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Multi-weight, Weighted Weak Type Estimates for the Multilinear Calderón-Zygmund Operators

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Abstract Let m be an integer and T be an m-linear Calderón-Zygmund operator, u, $v_1, \dots, v_m$ be weights. In this paper, the authors give some sufficient conditions on the weights (u, v_k) with $1 \le k \le m$, such that T is bounded from $L^{p_1}(\mathbb{R}^n, v_1) \times \cdots \times L^{p_m}(\mathbb{R}^n, v_m)$ to $L^{p,\infty}(\mathbb{R}^n, u)$.

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1. Introduction

In their remarkable works [1, ,2], Coifman and Meyer introduced the multilinear Calderón-Zygmund operator. Let $m \ge 1$, $K(x; y_1, \dots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = y_2 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, $A > 0$ and $\gamma \in (0, 1]$ be two constants. We say that K is a kernel in m-CZK(A, γ) if it satisfies the size condition that for all $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_j$ for some $1 \le j \le m$,

$$|K(x; y_1, \dots, y_m)| \le \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \tag{1.1}$$

and satisfies the regularity conditions that

$$|K(x; y_1, \dots, y_m) - K(x'; y_1, \dots, y_m)| \le \frac{A|x - x'|^\gamma}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\gamma}} \tag{1.2}$$

whenever $\max_{1 \le k \le m} |x - y_k| \ge 2|x - x'|$, and also that for each fixed k with $1 \le k \le m$,

$$|K(x; y_1, \dots, y_k, \dots, y_m) - K(x; y_1, \dots, y_k', \dots, y_m)| \le \frac{A|y_k - y_k'|^\gamma}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\gamma}} \tag{1.3}$$

whenever $\max_{1 \le j \le m} |x - y_j| \ge 2|y_k - y_k'|$. An operator T defined on m-fold product of Schwartz spaces and taking values in the space of tempered distributions, is said to be an m-linear Calderón-Zygmund operator with kernel K, if T is m-linear, bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times$

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$L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $q_1, ..., q_m \in [1, \infty]$ and $q \in (0, \infty)$ with $1/q = \sum_{k=1}^m 1/q_k$, and for some m-$CZK(A, \gamma)$ kernel K with positive constants A and γ,

$$T(f_1, ..., f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x; y_1, ..., y_m) \prod_{k=1}^m f_k(y_k)\, \mathrm{d}y_1, ..., \mathrm{d}y_m, \tag{1.4}$$

when $f_1, ..., f_m \in L^2(\mathbb{R}^n)$ with compact supports and $x \notin \cap_{k=1}^m \operatorname{supp} f_k$. It is obvious that when $m = 1$, this operator is just the classical Calderón-Zygmund operator and when $m \ge 2$, this operator has intimate connection with operator theory and partial differential equations. Grafakos and Torres [5] developed the idea used in Kenig and Stein [8], considered the behavior on $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ for the operator T, and proved that an m-linear Calderón-Zygmund operator is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $p_1, ..., p_m \in (1, \infty]$ and $p \in (0, \infty)$ with $1/p = \sum_{1 \le k \le m} 1/p_k$. Fairly recently, Lerner et al. [9] introduced a new maximal operator and a multilinear $A_p(\mathbb{R}^n)$ weight condition, and obtained some interesting weighted estimates for multilinear Calderón-Zygmund operators and the corresponding commutators. For other works about the multilinear Calderón-Zygmund operator, see [4], [6] and [7].

The purpose of this paper is to establish some multi-weight, weighted weak type norm inequalities for the multilinear Calderón-Zygmund operator T, in analogy with the two-weight, weighted estimate for classical Calderón-Zygmund operator established by Cruze-Uribe, SFO and Pérez [3]. To state our results, we first recall some notation.

