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Property (ω') and Its Perturbation

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Abstract In this note we define the property (ω'), a variant of Weyl's theorem, and establish for a bounded linear operator defined on a Hilbert space the necessary and sufficient conditions for which property (ω') holds by means of the variant of the essential approximate point spectrum $\sigma_1(\cdot)$ and the spectrum defined in view of the property of consistency in Fredholm and index. In addition, the perturbation of property (ω') is discussed.

Keywords property (ω'); spectrum; Weyl's theorem.

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1. Introduction

Weyl [1] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This "Weyl's theorem" has been considered by many authors. Variants have been discussed by Harte and Lee [2] and Rakočevič [3, 4]. In this note, we introduce a new variant of Weyl's theorem called property (ω') and show how property (ω') follows from properties of the variant (σ_1) of the essential approximate point spectrum and the spectrum defined in view of the property of consistency in Fredholm and index (defined in Section 2). In addition, the perturbation of property (ω') is discussed.

Throughout this note, let $B(H)$ ($K(H)$) denote the algebra of bounded linear operators (compact operators) acting on a complex, infinite dimensional Hilbert space H. If $T \in B(H)$, write $N(T)$ and $R(T)$ for the null space and the range of T; $\sigma(T)$ for the spectrum of T; $\pi_{00}(T) = \pi_0(T) \cap \operatorname{iso} \sigma(T)$, where $\pi_0(T) = \{\lambda \in \mathbb{C} : 0 < \dim N(T - \lambda I) < \infty\}$ are the eigenvalues of finite multiplicity. An operator $T \in B(H)$ is called upper semi-Fredholm if it has closed range with finite dimensional null space and if $R(T)$ has finite co-dimension, $T \in B(H)$ is called a lower semi-Fredholm operator. We call $T \in B(H)$ Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. For a semi-Fredholm operator, let

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$n(T) = \dim N(T)$ and $d(T) = \dim H/R(T) = \operatorname{codim} R(T)$. The index of a Fredholm operator $T \in B(H)$ is given by $\operatorname{ind}(T) = \dim N(T) - \dim H/R(T) = n(T) - d(T)$. The ascent of T, $\operatorname{asc}(T)$, is the least non-negative integer n such that $N(T^n) = N(T^{n+1})$ and the descent, $\operatorname{dsc}(T)$, is the least non-negative integer n such that $R(T^n) = R(T^{n+1})$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. And $T \in B(H)$ is called Browder if it is Fredholm "of finite ascent and descent": equivalently if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, the Browder spectrum $\sigma_b(T)$, the upper semi-Fredholm spectrum $\sigma_{SF_+}(T)$ and the lower semi-Fredholm spectrum $\sigma_{SF_-}(T)$ of $T \in B(H)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\},$$

$$\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm}\},$$

$$\sigma_{SF_-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Fredholm}\}.$$

The property (ω') which we will define has close relations with Weyl's theorem. The rest of this paper is organized as follows. In Section 2, by defining two new spectrums, we give the definition of property (ω') and the necessary and sufficient conditions for T such that property (ω') holds. As a consequence of the main result, the perturbation of property (ω') is discussed.

2. CFI operator and Property (ω')

We begin with a definition and a lemma [5]:

Definition 2.1 *We say $T \in B(H)$ is consistent in Fredholm and index (abbrev. a CFI operator), if for each $B \in B(H)$, one of the following cases occurs:*

(1) TB and BT are Fredholm together and $\operatorname{ind}(TB) = \operatorname{ind}(BT) = \operatorname{ind}(B)$;

(2) Both TB and BT are not Fredholm.

Lemma 2.1 *$T \in B(H)$ is a CFI operator if and only if one of the following three mutually disjoint cases occurs:*

(1) T is Weyl;

(2) $R(T)$ is not closed;

(3) $R(T)$ is closed and $\dim N(T) = \operatorname{codim} R(T) = \infty$.

