Journal of Mathematical Research & Exposition
Sept., 2011, Vol. 31, No. 5, pp. 814–818
DOI:10.3770/j.issn:1000-341X.2011.05.006
Http://jmre.dlut.edu.cn

Essential Closed Surfaces in a Class of Surface Sum of I-Bundle of Closed Surfaces

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Abstract In this paper, we will characterize all types of essential closed surfaces in a class of surface sum of I-bundle of closed surfaces, and give an application of the classification in the surface sum of two 3-manifolds.

Keywords essential surface; surface sum; length of thin position.

Document code A
MR(2010) Subject Classification 57M50; 57N10
Chinese Library Classification O189.21; O189.31

1. Introduction

All 3-manifolds in this paper are assumed to be compact and orientable.

Let F be either a properly embedded connected surface in a 3-manifold M or a connected subsurface of ∂M. If there is an essential simple closed curve on F which bounds a disk in M or F is a 2-sphere which bounds a 3-ball in M, then we say F is compressible, otherwise, F is said to be incompressible. If F cuts off a 3-manifold which is homeomorphic to $F\times I$, then we say F is ∂-parallel in M. If F is an incompressible surface and not ∂-parallel, then F is said to be essential. If M contains an essential 2-sphere, then M is said to be reducible, otherwise, M is said to be irreducible.

Let M be a 3-manifold. If $M \cong S \times I$, where S is a connected, orientable, closed surface, then M is said to be an I-bundle of closed surface.

A compression body C is a 3-manifold obtained by adding 2-handles to $S \times I$, where S is a connected closed surface, along a collection of pairwise disjoint simple closed curves on $S \times \{0\}$, then capping of any resulting 2-sphere boundary components with 3-balls. Denote by $\partial{+}C$ the surface $S \times \{1\}$ in ∂C, and $\partial_- C = \partial C - \partial_+ C$. When $\partial_- C = \emptyset$, C is a handlebody. When $C = S \times I$, C is a trivial compression body, i.e. an I–bundle of S.

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S = \partial_+ W = \partial_+ V$, then we say M has a Heegaard splitting, denoted by $M = V \cup_S W$, and S is called a Heegaard surface of M. Moreover, if the genus $g(S)$ of S is

Received August 4, 2010; Accepted January 18, 2011
Supported by the National Natural Science Foundation of China (Grant Nos. 10625102; 10901029).
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minimal among all the Heegaard surfaces of M, then $g(S)$ is called the Heegaard genus of M, denoted by $g(M)$. If there are essential disks $B \subset V$ and $D \subset W$ such that $\partial B \cap \partial D = \emptyset$, then $V \cup_S W$ is said to be weakly reducible. Otherwise, it is said to be strongly irreducible.

Let $M = V \cup_S W$ be a Heegaard splitting. Then $V \cup_S W$ has a thin position as

$$V \cup_S W = (V_1' \cup_{S_1'} W_1') \cup_{H_1} \ldots \cup_{H_{n-1}} (V_n' \cup_{S_n'} W_n')$$

where $n \geq 2$, each component of $H_1, \ldots, H_{n-1}$ is an incompressible closed surface in M and $V_i' \cup_{S_i'} W_i'$ is a strongly irreducible Heegaard splitting for $1 \leq i \leq n$. We call n the length of this thin position [1].

The distance of a Heegaard splitting was first introduced in [2]:

Let $M = V \cup_S W$ be a Heegaard splitting. The distance between two essential simple closed curves α and β on S, denoted by $d(\alpha, \beta)$, is the smallest integer $n \geq 0$ so that there is a sequence of essential simple closed curves $\alpha_0 = \alpha, \ldots, \alpha_n = \beta$ on S such that α_{i-1} is disjoint from α_i for $1 \leq i \leq n$. The distance of the Heegaard splitting $V \cup_S W$ is $d(S) = \mathrm{Min}\big\{d(\alpha, \beta)\big\}$, where α bounds a disk in V and β bounds a disk in W.