By a weight w we mean that w is a nonnegative and locally integrable function. For a measurable set E and a weight w, $w(E)$ denotes the integral $\int_E w(x)\mathrm{d}x$. For $p \in (0, \infty)$, $L^p(\mathbb{R}^n, w)$ denotes the usual weighted L^p space with weight w and $L^{p,\infty}(\mathbb{R}^n, w)$ denotes the weighted weak L^p norm with respect to the weight w, that is,

$$L^{p,\infty}(\mathbb{R}^n, w) = \{f : \|f\|_{L^{p,\infty}(\mathbb{R}^n, w)} < \infty\},$$

where and in the following,

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n, w)} = \sup_{\lambda > 0} \lambda \Big(w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) \Big)^{1/p}.$$

Given a cube Q, $p \ge 1$, $\delta \in \mathbb{R}$ and a suitable function f, set

$$\|f\|_{L(\log L)^\delta, Q} = \inf \Big\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|f(x)|}{\lambda} \log^\delta \Big(\mathrm{e} + \frac{|f(x)|}{\lambda} \Big) \mathrm{d}x \le 1 \Big\}.$$

Define the maximal operator $M_{L(\log L)^\delta}$ by

$$M_{L(\log L)^\delta} f(x) = \sup_{Q \ni x} \|f\|_{L(\log L)^\delta, Q},$$

where the supremum is taken over all cubes containing x. Note that when $\delta = 0$, $M_{L(\log L)^\delta}$ is just the standard Hardy-Littlewood maximal operator M.

Let u, v be a pair of weights on $\mathbb{R}^n$. For $\sigma \ge 0$, we say that $(u, v) \in A_{p,(\log L)^\sigma}(\mathbb{R}^n)$, if there exists a positive constant C such that for any cube Q,

$$\|u\|_{L(\log L)^\sigma, Q} \Big(\frac{1}{|Q|} \int_Q v^{-p'/p}(x) \mathrm{d}x \Big)^{p-1} \le C.$$

For the case of $\sigma = 0$, we denote $(u, v) \in A_p(\mathbb{R}^n)$ (see [3]).

Our results can be stated as follows.

Theorem 1.1 *Let m and ℓ be integers with $1 \le \ell \le m$, T be an m-linear Calderón-Zygmund operator, $u, v_1, ..., v_m$ be weights. Suppose that $p_1, ..., p_\ell \in (1, \infty)$, $p_{\ell+1}, ..., p_m \in (1, 1+\gamma/n)$, and for some $\delta > 0$, $(u, v_k) \in A_{p_k, (\log L)^{p_k-1+\delta}}(\mathbb{R}^n)$ for $1 \le k \le \ell$ and $(u, v_k) \in A_{p_k}(\mathbb{R}^n)$ for $\ell+1 \le k \le m$, then there exists a positive constant C, such that for all bounded functions $f_1, ..., f_m$ with compact supports,*

$$\|T(f_1, ..., f_m)\|_{L^{p,\infty}(\mathbb{R}^n, u)} \le C \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\mathbb{R}^n, v_k)}. \tag{1.5}$$

We mow make some conventions. Throughout this paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. For a measurable set E, χ_E denotes the characteristic function of E. Given $\lambda > 0$ and a cube Q, λQ denotes the cube with the same center as Q and whose side length is λ times that of Q. For a locally integrable function f on $\mathbb{R}^n$ and bounded measurable set E, $(f)_E$ denotes the mean value of f over E, that is, $(f)_E = \frac{1}{|E|}\int_E f(x)\mathrm{d}x$. For a fixed p with $p \in [1, \infty)$, p' denotes the dual exponent of p, namely, $p' = p/(p-1)$.

2. Proof of Theorem 1.1

We begin with some preliminary lemmas.

