Delta}\Big) < +\infty.
\end{aligned}
$$

From the absolute continuity of integral, we know $T(D)$ is equicontinuous on $[0,1]$. Thus according to the Ascoli-Arzela Theorem, $T(D)$ is a relatively compact set.

At the end, from the continuity of f, it is easy to check that $T: P\to P$ is continuous. Therefore, $T: P\to P$ is a completely continuous operator. This completes the proof of Lemma 2.4. □

The following theorem plays an important role in proving our main results.

Theorem 2.1 (see Theorem 2.3.4 in [12]) *Let K be a cone in real Banach space X. Let Ω_1 and Ω_2 be two bounded open subsets in X such that $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let operator $A: K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be completely continuous. Suppose that one of two conditions*

(i) $\|Au\|_X\le\|u\|_X$, $\forall u\in K\cap\partial\Omega_1$ and $\|Au\|_X\ge\|u\|_X$, $\forall u\in K\cap\partial\Omega_2$;

(ii) $\|Au\|_X\ge\|u\|_X$, $\forall u\in K\cap\partial\Omega_1$ and $\|Au\|_X\le\|u\|_X$, $\forall u\in K\cap\partial\Omega_2$

is satisfied. Then A has at least one fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$. Here 0 denotes the zero element of X, and $\|v\|_X$ denotes the norm of element v in X.

3. Main results

In this section, we give our main results and an example to demonstrate their applications.

Theorem 3.1 *Suppose that (H_1)–(H_3) are satisfied. Assume that there exists a constant Υ satisfying*

$$\Upsilon \geq \Big[\frac{\beta\lambda(2-\lambda)}{2}\int_\theta^\vartheta G(\theta,s)\mathrm{d}s\Big]^{-1}$$

such that

$$\min_{t\in[\theta,\vartheta]}\frac{f(t,u)}{u} \geq \Upsilon \quad \text{for} \quad u \geq M. \tag{3.1}$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \in (0,\lambda_0]$, the BVP (1.1) has at least one positive solution $u^* \in P$, where λ_0 satisfies

$$\lambda_0 = \min\Big\{1,\ \frac{Vr_1}{\beta\cdot r_2\cdot[(\max\{M/\beta,1\})^\mu+1]}\Big\},$$

here r_1 and r_2 are defined in (H_2).

Proof For any $l > 0$, we set

$$\Omega_l := \{u \in P : \|u\| < l\}, \quad \partial\Omega_l := \{u \in P : \|u\| = l\}.$$

Let

$$r = \frac{M}{\beta}, \quad \lambda_0 = \min\Big\{1,\ \frac{Vr_1}{\beta\cdot r_2\cdot[(\max\{M/\beta,1\})^\mu+1]}\Big\},$$

where r_1 and r_2 are defined in (H_2). Since $u(t) \geq \beta\|u\| = \beta r$ for any $u \in \partial\Omega_r$, by (2.8), we have

$$u(t) - w(t) \geq \beta r - \lambda M = M - \lambda M \geq M(1-\lambda_0) \geq 0 \quad \text{for any} \quad u \in \partial\Omega_r \quad \text{and} \quad \lambda \in (0,\lambda_0].$$

Noting that $0 \leq u(s) - w(s) \leq u(s) \leq \|u\| = r \leq \max\{r,1\}$, by (2.2) and Lemma 2.2, we get

$$f(s,[u(s)-w(s)]^*) \leq f(s,\max\{r,1\}) \leq (\max\{r,1\})^\mu f(s,1) \quad \text{for any} \quad u \in \partial\Omega_r. \tag{3.2}$$

Hence for any $t \in [0,1]$, $u \in \partial\Omega_r$ and $\lambda \in (0,\lambda_0]$, by (2.6), (3.2) and Lemma 2.2, we obtain