Let

$$\rho_2(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is a CFI operator}\}$$

and let $\sigma_2(T) = \mathbb{C} \backslash \rho_2(T)$. Clearly, $\lambda_0 \in \sigma_2(T)$ if and only if $T - \lambda_0 I$ is a semi-Fredholm operator but $\operatorname{ind}(T - \lambda_0 I) \neq 0$. By perturbation theorem of semi-Fredholm operator, $\sigma_2(T)$ is an open set in the spectrum $\sigma(T)$ of operator T. Let $H(T)$ be the class of complex-valued functions which are analytic in a neighborhood of $\sigma(T)$ and are not constant on any neighbourhood of any component of $\sigma(T)$.

Remark 2.1 (1) If $\operatorname{int}\sigma(T)=\emptyset$, then $\sigma_2(T)=\emptyset$;

(2) It is easy to prove that $\sigma_2(f(T))\subseteq f(\sigma_2(T))$ for any $f\in H(T)$; But in general the converse inclusion fails.

For example, suppose $A_1, A_2\in B(\ell^2)$ are defined by:

$$A_1(x_1,x_2,x_3,\ldots)=(0,x_1,0,x_2,0,x_3,\ldots),$$

$$A_2(x_1,x_2,x_3,\ldots)=(x_1,0,0,\ldots).$$

Let $T=\begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}$ and $p(T)=T(\frac{I}{2}-T)$. Then $p(T)$ is a CFI operator, that is, $0\notin\sigma_2(p(T))$. But since $\frac{1}{2}\in\sigma_2(T)$, we know that $0\in p(\sigma_2(T))$, which means that the inclusion $f(\sigma_2(T))\subseteq\sigma_2(f(T))$ fails.

(3) For any $f\in H(T)$, $\sigma_2(f(T))=f(\sigma_2(T))$ if and only if $\sigma_2(T)=\emptyset$.

Suppose $\sigma_2(T)=\emptyset$, then $f(\sigma_2(T))=\emptyset$. Since $\sigma_2(f(T))\subseteq f(\sigma_2(T))$, $\sigma_2(f(T))=\emptyset$. Then $\sigma_2(f(T))=f(\sigma_2(T))$ for any $f\in H(T)$.

Conversely, suppose that spectrum mapping theorem holds for $\sigma_2(\cdot)$. If $\sigma_2(T)\neq\emptyset$, let $\lambda_0\in\sigma_2(T)$, that is $T-\lambda_0 I$ is a semi-Fredholm operator. Since $\sigma_{SF_+}(T)\cap\sigma_{SF_-}(T)\neq\emptyset$, take $\mu_0\in\sigma_{SF_+}(T)\cap\sigma_{SF_-}(T)$, and let $f(T)=(T-\lambda_0 I)(T-\mu_0 I)$. If $R(T-\mu_0 I)$ is not closed, $R(f(T))$ must not be closed, in this case $f(T)$ is a CFI operator. In the following we suppose $R(T-\mu_0 I)$ is closed, then $n(T-\mu_0 I)=d(T-\mu_0 I)=\infty$. Using the fact that $R(f(T))=R(T-\lambda_0 I)\cap R(T-\mu_0 I)$ and $N(T-\mu_0 I)\subseteq N(f(T))$, we know that $R(f(T))$ is closed and $n(f(T))=d(f(T))=\infty$, which means that $f(T)$ is a CFI operator again, that is $0\notin\sigma_2(f(T))(=f(\sigma_2(T)))$. Then $\lambda_0\notin\sigma_2(T)$, it is in contradiction to the fact that $\lambda_0\in\sigma_2(T)$.

(4) $\sigma_2(T)=\emptyset$ if and only if $\sigma_{SF_+}(T)=\sigma_{SF_-}(T)=\sigma_w(T)$.