Lemma 2.1 ([3, Theorem 1.2]) *Let T be a Calderón-Zygmund operator. Given a pair of weights (u, v) and p, $1 < p < \infty$, suppose that for some $\delta > 0$, $(u, v) \in A_{p, (\log L)^{p-1+\delta}}(\mathbb{R}^n)$. Then T is bounded from $L^p(\mathbb{R}^n, v)$ to $L^{p,\infty}(\mathbb{R}^n, u)$.*

Lemma 2.2 *Let $m \ge 2$, T be an m-linear Calderón-Zygmund operator with kernel K in m-$CZK(A, \gamma)$ for some $A, \gamma > 0$. Then for all positive integer l with $1 \le l < m$ and all bounded functions $f_1, ..., f_{m-l}$ with compact supports, the operator $T_{f_1,...,f_{m-l}}$ defined by*

$$T_{f_1,...,f_{m-l}}(f_{m-l+1}, ..., f_m)(x) = T(f_1, ..., f_m)(x)$$

is an l-linear Calderón-Zygmund operator with kernel K in l-$CZK(A\prod_{k=1}^{m-l}\|f_k\|_{L^\infty(\mathbb{R}^n)}, \gamma)$.

This lemma is a combination of Lemma 3 and Theorem 2 in [5].

Lemma 2.3 ([3, p. 424)] *Let $q \in (1, \infty)$, $(u, v) \in A_{q, (\log L)^{q-1+\sigma}}(\mathbb{R}^n)$ for some $\sigma > 0$. Then for any $\delta \in [0, \sigma/q)$, there exists a positive constant C such that*

$$\|M_{L(\log L)^\delta} f\|_{L^{q'}(\mathbb{R}^n, v^{-q'/q})} \le C\|f\|_{L^{q'}(\mathbb{R}^n, u^{-q'/q})}.$$

Proof of Theorem 1.1 First, we prove the case that $\ell = m$. We will proceed by an inductive argument on m. By Lemma 2.1 we know that (1.5) holds for the case $m = 1$. Let $m \ge 2$ be a positive integer. We assume that (1.5) holds if T is an l-linear Calderón-Zygmund operator with

$1 \le l \le m-1$. Let $f_1, ..., f_m$ be bounded functions with compact supports and

$$\|f_1\|_{L^{p_1}(\mathbb{R}^n,\, v_1)} = \|f_2\|_{L^{p_2}(\mathbb{R}^n,\, v_2)} = \cdots = \|f_m\|_{L^{p_m}(\mathbb{R}^n,\, v_m)} = 1.$$

Our goal is to prove that there exists a positive constant C such that for any $\lambda > 0$,

$$u(\{x \in \mathbb{R}^n : |T(f_1, ..., f_m)(x)| > \lambda\}) \le C\lambda^{-p}. \tag{2.1}$$

For each fixed $\lambda > 0$, applying the Calderón-Zygmund decomposition to $|f_m|^{p_m}$ at the level λ^p, we then obtain sequences of cubes $\{Q_m^j\}_j$ with disjoint interiors, such that

(i) For any fixed j,

$$\lambda^{p/p_m} < \frac{1}{|Q_m^j|}\int_{Q_m^j} |f_m(y)|\,\mathrm{d}y \le 2^n\lambda^{p/p_m}. \tag{2.2}$$

(ii) $|f_m(x)| \le C\lambda^{p/p_m}$ a. e. $x \in \mathbb{R}^n \setminus \cup_j Q_m^j$.

Set

$$g_m(x) = f_m(x)\chi_{\mathbb{R}^n\setminus\cup_j Q_m^j}(x) + \sum_j (f_m)_{Q_m^j}\chi_{Q_m^j}(x),$$

and

$$b_m(x) = \sum_j \big(f_m(x) - (f_m)_{Q_m^j}\big)\chi_{Q_m^j}(x) = \sum_j b_m^j(x).$$