$$\begin{aligned}
&(Tu)(t)\\
&= \int_0^1 \lambda G(t,s)[f(s,[u(s)-w(s)]^*) + q_+(s)]\mathrm{d}s + \lambda[A(\widehat{f}+q_+)\psi(t) + B(\widehat{f}+q_+)\varphi(t)]\\
&\leq \lambda_0\int_0^1 G(s,s)[(\max\{r,1\})^\mu f(s,1) + q_+(s)]\mathrm{d}s + \lambda_0\psi(1)\cdot A(\widehat{f}+q_+) + \lambda_0\varphi(0)\cdot B(\widehat{f}+q_+)\\
&\leq \lambda_0 K_0\int_0^1 G(s,s)[f(s,1) + q_+(s)]\mathrm{d}s + \lambda_0\psi(1)\cdot A(\widehat{f}+q_+) + \lambda_0\varphi(0)\cdot B(\widehat{f}+q_+)
\end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_0 K_0 r_2 + \frac{\lambda_0\psi(1)}{\Delta}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)[(\max\{r,1\})^\mu f(s,1)+q_+(s)]\mathrm{d}s & \rho-\sum\limits_{i=1}^{m-2}a_i\varphi(\xi_i) \\ \sum\limits_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)[(\max\{r,1\})^\mu f(s,1)+q_+(s)]\mathrm{d}s & -\sum\limits_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix}+ \\
&\quad \frac{\lambda_0\varphi(0)}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2}a_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)[(\max\{r,1\})^\mu f(s,1)+q_+(s)]\mathrm{d}s \\ \rho-\sum\limits_{i=1}^{m-2}b_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)[(\max\{r,1\})^\mu f(s,1)+q_+(s)]\mathrm{d}s\end{vmatrix} \\
&\leq \lambda_0 K_0 r_2 + \frac{\lambda_0\psi(1)}{\Delta}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i \int_0^1 K_0G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s & \rho-\sum\limits_{i=1}^{m-2}a_i\varphi(\xi_i) \\ \sum\limits_{i=1}^{m-2} b_i \int_0^1 K_0G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s & -\sum\limits_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix}+ \\
&\quad \frac{\lambda_0\varphi(0)}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2}a_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} a_i \int_0^1 K_0G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s \\ \rho-\sum\limits_{i=1}^{m-2}b_i\psi(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \int_0^1 K_0G(s,s)[f(s,1)+q_+(s)]\mathrm{d}s\end{vmatrix} \\
&= \lambda_0 K_0 r_2 + \frac{\lambda_0\psi(1)}{\Delta}\begin{vmatrix} K_0r_2\sum\limits_{i=1}^{m-2} a_i & \rho-\sum\limits_{i=1}^{m-2}a_i\varphi(\xi_i) \\ K_0r_2\sum\limits_{i=1}^{m-2} b_i & -\sum\limits_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix}+ \\
&\quad \frac{\lambda_0\varphi(0)}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2}a_i\psi(\xi_i) & K_0r_2\sum\limits_{i=1}^{m-2} a_i \\ \rho-\sum\limits_{i=1}^{m-2}b_i\psi(\xi_i) & K_0r_2\sum\limits_{i=1}^{m-2} b_i\end{vmatrix} \\
&= \lambda_0 K_0 r_2 + \frac{\lambda_0\psi(1)r_2K_0I}{\Delta}+\frac{\lambda_0\varphi(0)r_2K_0J}{\Delta} = r_2K_0\lambda_0\Big(1+\frac{\psi(1)I}{\Delta}+\frac{\varphi(0)J}{\Delta}\Big) \\
&\leq r_2K_0\Big(1+\frac{\psi(1)I}{\Delta}+\frac{\varphi(0)J}{\Delta}\Big)\cdot\frac{Vr_1}{\beta\cdot r_2\cdot[(\max\{M/\beta,1\})^\mu+1]} \\
&= \frac{1}{\beta}\cdot\Big(Vr_1+\frac{Vr_1\psi(1)I}{\Delta}+\frac{Vr_1\varphi(0)J}{\Delta}\Big)=\frac{M}{\beta}=r=\|u\|,
\end{aligned}
$$

where $K_0 = (\max\{M/\beta, 1\})^\mu + 1$. Thus for any $\lambda \in (0, \lambda_0]$, we have

$$\|Tu\| \leq \|u\| \quad \text{for any} \quad u \in \partial\Omega_r. \tag{3.3}$$

Let $R > 2r$. Then $R > \frac{2M}{\beta}$ and $M < \frac{\beta R}{2}$. For any $s \in [\theta, \vartheta]$, $u \in \partial\Omega_R$ and $\lambda \in (0, \lambda_0]$, by (2.8), we have

$$u(s) - w(s) \geq \beta R - \lambda M > 2M - \lambda M = (2-\lambda)M > M \tag{3.4}$$

and

$$u(s) - w(s) \geq \beta R - \lambda M \geq \beta R - \frac{\lambda\beta R}{2} = \frac{(2-\lambda)\beta R}{2}. \tag{3.5}$$

Hence for any $u \in \partial\Omega_R$ and $\lambda \in (0, \lambda_0]$, by (3.1) and (3.4)–(3.5), we obtain

$$\begin{aligned}(Tu)(\theta) &= \int_0^1 \lambda G(\theta,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s + \lambda[A(\widehat{f}+q_+)\psi(\theta)+B(\widehat{f}+q_+)\varphi(\theta)] \\ &\geq \int_0^1 \lambda G(\theta,s)[f(s,[u(s)-w(s)]^*)+q_+(s)]\mathrm{d}s \\ &\geq \int_\theta^\vartheta \lambda G(\theta,s) f(s,[u(s)-w(s)])\mathrm{d}s \geq \int_\theta^\vartheta \lambda G(\theta,s)\cdot\Upsilon\cdot[u(s)-w(s)]\cdot\mathrm{d}s \\ &\geq \int_\theta^\vartheta \lambda G(\theta,s)\cdot\Upsilon\cdot\frac{(2-\lambda)\beta R}{2}\cdot\mathrm{d}s = R\cdot\Upsilon\cdot\frac{\beta\lambda(2-\lambda)}{2}\int_\theta^\vartheta G(\theta,s)\mathrm{d}s \geq R = \|u\|.\end{aligned}$$