Let M_1 and M_2 be two 3-manifolds, P_i be one component of ∂M_i and F_i be a connected incompressible subsurface on P_i for $i = 1, 2$. Let $f : F_1 \to F_2$ be a homeomorphism. Then the manifold M obtained by gluing M_1 and M_2 along F_1 and F_2 via f is called the surface sum of M_1 and M_2 along F_1 and F_2, and is denoted by $M = M_1 \cup_f M_2$. Specially, we denote $M = M_1 \cup_F M_2$, where F is the surface F_i $(i = 1, 2)$ in M. Let $P_i \times [0, 1]$ be a regular neighborhood of P_i in M_i. Denote $P_i = P_i \times \{0\}$, $P^i = P_i \times \{1\}$, $M^i = M_i - P_i \times [0, 1)$ for $i = 1, 2$, and $M^0 = (P_1 \times I) \cup_F (P_2 \times I)$, where $I = [0, 1]$. Then $M = M^1 \cup_{P^1} M^0 \cup_{P^2} M^2$ and M^0 is the surface sum of I-bundle of closed surfaces P_1 and P_2 along F.

There are some results about surface sum of 3-manifolds [3, 4]. In this paper, we will characterize all types of essential closed surfaces in a class of surface sum of I-bundle of closed surfaces, and give an application of the classification in the surface sum of two 3-manifolds. Note that essential closed surfaces in the annular sum of I-bundle of closed surfaces have been characterized in [3], so we assume F is not an annular in this paper. The main results are the following.

Theorem 1 *Let $M^0 = (P_1 \times I) \cup_F (P_2 \times I)$, where P_i is a connected, orientable, closed surface with $g(P_i) \geq i$ and F is a connected incompressible planar surface on P_i for $i = 1, 2$. Suppose F is separating on one of P_1 and P_2, say P_2, and each curve of ∂F is separating on P_2. Then M^0 contains exactly $2^n - 2$ types of essential closed surfaces up to isotopy and any two types of the essential closed surfaces must intersect with each other, where n is the component number of $P_2 \setminus \mathrm{int}\, F$.*

Corollary 1 *Let $M = M_1 \cup_{P^1} \cup M^0 \cup_{P^2} M_2$, where M_i is an irreducible, ∂-irreducible 3-manifold, P^i is a component of ∂M_i for $i = 1, 2$, M^0 satisfies the conditions of Theorem 1. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $d(S_i) \geq 2(g(M_i) + 1)$ for $i = 1, 2$. Then any minimal Heegaard splitting of M has length at most 4.*

Remark 1 In fact, by Corollary 1 we can argue whether Heegaard genus of M_1 and M_2 is additive under the corresponding surface sum, so the classification in Theorem 1 makes sense.

Definitions and terms which are not defined here are standard [6, 7].

2. The proof of Theorem 1 and Corollary 1

Recalling the definitions of F, P_i, P^i and M^0 in Section 1, and denote $P_i \times I$ by N_i for $i = 1, 2$.

Proof of Theorem 1 Without loss of generality, we may suppose $P_2 \setminus \operatorname{int} F = P_2^1 \cup P_2^2 \cup \cdots \cup P_2^n$, where n is the component number of ∂F. Let S be an essential separated closed surface in M^0. If S is disjoint from F, then S is parallel to P^1 or P^2, a contradiction. Hence $S \cap F \neq \emptyset$. Since F is an incompressible surface on P_i and N_i $(i = 1, 2)$ is irreducible, S can be isotoped such that each component of $S \cap F$ is essential on both S and F. Furthermore, we may assume that $|S \cap F|$ is minimal up to isotopy of S in M^0. Let $S_i = S \cap N_i$ for $i = 1, 2$. Then each component of S_1 and S_2 is incompressible. Suppose H is any component of S_i, by Lemma 2.3 in [5], H is ∂-parallel in N_i for $i = 1, 2$. By the minimality of $|S \cap F|$, H is not ∂-parallel to F if H is an outermost component. Let S_1^* be the outermost component of S_1, $P_1^* \subset P_1$ be the subsurface to which S_1^* is ∂-parallel and $F^* = P_1^* \cap F$.