Lemma 2.2, together with the fact that $\|g_m\|_{L^\infty(\mathbb{R}^n)} \le C\lambda^{p/p_m}$ and the inductive hypothesis, tells us that

$$\begin{aligned} u(\{x \in \mathbb{R}^n : |T(f_1, ..., f_{m-1}, g_m)(x)| > \lambda/2\}) &\le C\lambda^{-\tilde{p}}\|g_m\|^{\tilde{p}}_{L^\infty(\mathbb{R}^n)}\prod_{k=1}^{m-1}\|f_k\|^{\tilde{p}}_{L^{p_k}(\mathbb{R}^n,\, v_k)} \\ &\le C\lambda^{-p}, \end{aligned}$$

where $\tilde{p} \in (0, \infty)$ with $1/\tilde{p} = \sum_{k=1}^{m-1} 1/p_k$. For any j, a trivial computation involving the Hölder inequality in (2.2), shows that

$$\begin{aligned} &\Big(\int_{Q_m^j} v_m^{-p'_m/p_m}(x)\mathrm{d}x\Big)^{1/p'_m} \\ &\le \lambda^{-p/(p_m p'_m)}\Big(\int_{Q_m^j}|f_m(x)|\mathrm{d}x\Big)^{1/p'_m}\Big(\frac{1}{|Q_m^j|}\int_{Q_m^j} v_m^{-p'_m/p_m}(x)\mathrm{d}x\Big)^{1/p'_m} \\ &\le \lambda^{-p/(p_m p'_m)}\Big(\int_{Q_m^j}|f_m(x)|^{p_m}v_m(x)\mathrm{d}x\Big)^{1/(p_m p'_m)}\times \\ &\quad\times\Big(\frac{1}{|Q_m^j|}\int_{Q_m^j} v_m^{-p'_m/p_m}(x)\mathrm{d}x\Big)^{1/p'_m}\Big(\int_{Q_m^j} v_m^{-p'_m/p_m}(x)\mathrm{d}x\Big)^{1/(p'_m p'_m)}, \end{aligned}$$

and so

$$\begin{aligned} \Big(\int_{Q_m^j} v_m^{-p'_m/p_m}(x)\mathrm{d}x\Big)^{1/p'_m} \le& \lambda^{-p/p'_m}\Big(\int_{Q_m^j}|f_m(x)|^{p_m}v_m(x)\mathrm{d}x\Big)^{1/p'_m}\times \\ &\Big(\frac{1}{|Q_m^j|}\int_{Q_m^j} v_m^{-p'_m/p_m}(x)\mathrm{d}x\Big)^{p_m/p'_m}. \end{aligned} \tag{2.3}$$

Let $\Omega = \bigcup_j 4nQ_m^j$. The estimate (2.3), via the Hölder inequality, leads to that

$$\begin{aligned}
u(\Omega) \leq & C\lambda^{-p/p_m} \sum_j \frac{u(4nQ_m^j)}{|4nQ_m^j|} \int_{Q_m^j} |f_m(x)|\mathrm{d}x \\
\leq & C\lambda^{-p/p_m} \sum_j \frac{u(4nQ_m^j)}{|4nQ_m^j|} \Big(\int_{Q_m^j} |f_m(x)|^{p_m} v_m(x)\mathrm{d}x \Big)^{1/p_m} \Big(\int_{Q_m^j} v_m^{-p_m'/p_m}(x)\mathrm{d}x \Big)^{1/p_m'} \\
\leq & C\lambda^{-p/p_m}\lambda^{-p/p_m'} \sum_j \int_{Q_m^j} |f_m(x)|^{p_m} v_m(x)\mathrm{d}x \times \\
& \frac{u(4nQ_m^j)}{|4nQ_m^j|} \Big(\frac{1}{|Q_m^j|} \int_{Q_m^j} v_m^{-p_m'/p_m}(x)\mathrm{d}x \Big)^{p_m/p_m'} \\
\leq & C\lambda^{-p/p_m}\lambda^{-p/p_m'} \sum_j \int_{Q_m^j} |f_m(x)|^{p_m} v_m(x)\mathrm{d}x \leq C\lambda^{-p}.
\end{aligned} \tag{2.4}$$

If we can prove that

$$u(\{x \in \mathbb{R}^n \backslash \Omega : \, |T(f_1, ..., \, f_{m-1}, b_m)(x)| > \lambda/2\}) \leq C\lambda^{-p}, \tag{2.5}$$

the inequality (2.1) then follows from (2.2), (2.3) and (2.4) directly.