Thus for $\lambda \in (0, \lambda_0]$, we get

$$\|Tu\| \geq (Tu)(\theta) \geq \|u\| \quad \text{for any} \quad u \in \partial\Omega_R. \tag{3.6}$$

By (3.3), (3.6) and Lemma 2.4, according to Theorem 2.1, we know that T has at least a fixed point $u^* \in \overline{\Omega}_R \setminus \Omega_r$. Thus for any $\lambda \in (0, \lambda_0]$, by (2.8), we have

$$u^*(t) - w(t) \geq \beta\cdot\|u^*\| - \lambda M \geq \beta\cdot r - \lambda M = M - \lambda M \geq 0.$$

It follows from Lemma 2.3 that $u^*(t)-w(t)$ is a positive solution of the BVP (1.1). This completes the proof of Theorem 3.1. □

Corollary 3.1 *Suppose that* (H_1)–(H_3) *hold. Assume that there exist constants* $0 < \theta_1 < \vartheta_1 < 1$ *such that*

$$\lim_{\|u\|\to+\infty} \min_{t\in[\theta_1,\vartheta_1]} \frac{f(t,u)}{u} = +\infty. \tag{3.7}$$

Then for λ *sufficiently small, the BVP (1.1) has at least one positive solution* $u^* \in P$.

Proof Obviously (3.7) implies that (3.1) is satisfied. Thus by Theorem 3.1, we know that Corollary 3.1 holds. This completes the proof of Corollary 3.1. □

Example 3.1 Consider the following singular second order BVP

$$\begin{cases} x''(t) + \lambda\Big[t^2(1-t)(x^{1/2}+x^{3/2}) - \dfrac{1}{8}\displaystyle\sum_{i=1}^3 \frac{1}{(t-1/i)^{2/3}}\Big] = 0 \quad \text{for a.e. } t \in (0,1), \\ x(0) - x'(0) = x(\frac{1}{2}), \quad x(1) + x'(1) = \frac{1}{2}x(\frac{1}{2}), \end{cases} \tag{3.8}$$

where λ is a positive parameter, $a = b = c = d = a_1 = 1$, $b_1 = 1/2$, $\xi_1 = 1/2$,

$$q(t) = -\frac{1}{8}\sum_{i=1}^3 \frac{1}{(t-1/i)^{2/3}}, \quad f(t,x) = t^2(1-t)(x^{1/2}+x^{3/2}).$$

Let $\gamma = \frac{1}{2}$, $\mu = \frac{3}{2}$. Then (H_1) is satisfied. By calculation, it is easy to obtain that

$$r_1 = \int_0^1 q_-(s)\mathrm{d}s = \frac{1}{8}\int_0^1 \sum_{i=1}^3 \frac{1}{(s-1/i)^{2/3}}\mathrm{d}s = \frac{1}{8}\Big(3 + \frac{6}{\sqrt[3]{2}} + \frac{3\sqrt[3]{2}+3}{\sqrt[3]{3}}\Big) \approx 1.558,$$

$$r_2 = \int_0^1 G(s,s)[f(s,1)+q_+(s)\mathrm{d}s] = \frac{2}{3}\int_0^1 (s+1)(2-s)s^2(1-s)\mathrm{d}s = \frac{11}{90}.$$

Thus (H_2) holds. By direct computation, we get

$$\rho = ac + bc + ad = 3 > 0,\ \rho - a_1\varphi(\xi_1) = \frac{3}{2} > 0,\ \rho - b_1\psi(\xi_1) = \frac{9}{4} > 0,\ \Delta = -\frac{9}{4} < 0.$$

Hence (H_3) is satisfied. Take $\theta = 1/4$, $\vartheta = 3/4$, then we obtain that

$$\beta = \min\{\frac{\psi(\theta)}{\psi(1)},\ \frac{\varphi(\vartheta)}{\varphi(0)}\} = \frac{5}{8},\quad V = \max\{G(t,s)|\ 0 \le t \le 1,\ 0 \le s \le 1\} = \frac{3}{4},$$
$$I = -\frac{3}{2},\quad J = -3,\quad M = Vr_1 + \frac{\psi(1)r_1VI}{\Delta} + \frac{\varphi(0)r_1VJ}{\Delta} \approx 5.8425.$$

So we have

$$r = \frac{M}{\beta} \approx 9.348,\quad \lambda_0 = \min\Big\{1, \frac{Vr_1}{\beta \cdot r_2 \cdot [(\max\{M/\beta, 1\})^{\mu} + 1]}\Big\} \approx 0.5171.$$

Since

$$\lim_{\|x\|\to+\infty} \min_{t\in[\theta,\vartheta]} \frac{f(t,x)}{x} = +\infty,$$

for any $\lambda \in (0, \lambda_0]$, by Corollary 3.1, we know that the BVP (3.8) has at least one positive solution $x^* \in C[0,1] \cap C^2(0,1) \cap P$ for a.e. $t \in [0,1]$ with $\|x^*\| \ge 9.348$.

Remark 3.1 This paper generalizes and improves some well-known results [10–11].

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