Weyl's theorem for an operator says that the complement in the spectrum of the Weyl spectrum coincides with the isolated points of the spectrum which are eigenvalues of finite multiplicity. Weyl [1] discovered that this property holds for hermitian operators and it has been extended to many other operators. In recent years, a number of researchers have considered the Weyl's theorem for operators and operator matrices (such as [2, 6–9], et.) In the following, we consider a variant of Weyl's theorem called property (ω').

We say that the Weyl's theorem holds for $T\in B(H)$ if there is equality

$$\sigma(T)\backslash\sigma_w(T)=\pi_{00}(T).$$

Let $SF_+^-(H)=\{A\in B(H): A$ is an upper semi-Fredholm operator with $\operatorname{ind}(A)\leq 0\}$. The essential approximate point spectrum $\sigma_{ea}(T)$ of T is defined by: $\sigma_{ea}(T)=\{\lambda\in\mathbb{C}: T-\lambda I\notin SF_+^-(H)\}$. Rakočevič has looked at variants of " Weyl's theorem " in which the spectrum is replaced by the approximate point spectrum: "the a-Weyl's theorem holds " for T if

$$\sigma_a(T)\backslash\sigma_{ea}(T)=\pi_{00}^a(T),$$

where we write $\sigma_a(T)$ for the approximate point spectrum of T, $\pi_{00}^a(T)$ for the set of all $\lambda \in \mathbb{C}$ such that λ is isolated point in $\sigma_a(T)$ and $0 < \dim N(T-\lambda I) < \infty$.

Definition 2.2 *$T \in B(H)$ is said to satisfy property (ω') if*

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}^a(T).$$

Remark 2.2 (1) Property (ω') implies Weyl's theorem, but the converse is not true.

For example, let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots),$$

$$B(x_1, x_2, x_3, \ldots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots),$$

and $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then $\sigma(T) = \sigma_w(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\pi_{00}(T) = \emptyset$, $\pi_{00}^a(T) = \{0\}$, which means that Weyl's theorem holds for T but property (ω') fails for T.

(2) Property (ω') cannot induce a-Weyl's theorem.

For example, let $A, B \in B(\ell^2)$ be defined by:

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots),$$

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

and let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then $\sigma(T) = \sigma_w(T) = \sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, $\pi_{00}^a(T) = \emptyset$ and $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, thus T has property (ω'), but a-Weyl's theorem is not true for T.

(3) a-Weyl's theorem cannot induce Property (ω').

Let $T \in B(\ell^2)$ be defined by:

$$T(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, x_3, x_4, \ldots).$$

Then

(a) $\sigma_a(T) = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and $\pi_{00}^a(T) = \{0\}$;

(b) $\sigma(T) = \sigma_w(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

This shows that a-Weyl's theorem holds for T, but property (ω') fails for T.

(4) Property (ω') holds for $T \Leftrightarrow$ Weyl's theorem holds for T and $\pi_{00}(T) = \pi_{00}^a(T) \Leftrightarrow$ Weyl's theorem holds for T and $\sigma_w(T) \cap \pi_{00}^a(T) = \emptyset \Leftrightarrow \sigma(T) = \sigma_w(T) \cup \pi_{00}^a(T)$ and $\sigma_w(T) \cap \pi_{00}^a(T) = \emptyset$.

Let $\rho_1(T) = \{\lambda \in \mathbb{C} : n(T-\lambda I) < \infty$ and there exists $\epsilon > 0$ such that $T - \mu I \in SF_+^-(H)$ and

$$N(T-\mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\mu I)^n] \text{ if } 0 < |\mu - \lambda| < \epsilon\}$$

and let $\sigma_1(T) = \mathbb{C}\backslash\rho_1(T)$. Clearly, $\sigma_1(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_a(T)$ and $\sigma_1(T) \subseteq \sigma_b(T)$. T is called a-isoloid if $\lambda \in \operatorname{iso}\sigma_a(T) \Rightarrow N(T-\lambda I) \neq \{0\}$. The following theorems give the relation between property (ω') and property of consistency in Fredholm and index.