Claim 1 S_1 is connected.

Proof Otherwise, since F is non-separating on P_1, all components of S_1 are nested. As S is a closed surface in M^0, there must exist one component S_2^* of S_2 which is ∂-parallel to P_2 and connects with S_1^* and another component S_1^{**} of S_1. Since F and each component of ∂F are separating on P_2, S_2^* is either ∂-parallel to F or there is a boundary compressing disk for S_2 in N_2, thus $|S \cap F|$ will be reduced, a contradiction with that $|S \cap F|$ is minimal up to isotopy of S in M^0. $\square$

Claim 2 Each outermost component of S_2 cuts off an annular component from F.

Proof Otherwise, suppose there is an outermost component S_2^* of S_2 which cuts off a non-annular component from F. Let $P^* \subset P_2$ be the subsurface to which S_2^* is ∂-parallel. Then we can take an essential arc b in $P^* \cap F$ such that ∂b lies in ∂P^*. As S_1 and S_2^* are ∂-parallel, there exists a disk D_i in N_i such that $\partial D_i = b_i \cup b$ for $i = 1, 2$, where b_1 is an essential arc in S_1, b_2 is an essential arc in S_2^*. Let $D = D_1 \cup_b D_2$. Thus D is a disk in M^0 and $D \cap S = \partial D = b_1 \cup b_2$ is an essential simple closed curve on S, hence S is compressible in M^0, a contradiction. $\square$

Claim 3 Not all components of S_2 are outermost.

Proof By Claims 1 and 2, if all components of S_2 are outermost, then each component of $P_1^* \cap F$ is annular, thus, S is ∂-parallel to $(P_1 \setminus \operatorname{int} F) \cup (P_2 \setminus \operatorname{int} F)$, a contradiction. $\square$

Let H be the union of all outermost components of S_2, then each component of H cuts off an annular component from F and is ∂-parallel to one component of $P_2 \setminus \operatorname{int} F$. By the minimality of $|S \cap F|$, $\partial S_1 \setminus \partial H$ is connected. By Claim 3 not all components of $P_2 \setminus \operatorname{int} F$ are contained by S_2, so we may suppose H contains k components of $P_2 \setminus \operatorname{int} F$. Without loss of generality, let $H = S_2^1 \cup S_2^2 \cup \cdots \cup S_2^k$ where $1 \le k \le n-1$, S_2^i is ∂-parallel to P_2^i and $\partial S_2^i = c_i$ for $1 \le i \le k$. Since $S_2 \setminus H$ is ∂-parallel in N_2, S_1 is ∂-parallel in N_1, by boundary compress $S_2 \setminus H$ in N_2, we can always isotopy S into a standard position such that $P_1^* \cap F$ is the complement of k essential annuli in F. We also denote the new surface by S. Let $H_1 = S_2 \setminus H = (S_2^1)^* \cup (S_2^2)^* \cdots (S_2^k)^*$, where $(S_2^i)^*$ is ∂-parallel to P_2^i and has the same form as S_2^i. For simplification, we denote $P_1^* \cap F = F_1 \cup F_0 \cup F_2$, where F_0 and F_1 are both the union of k essential annulus in F. F_2 is homeomorphic to F. As S is separating in M^0, let $M^0 = A \cup_S B$, where B contains $F_1 \cup F_2$. See Figure 1.

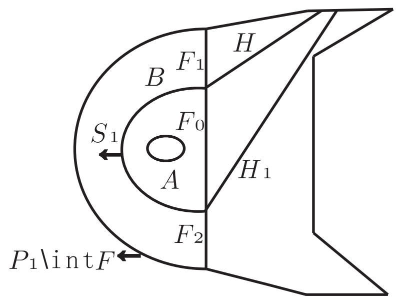

Figure 1 A position of S with H and H_1 have the same form

Claim 4 S is incompressible in A.