We now prove (2.5). Note that for any $\sigma > 0$,

$$\int_{\mathbb{R}^n} \frac{1}{(|x-y_1| + \sum_{k=2}^m |x-y_k|)^{n+\sigma}} |f(y_1)|\mathrm{d}y_1 \leq \frac{C}{(\sum_{k=2}^m |x-y_k|)^{\sigma}} Mf(x).$$

By the vanishing moment of b_m^j and the regularity (1.3), we see that for $x \in \mathbb{R}^n \backslash \Omega$,

$$\begin{aligned}
& |T(f_1, ..., \, f_{m-1}, \, b_m)(x)| \sum_j \Big| \int_{(\mathbb{R}^n)^m} K(x; y_1, ..., \, y_m) f_1(y_1) \cdots f_{m-1}(y_{m-1}) b_m^j(y_m)\mathrm{d}y_m \Big| \\
& \leq \sum_j \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^{m-1}} \frac{|y_m - c_m^j|^{\gamma}}{(\sum_{k=1}^m |x-y_k|)^{mn+\gamma}} \prod_{k=1}^{m-1} |f_k(y_k)|\mathrm{d}y_1 \cdots \mathrm{d}y_{m-1} |b_m^j(y_m)|\mathrm{d}y_m \\
& \leq C \sum_j \prod_{k=1}^{m-1} Mf_k(x) \int_{\mathbb{R}^n} \frac{|y_m - c_m^j|^{\gamma}}{|x-y_m|^{n+\gamma}} |b_m^j(y_m)|\mathrm{d}y_m \\
& \leq C \prod_{k=1}^{m-1} Mf_k(x) \mathcal{M}_m(x),
\end{aligned} \tag{2.6}$$

where for each fixed j, c_m^j and $l(Q_m^j)$ are the center and side length of Q_m^j and $\mathcal{M}_m$ is the Marcinkiewicz function defined by

$$\mathcal{M}_m(x) = \sum_j \|b_m^j\|_{L^1(\mathbb{R}^n)} \frac{\{l(Q_m^j)\}^{\gamma}}{|x - c_m^j|^{n+\gamma}} \chi_{\mathbb{R}^n \backslash \Omega}(x).$$

It is well known that if $(u, \, v) \in A_r(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator is bounded from $L^r(\mathbb{R}^n, \, v)$ to $L^{r,\,\infty}(\mathbb{R}^n, \, u)$. Therefore,

$$u(\{x \in \mathbb{R}^n : \, Mf_k(x) > \lambda^{p/p_k}\}) \leq C\lambda^{-p} \int_{\mathbb{R}^n} |f_k(x)|^{p_k} v_k(x)\mathrm{d}x. \tag{2.7}$$

On the other hand, an application of the Hölder inequality shows that for any weight w,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\Omega}\mathcal{M}_m(x)w(x)\mathrm{d}x &\leq \sum_j \|b_m^j\|_{L^1(\mathbb{R}^n)}\{l(Q_m^j)\}^\gamma\int_{\mathbb{R}^n\setminus 4nQ_m^j}\frac{w(x)}{|x-c_m^j|^{n+\gamma}}\mathrm{d}x\\ &\leq C\sum_j \|b_m^j\|_{L^1(\mathbb{R}^n)}\inf_{y\in Q_m^j} Mw(y)\\ &\leq C\int_{\mathbb{R}^n}|f_m(x)|Mw(x)\mathrm{d}x\\ &\leq C\|f_m\|_{L^{p_m}(\mathbb{R}^n,\, v_m)}\|Mw\|_{L^{p_m'}(\mathbb{R}^n,\, v_m^{-p_m'/p_m})}.\end{aligned}$$