Theorem 2.1 *$T \in B(H)$ is a-isoloid and property (ω') holds for T if and only if $\sigma_b(T) =$*

$\sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]$.

Proof Suppose that $\sigma_b(T) = \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]$. Let $\lambda_0 \in \sigma(T) \backslash \sigma_w(T)$. Then $\lambda_0 \notin [\sigma_1(T) \cup \rho_a(T)]$. By the perturbation theorem of semi-Fredholm operators, we know that $\lambda_0 \notin \overline{\sigma_2(T)}$, then $\lambda_0 \notin [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)]$. Thus $\lambda_0 \notin \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]$, which means that $T - \lambda_0 I$ is Browder, and hence $\lambda_0 \in \pi_{00}^a(T)$. For the converse, let $\lambda_0 \in \pi_{00}^a(T)$. It is easy to see that $\lambda_0 \notin \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]$. Then $\lambda_0 \notin \sigma_b(T)$, that is $\lambda_0 \in \sigma(T) \backslash \sigma_w(T)$. This shows that $\sigma(T) \backslash \sigma_w(T) = \pi_{00}^a(T)$ and property (ω') holds for T. For the a-isoloid, let $\lambda_0 \in \mathrm{iso}\,\sigma_a(T)$ and suppose that $n(T - \lambda_0 I) = 0$, then $\lambda_0 \notin \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]$. This induces that $T - \lambda_0 I$ is Browder and $n(T - \lambda_0 I) = 0$. Thus $T - \lambda_0 I$ is invertible. It is in contradiction to the fact that $\lambda_0 \in \sigma(T)$.

Suppose that T is a-isoloid and property (ω') holds for T. The inclusion $\sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)] \subseteq \sigma_b(T)$ is clear. For the converse inclusion, let $\lambda_0 \notin \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]$. Then $n(T - \lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $T - \lambda I \in SF_+^-(H)$ and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Also, $\lambda_0 \notin [\rho_a(T) \cap \sigma(T)]$ and $\lambda_0 \notin [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)]$. Without loss of generality, we suppose that $\lambda_0 \notin \rho_a(T)$, that is $\lambda_0 \in \sigma_a(T)$. There are two cases to consider.

Case 1 Suppose $\lambda_0 \notin \overline{\sigma_2(T)}$. Then $T - \lambda I$ is CFI operator if $0 < |\lambda - \lambda_0|$ is small sufficiently. But since $T - \lambda I \in SF_+^-(H)$ and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Then $T - \lambda I$ is Weyl if $0 < |\lambda - \lambda_0|$ is sufficiently small by Lemma 2.1. Since property (ω') holds for T, it follows that $T - \lambda I$ is Browder. Then $N(T - \lambda I) = N(T - \lambda I) \cap \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n] = \{0\}$ (see [10, Lemma 3.4]), which means that $T - \lambda I$ is invertible. This proves that $\lambda_0 \in \mathrm{iso}\,\sigma(T)$. Using the fact that T is a-isoloid and $n(T - \lambda_0 I) < \infty$ we know that $\lambda_0 \in \pi_{00}^a(T)$. Since property (ω') holds for T, it follows that $T - \lambda_0 I$ is Browder. Then $\lambda_0 \notin \sigma_b(T)$.

Case 2 Suppose $\lambda_0 \notin \mathrm{acc}\,\sigma_a(T)$. Then $\lambda_0 \in \mathrm{iso}\,\sigma_a(T)$ and hence $\lambda_0 \in \pi_{00}^a(T)$. Using the same way, we know that $\lambda_0 \notin \sigma_b(T)$. □

In the same way, we can prove

Corollary 2.1 *$T \in B(H)$ satisfies property (ω') $\Leftrightarrow$ $\sigma_b(T) = \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.*

If $\sigma_2(T) = \emptyset$, then $\rho_a(T) \cap \sigma(T) = \emptyset$. Thus:

Corollary 2.2 *Suppose $\sigma_2(T) = \emptyset$, then*

(1) T is a-isoloid and property (ω') holds for T $\Leftrightarrow$ $\sigma_b(T) = \sigma_1(T)$;

(2) Property (ω') holds for T $\Leftrightarrow$ $\sigma_b(T) = \sigma_1(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

"T is a-isoloid" is essential in Theorem 2.1. For example, $T \in B(\ell^2)$ is defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots),$$

then $\sigma(T) = \sigma_w(T) = \sigma_a(T) = \{0\}$ and $\pi_{00}^a(T) = \emptyset$, which shows that T has property (ω') and T is not a-isoloid. But we know that $\sigma_b(T) = \{0\}$ and $\sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \mathrm{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)] = \emptyset$, that is $\sigma_b(T) \neq \sigma_1(T) \cup [\overline{\sigma_2(T)} \cap \text{acc}\,\sigma_a(T)] \cup [\rho_a(T) \cap \sigma(T)]$.

Theorem 2.2 *Suppose $T \in B(H)$ is a-isoloid and property (ω') holds for T, then the following statements are equivalent:*

(1) For any $f \in H(T)$, property (ω') holds for $f(T)$;

(2) For any $f \in H(T)$, $\sigma_w(f(T)) = f(\sigma_w(T))$, and $\sigma(T) = \sigma_w(T)$ or $\sigma(T) = \sigma_a(T)$;

(3) For each pair λ, $\mu \in \mathbb{C}\backslash\sigma_e(T)$, $\text{ind}(T-\lambda I)\text{ind}(T-\mu I) \geq 0$, and $\sigma(T) = \sigma_w(T)$ or $\sigma(T) = \sigma_a(T)$.

Proof (1) $\Rightarrow$ (2). $\sigma_w(f(T))) \subseteq f(\sigma_w(T))$ is clear. We need to prove $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Suppose $\mu_0 \notin \sigma_w(f(T))$, then $f(T) - \mu_0 I$ is Weyl. Let

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Thus $T - \lambda_i I$ is Fredholm operator and $\mu_0 \notin \sigma(f(T))$ or $\mu_0 \in \sigma(f(T))\backslash\sigma_w(f(T))$. If $\mu_0 \notin \sigma(f(T))$, then $f(T) - \mu_0 I$ is invertible, which means that each $T - \lambda_i I$ is invertible. Then $\mu_0 \notin f(\sigma_w(T))$. If $\mu_0 \in \sigma(f(T))\backslash\sigma_w(f(T))$, since property (ω') holds for $f(T)$, we know that $f(T) - \mu_0 I$ is Browder. Hence $T - \lambda_i I$ is Browder and $\lambda_i \notin \sigma_w(T)$. Then $\mu_0 \notin f(\sigma_w(T))$.

Next we will prove if $\sigma(T) \neq \sigma_w(T)$, then $\sigma(T) = \sigma_a(T)$. Let $\lambda_0 \in \sigma(T)\backslash\sigma_w(T)$. Then $T - \lambda_0 I$ is Browder because property (ω') holds for T. For any $\mu_0 \notin \sigma_a(T)$, let $f(T) = (T - \mu_0 I)(T - \lambda_0 I)$. Then $f(T)$ is an upper semi-Fredholm operator with $asc(f(T)) < \infty$ and $n(f(T)) > 0$. Thus $0 \in \pi^a_{00}(f(T))$. Since $f(T)$ satisfies property (ω'), $f(T)$ is Browder. This implies that $T - \mu_0 I$ is Browder. Using the fact that $T - \mu_0 I$ is bounded from below, we know that $T - \mu_0 I$ is invertible. Then we prove that $\sigma(T) = \sigma_a(T)$ if $\sigma(T) \neq \sigma_w(T)$.