Proof Let $F_1 = B_1 \cup B_2 \cup \cdots \cup B_k$, where ∂B_i are parallel to c_i $(1 \le i \le k)$. If S is compressible in A, let D be a compressing disk for S in A such that $|D \cap F_0|$, the component number of $D \cap F_0$ is minimal among all compressing disk for S in A. Since S_1, H and H_1 are incompressible in the respective 3-manifolds, $D \cap F_0 \neq \emptyset$. As each component of F_0 is an essential annular on F, by the minimality of $|D \cap F_0|$, each component of $D \cap F_0$ is an essential arc in F_0. Let α be an outermost component of $D \cap F_0$ in D. Then α lies in a component of F_0. Since $H \cup H_1$ is not connected, α cuts off a disk D^* from D such that $D^* \cap F = \emptyset$, $\partial D^* \setminus \alpha$ lies in S_1 and D^* lies in N_1. By Claim 2, P_1 is compressible in N_1, a contradiction. □

Claim 5 S is incompressible in B.

Proof Otherwise, let D be a compressing disk for S in B such that $|D \cap (F_1 \cup F_2)|$ is minimal among all compressing disks for S in B. By the proof of Claim 4, $D \cap (F_1 \cup F_2) \neq \emptyset$. Since each component of F_1 contains one boundary component lying in the boundary of M^0, D can only intersect each component of F_1 in inessential arcs. By the minimality of $|D \cap (F_1 \cup F_2)|$, $D \cap F_1 = \emptyset$ and each component of $D \cap F_2$ in F_2 is an essential arc. Suppose α is an outermost component of $D \cap F_2$ in D. Since each component of H_1 is ∂-parallel, α cuts off a disk D^* from D and D^* lies in N_1. Now we consider $D \cap F_2$ in D. If all components of $D \cap F_2$ in D are outermost, by the finiteness of $D\cap F_2$, let $D\cap F_2=\alpha_1\cup\alpha_2\cup\cdots\cup\alpha_n$, where α_i cuts off a disk D_i from D, $\partial D_i=\alpha_i\cup\beta_i$ for $1\leq i\leq n$. Let $D'=cl(D\setminus\cup_{i=1}^{i=n}D_i)$ and $\partial D'=c'$. Then D' lies in N_2 and c' lies in $H_1\cup F_2$ and c' is essential in $H_1\cup F_2$, so P_2 is compressible in N_2, a contradiction. If the components of $D\cap F_2$ are not all outermost, we can always find a non-outermost arc β of $D\cap F_2$ in D such that β cuts off a disk D_β from D and each component of $D_\beta\cap F_2$ is an outermost component of $D\cap F_2$. Using the same arguments as above, we get a contradiction. □

By the above arguments, S is an essential closed surface in M^0. Since H can be any non-empty and peoper subset of $P_2\setminus\operatorname{int}F$, and any two types of the essential closed surfaces in M^0 either intersect or ∂-parallel with each other. Then M^0 contains $(C_n^1+C_n^2+\ldots+C_n^{n-1})=2^n-2$ types of essential closed surfaces up to isotopy, where n is the component number of $P_2\setminus\operatorname{int}F$.

Proof of Corollary 1 By the proof of Theorem 1 in [4], any minimal Heegaard splitting of M is weakly reducible, so any minimal Heegaard splitting of M has a thin position. Since M_i $(i=1,2)$ has a high distance Heegaard splitting, by a combinational argument we can deduce any incompressible closed surface which appears in the corresponding thin position can be isotoped into M^0. By Theorem 1, any collection number of non-disjoint, non-isotopic essential closed surfaces in M^0 is 1. As P^1 and P^2 are essential in M, the thin position of any minimal Heegaard splitting of M has length at most 4.

Remark 2 In a following paper, we hope to give a complete classification of essential closed surfaces in the surface sum of I-bundle of closed surfaces, but we have to deal with a complicated case in the combinational argument.

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