We thus have by a standard duality argument and Lemma 2.3 that

$$\begin{aligned}u(\{x\in\mathbb{R}^n\setminus\Omega:\, \mathcal{M}_m(x)>\lambda^{p/p_m}\}) &\leq C\lambda^{-p}\int_{\mathbb{R}^n\setminus\Omega}(\mathcal{M}_m(x))^{p_m}u(x)\mathrm{d}x\\ &= C\lambda^{-p}\Big(\sup_{\|w\|_{L^{p_m'}(\mathbb{R}^n,\, u^{-p_m'/p_m})}\leq 1}\int_{\mathbb{R}^n\setminus\Omega}\mathcal{M}_m(x)w(x)\mathrm{d}x\Big)^{p_m}\\ &\leq C\lambda^{-p}\|f_m\|^{p_m}_{L^{p_m}(\mathbb{R}^n,\, v_m)}.\end{aligned}\tag{2.8}$$

Combining the inequalities (2.6), (2.7) and (2.8) yields

$$\begin{aligned}u(\{x\in\mathbb{R}^n\setminus\Omega:\, |T(f_1,\dots,f_{m-1},b_m)(x)|>\lambda/2\}) &\leq \sum_{k=1}^{m-1}u(\{x\in\mathbb{R}^n:\, Mf_k>\lambda^{p/p_k}\})+\\ &\quad u(\{x\in\mathbb{R}^n\setminus\Omega:\mathcal{M}_m(x)>\lambda^{p/p_m}/2\})\\ &\leq C\lambda^{-p}\end{aligned}$$

and then establishes (2.5).

Now, we turn our attention to the case $1\leq \ell<m$. Let $p_k\in(1,\,1+\gamma/n)$ with $\ell\leq k\leq m$, $f_1,\dots,\,f_m$ be bounded functions with compact supports and

$$\|f_1\|_{L^{p_1}(\mathbb{R}^n,\, v_1)}=\|f_2\|_{L^{p_2}(\mathbb{R}^n,\, v_2)}=\cdots=\|f_m\|_{L^{p_m}(\mathbb{R}^n,\, v_m)}=1.$$

For each k with $\ell+1\leq k\leq m$ and each fixed $\lambda>0$, applying Calderón-Zygmund decomposition to $|f_k|$ at the level λ^{p/p_k}, we obtain sequences of cubes $\{Q_k^j\}_j$, g_k, b_k, and b_k^j which are similar to that of the case $\ell=m$. Then, Lemma 2.2 and (1.5) with $\ell=m$ give us that

$$\begin{aligned}&u(\{x\in\mathbb{R}^n:\, |T(f_1,\dots,\,f_\ell,\,g_{\ell+1},\dots,\,g_m)(x)|>\lambda/2\})\\ &\leq C\lambda^{-\widetilde{p}_\ell}\prod_{k=1}^{\ell}\|f_k\|^{\widetilde{p}_\ell}_{L^{p_k}(\mathbb{R}^n,\, v_k)}\prod_{\ell+1}^{m}\|g_k\|^{\widetilde{p}_\ell}_{L^\infty(\mathbb{R}^n)}\\ &\leq C\lambda^{-\widetilde{p}_\ell}\prod_{\ell+1}^{m}\lambda^{\widetilde{p}_\ell p/p_k}\leq C\lambda^{-p},\end{aligned}$$

where $\widetilde{p}_\ell\in(0,\,\infty)$ with $1/\widetilde{p}_\ell=\sum_{k=1}^{\ell}1/p_k$. Set $E=\bigcup_{\ell+1\leq k\leq m}\bigcup_j 4nQ_k^j$. It is proved that $u(E)\leq C\lambda^{-p}$. Thus, the proof of (1.5) in this case is reduced to proving

$$u(\{x\in\mathbb{R}^n\setminus E:\, |T(f_1,\dots,\,f_\ell,\,h_{\ell+1},\dots,\,h_m)(x)|>\lambda/2\})\leq C\lambda^{-p},\tag{2.9}$$

where $h_k\in\{g_k,\,b_k\}$ for k with $\ell+1\leq k\leq m$, and at least one $h_k=b_k$.