(2) $\Rightarrow$ (1). If $\sigma(T) = \sigma_w(T)$, then $\pi^a_{00}(T) = \emptyset$ since T satisfies property (ω'). In this case, $\sigma(f(T)) = f(\sigma(T)) = f(\sigma_w(T)) = \sigma_w(f(T))$ and $\pi^a_{00}(f(T)) = \emptyset$. Then $\sigma(f(T))\backslash\sigma_w(f(T)) = \pi^a_{00}(f(T))$, which means that $f(T)$ satisfies property (ω'). In the following, we suppose that $\sigma(T) \neq \sigma_w(T)$, then $\sigma(T) = \sigma_a(T)$. Let $\mu_0 \in \sigma(f(T))\backslash\sigma_w(f(T))$. Then $f(T) - \mu_0 I$ is Weyl and $n(f(T) - \mu_0 I) > 0$. Let

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Since $\sigma_w(f(T)) = f(\sigma_w(T))$ and $\mu_0 \notin \sigma_w(f(T))$, it follows that $\lambda_i \notin \sigma_w(T)$. Then $T - \lambda_i I$ is Weyl. Since property (ω') holds for T, it follows that $T - \lambda_i I$ is Browder. Then $f(T) - \mu_0 I$ is Browder, thus $\mu_0 \in \pi^a_{00}(f(T))$. Conversely, let $\mu_0 \in \pi^a_{00}(f(T))$ and let $f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T)$, where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Let $T - \lambda_i I$ is bounded from below if $1 \leq i \leq j$ and $\lambda_i \in \sigma_a(T)$ if $j < i \leq k$. Then $T - \lambda_i I$ is invertible if $1 \leq i \leq j$ since $\sigma(T) = \sigma_a(T)$. If $j < i \leq k$, then $\lambda_i \in \pi^a_{00}(T)$. Since T has property (ω'), $T - \lambda_i I$ is Browder. Thus $f(T) - \mu_0 I$ is Browder and $\mu_0 \in \sigma(f(T))\backslash\sigma_w(f(T))$. Hence property (ω') holds for $f(T)$.

(2) $\Leftrightarrow$ (3). By Theorem 2 in [2], we can get the result. □

If $\sigma_1(T) = \sigma_b(T)$, using the perturbation theorem of semi-Fredholm operators, we can prove that $\sigma_a(T) = \sigma(T)$ and $\text{ind}(T - \lambda I) \geq 0$ for any $\lambda \in \mathbb{C}\backslash\sigma_e(T)$. By Corollary 2.2 and Theorem 2.2, we get:

Corollary 2.3 *If* $\sigma_1(T) = \sigma_b(T)$, *then property* (ω') *holds for* $f(T)$ *for any* $f \in H(T)$.

From Theorem 1.10 in [11], for each pair λ, $\mu \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\text{ind}(T - \lambda I)\text{ind}(T - \mu I) \geq 0$ if and only if for any $f \in H(T)$, $f(\sigma_1(T)) \subseteq \sigma_1(f(T))$.

Corollary 2.4 *Suppose that* $T \in B(H)$ *is a-isoloid and property* (ω') *holds for* T. *If for any* $f \in H(T)$, $f(\sigma_1(T)) \subseteq \sigma_1(f(T))$, *then for any* $f \in H(T)$, *property* (ω') *holds for* $f(T)$ *if and only if* $\sigma(T) = \sigma_w(T)$ *or* $\sigma(T) = \sigma_a(T)$.

If $\sigma_2(T) = \emptyset$, we have that $\sigma_e(T) = \sigma_w(T)$ and $\sigma(T) = \sigma_a(T)$. Then $f(\sigma_1(T)) \subseteq \sigma_1(f(T))$ for any $f \in H(T)$. In this case, if $T \in B(H)$ is a-isoloid and property (ω') holds for T, by Corollary 2.2, $\sigma_b(T) = \sigma_1(T)$. Then for any $f \in H(T)$, property (ω') holds for $f(T)$ (Corollary 2.3) and $f(\sigma_1(T)) = f(\sigma_b(T)) = \sigma_b(f(T)) \supseteq \sigma_1(f(T))$.

Corollary 2.5 *Suppose that* $T \in B(H)$ *is a-isoloid and property* (ω') *holds for* T. *If* $\sigma_2(T) = \emptyset$, *then*

(1) $f(\sigma_1(T)) = \sigma_1(f(T))$ *for any* $f \in H(T)$;

(2) For any $f \in H(T)$, *property* (ω') *holds for* $f(T)$.

Oberai [12] has examples showing that the Weyl's theorem for T is not sufficient for the Weyl's theorem for $T + F$ with finite rank F. For property (ω'), it has the same case. For example, let $T = A \oplus I$ acting on $H \oplus H$ with A an injective quasinilpotent operator. It is clear that T satisfies property (ω'). Take any finite rank projection $P \in B(H)$, and let $F = 0 \oplus (-P)$. Then $TF = FT$, but property (ω') fails for $T + F$ because $0 \in \pi^a_{00}(T + F) \cap \sigma_w(T + F)$.

Corollary 2.6 *Suppose that* $T \in B(H)$ *is a-isoloid and property* (ω') *holds for* T. *If* $F \in B(H)$ *is a finite rank operator commuting with* T *and* $\sigma_a(T) = \sigma_a(T + F)$, *then* $T + F$ *is a-isoloid and property* (ω') *holds for* $T + F$.

Proof By Theorem 2.1, we need to prove that $\sigma_b(T+F) \subseteq \sigma_1(T+F) \cup [\overline{\sigma_2(T + F)} \cap \text{acc}\,\sigma_a(T+F)] \cup [\rho_a(T+F) \cap \sigma(T+F)]$. Let $\lambda_0 \notin \sigma_1(T+F) \cup [\overline{\sigma_2(T + F)} \cap \text{acc}\,\sigma_a(T+F)] \cup [\rho_a(T+F) \cap \sigma(T+F)]$. Without loss of generality, we suppose that $\lambda_0 \in \sigma_a(T+F)$. Then $n(T+F-\lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $T + F - \lambda I \in SF^-_+(H)$ and $N(T + F - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T + F - \lambda I)^n]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Also $\lambda_0 \notin [\overline{\sigma_2(T + F)} \cap \text{acc}\,\sigma_a(T+F)]$. Then $n(T - \lambda_0 I) < \infty$. If $\lambda_0 \notin \overline{\sigma_2(T + F)}$, we can prove that $T + F - \lambda I$ is Weyl if $0 < |\lambda - \lambda_0|$ is small enough. Then $T - \lambda I$ is Weyl. Since property (ω') holds for T, we know that $T - \lambda I$ is Browder. This shows that $T + F - \lambda I$ is Browder. Then we can get that $T + F - \lambda I$ is invertible. Now we get $\lambda_0 \in \text{iso}\,\sigma(T + F)$. Thus $\lambda_0 \in \text{iso}\,\sigma_a(T)$. The fact that T is a-isoloid tells us that $\lambda_0 \in \pi^a_{00}(T)$. Since property (ω') holds for T, it follows that $T - \lambda_0 I$ is Browder. Then $T + F - \lambda_0 I$ is Browder, that is $\lambda_0 \notin \sigma_b(T + F)$. If $\lambda_0 \notin \text{acc}\,\sigma_a(T + F)$, we can prove that $\lambda_0 \in \text{iso}\,\sigma(T)$. Again, we get that $\lambda_0 \notin \sigma_b(T + F)$. □