We only prove (2.9) for the case $h_m = b_m$ since the other cases can be dealt with in a similar way. Again, we can easily obtain that for $x \in \mathbb{R}^n \backslash E$,

$$|T(f_1, ..., f_\ell, h_{\ell+1}, ..., h_{m-1}, b_m)(x)| \le C \prod_{k=1}^{\ell} Mf_k(x) \prod_{k=\ell+1}^{m-1} Mh_k(x)\mathcal{M}_m(x).$$

Note that for any fixed k with $\ell + 1 \le k \le m$,

$$|h_k(x)| \le |f_k(x)| + C_0\lambda^{p/p_k},$$

with C_0 a positive constant. It then follows that

$$\begin{aligned} u(\{x \in \mathbb{R}^n : Mh_k(x) > (C_0+1)\lambda^{p/p_k}\}) &\le u(\{x \in \mathbb{R}^n : Mf_k(x) > \lambda^{p/p_k}\}) \\ &\le C\lambda^{-p} \int_{\mathbb{R}^n} |f_k(x)|^{p_k} v_k(x)\mathrm{d}x. \end{aligned} \tag{2.10}$$

On the other hand, a straightforward computation, along with the Hölder inequality and the estimate (2.3), leads to that

$$\begin{aligned} \int_{\mathbb{R}^n\backslash E} \mathcal{M}_m(x)u(x)\mathrm{d}x \le& \sum_j \|b_m^j\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n\backslash E} \frac{|Q_m^j|^{\gamma/n}}{|x-c_m^j|^{n+\gamma}} u(x)\mathrm{d}x \\ \le& C\sum_j \int_{Q_m^j} |f_m(y)|\mathrm{d}y \sum_{l=1}^{\infty} \frac{|Q_m^j|^{\gamma/n}}{|2^l 4nQ_m^j|^{1+\gamma/n}} \int_{2^l 4nQ_m^j} u(x)\mathrm{d}x \\ \le& C\lambda^{-p/p_m'} \sum_j \int_{Q_m^j} |f_m(y)|^{p_m} v_m(y)\mathrm{d}y \times \\ & \Big(\frac{1}{|Q_m^j|} \int_{Q_m^j} v_m^{-p_m'/p_m}(x)\mathrm{d}x\Big)^{p_m/p_m'} \times \\ & \sum_{l=1}^{\infty} \frac{|Q_m^j|^{\gamma/n}}{|2^l 4nQ_m^j|^{1+\gamma/n}} \int_{2^l 4nQ_m^j} u(x)\mathrm{d}x \\ \le& C\lambda^{-p/p_m'} \sum_j \int_{Q_m^j} |f_m(y)|^{p_m} v_m(y)\mathrm{d}y \sum_{l=1}^{\infty} 2^{nl(p_m-1-\gamma/n)} \\ \le& C\lambda^{-p/p_m'} \int_{\mathbb{R}^n} |f_m(y)|^{p_m} v_m(y)\mathrm{d}y. \end{aligned}$$

This, via (2.10), in turn implies that

$$\begin{aligned} & u(\{x \in \mathbb{R}^n\backslash E : |T(f_1, ..., f_\ell, h_{\ell+1}, ..., h_{m-1}, b_m)(x)| > \lambda/2\}) \\ & \le \sum_{k=1}^{\ell} u(\{x \in \mathbb{R}^n : Mf_k(x) > \lambda^{p/p_k}\}) + \sum_{k=\ell+1}^{m-1} u(\{x \in \mathbb{R}^n : Mh_k(x) > \lambda^{p/p_k}\}) + \\ & \quad u(\{x \in \mathbb{R}^n\backslash E : \mathcal{M}_m(x) > (1+C_0)^{\ell+1-m}\lambda^{p/p_m}/2\}) \\ & \le C\lambda^{-p}. \end{aligned}$$

The proof of Theorem 1.1 is completed. □

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