For finite rank operator F commuting with T, we know $\sigma_b(T+F) = \sigma_b(T)$. If $\sigma_1(T) = \sigma_b(T)$, we claim that $\sigma_1(T+F) = \sigma_b(T+F)$. In fact, let $\lambda_0 \notin \sigma_1(T+F)$. Then $n(T+F-\lambda_0 I) < \infty$ and there exists $\epsilon > 0$ such that $T+F-\lambda I \in SF_+^-(H)$ and $N(T+F-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T+F-\lambda I)^n)]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Thus $T - \lambda I \in SF_+^-(H)$ and $n(T - \lambda_0 I) < \infty$. By $\sigma_1(T) = \sigma_b(T)$ we know that $T - \lambda I$ is Browder. This induces that $T + F - \lambda I$ is Browder if $0 < |\lambda - \lambda_0| < \epsilon$. Then $T + F - \lambda I$ is invertible. Now we get that $\lambda_0 \in [\mathrm{iso}\,\sigma(T+F) \cup \rho(T+F)]$. We may suppose $\lambda_0 \in \mathrm{iso}\,\sigma(T+F)$. Then $\lambda_0 \in \mathrm{iso}\,\sigma_a(T + F)$. Using Corollary 2.4 in [11], $\lambda_0 \in \mathrm{iso}\,\sigma_a(T) \cup \rho_a(T)$. Thus $\lambda_0 \notin \sigma_1(T)$. The fact that $\sigma_1(T) = \sigma_b(T)$ implies that $T - \lambda_0 I$ is Browder. Then $T + F - \lambda_0 I$ is Browder, that is $\lambda_0 \notin \sigma_b(T + F)$.

Corollary 2.7 *(1) Suppose $\sigma_1(T) = \sigma_b(T)$. If $F \in B(H)$ is a finite rank operator commuting with T, then $T + F$ is a-isoloid and property (ω') holds for $T + F$;*

(2) If $\sigma_2(T) = \emptyset$, T is a-isoloid and property (ω') holds for T, then for any finite rank operator $F \in B(H)$ commuting with T, $T + F$ is a-isoloid and property (ω') holds for $T + F$.

References

[1] WEYL H. *Über beschränkte quadratische Formen, deren Differenz vollstetig ist* [J]. Rend. Circ. Mat. Palermo, 1909, **27**: 373–392.

[2] HARTE R, LEE W Y. *Another note on Weyl's theorem* [J]. Trans. Amer. Math. Soc., 1997, **349**(5): 2115–2124.

[3] RAKOČEVIČ V. *Operators obeying a-Weyl's theorem* [J]. Rev. Roumaine Math. Pures Appl., 1989, **34**(10): 915–919.

[4] RAKOČEVIČ V. *On a class of operators* [J]. Mat. Vesnik, 1985, **37**(4): 423–426.

[5] CAO Xiaohong. *Weyl spectrum of the products of operators* [J]. J. Korean Math. Soc., 2008, **45**(3): 771–780.

[6] CAO Xiaohong, MENG Bin. *Essential approximate point spectra and Weyl's theorem for operator matrices* [J]. J. Math. Anal. Appl., 2005, **304**(2): 759–771.

[7] DUGGAL B P, KUBRUSLY C. *Weyl's theorems for posinormal operators* [J]. J. Korean Math. Soc., 2005, **42**(3): 529–541.

[8] DUGGAL B P, JEON I H, KIM I H. *On Weyl's theorem for quasi-class A operators* [J]. J. Korean Math. Soc., 2006, **43**(4): 899–909.

[9] LEE W Y. *Weyl's theorem for operator matrices* [J]. Integral Equations Operator Theory, 1998, **32**(3): 319–331.

[10] TAYLOR A E. *Theorems on ascent, descent, nullity and defect of linear operators* [J]. Math. Ann., 1966, **163**: 18–49.

[11] CAO Xiaohong, GUO Maozheng, MENG Bin. *Weyl spectra and Weyl's theorem* [J]. J. Math. Anal. Appl., 2003, **288**(2): 758–767.

[12] OBERAI K K. *On the Weyl spectrum (II)* [J]. Illinois J. Math., 1977, **21**(1): 84–90.