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Shrinking Projection Methods for a Family of Quasi-ϕ-Strict Asymptotically Pseudo-Contractions in Banach Spaces

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Abstract The purpose of this article is to propose a shrinking projection method and prove a strong convergence theorem for a family of quasi-ϕ-strict asymptotically pseudo-contractions. Its results hold in reflexive, strictly convex, smooth Banach spaces with the property (K). The results of this paper improve and extend the results of Matsushita and Takahashi, Marino and Xu, Zhou and Gao and others.

Keywords strong convergence; a family of quasi-ϕ-strict asymptotically pseudo-contractions; generalized projection; shrinking projection method; Banach spaces.

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1. Introduction

It is well known that, in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. Consequently, in order to obtain strong convergence, one has to modify the normal Mann's iteration algorithm, and the so called hybrid projection iteration method is such a modification.

The hybrid projection iteration algorithm (HPIA) was introduced initially by Haugazeau [1] in 1968. For 40 years, (HPIA) has received rapid developments. For details, the readers are referred to papers [2–8] and the references therein.

Recently, Zhou and Gao [9] proposed the following shrinking projection method which was simpler than (HPIA) for quasi-ϕ-strict pseudo-contractions in the setting of reflexive, strictly convex, smooth Banach spaces with the property (K). To be more precise, they proved the following theorem:

Theorem 1.1 ([9]) *Let X be a reflexive, strictly convex, smooth Banach space such that X and*

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X^* have the property (K). Assume C is a nonempty closed convex subset of X. Let $T : C \to C$ be a closed and quasi-ϕ-strict pseudo-contraction. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : \phi(x_n, Tx_n) \le \dfrac{2}{1-\kappa}\langle x_n - z, Jx_n - JTx_n\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \ge 0, \end{cases} \tag{1.1}$$

where $\kappa \in [0, 1)$. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_{F(T)}x_0$.

At this point, we put forth the following questions

Question 1.1 Can Theorem 1.1 be extended from a single quasi-ϕ-strict pseudo-contraction to a family of asymptotically quasi-ϕ-strict pseudo-contractions which are more general than quasi-ϕ-strict pseudo-contraction?

The purpose of this article is to give some affirmative answers to Question 1.1 mentioned above, introduce a shrinking projection method and prove a strong convergence theorem for quasi-ϕ-strict asymptotically pseudo-contractions by using new analysis techniques in the setting of reflexive, strictly convex, smooth Banach spaces with the property (K). The results of this paper mainly improve and extend the results of [3, 6, 7, 9] and others.

2. Preliminaries

In this paper, we denote by X and X^* a Banach space and the dual space of X, respectively. We denote by $\mathbb{N}$ the set of positive integers and by R the set of real numbers. We denote by $\to$ and $\rightharpoonup$ strong and weak convergence, respectively. Let C be a nonempty closed convex subset of X. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) = \{j \in X^* : \langle x, j\rangle = \|x\|^2 = \|j\|^2\},$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between X and X^*. It is well known that if X is reflexive and smooth, then $J : X \to X^*$ is single-valued, surjective and demi-continuous.

It is also well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [10] recently introduced a generalized projection operator Π_C in a Banach space X which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that X is a real smooth and strictly convex Banach space. Let us consider the functional defined as in [3] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for} \quad x, y \in X. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

The generalized projection $\Pi_C : X \to C$ is a map that assigns to an arbitrary point $x \in X$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.2}$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [10, 11]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2 \quad \text{for all} \quad x, y \in X \tag{2.3}$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \tag{2.4}$$

for all $x, y, z \in X$.

Remark 2.1 ([9]) If X is a reflexive, strictly convex and smooth Banach space, then for $x, y \in X$, $\phi(x, y) = 0$ if and only if $x = y$.

Let C be a closed convex subset of X, and T a mapping from C into itself. We use $F(T)$ to denote the fixed point set of T. A point p in C is said to be asymptotic fixed point of T (see [12]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive [3] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

T is said to be a quasi-ϕ-strict pseudo-contraction [9] if there exists a constant $\kappa \in [0, 1)$ and $F(T) \neq \emptyset$ such that

$$\phi(p, Tx) \le \phi(p, x) + \kappa\phi(x, Tx)$$

for all $x \in C$ and $p \in F(T)$ (in this case, we also say that T is a quasi-ϕ-κ-strict pseudo-contraction). In particular, T is said to be quasi-ϕ-nonexpansive if $\kappa = 0$ and T is said to be quasi-ϕ-pseudo-contractive if $\kappa = 1$.

T is said to be a quasi-ϕ-strict asymptotically pseudo-contraction if there exist a constant $\kappa \in [0, 1)$ and some real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ and $F(T) \neq \emptyset$ such that

$$\phi(p, T^n x) \le k_n^2 \phi(p, x) + \kappa\phi(x, T^n x)$$

for all $x \in C$, $p \in F(T)$ and all $n \in \mathbb{N}$, where $\{k_n\}$ is said to be an asymptotic sequence of T. In this case, we also say that T is a quasi-ϕ-κ-strict asymptotically pseudo-contraction.

Remark 2.2 It is clear that every quasi-ϕ-strict pseudo-contraction is a quasi-ϕ-strict asymptotically pseudo-contraction with a constant sequence $\{1\}$.

Remark 2.3 Every relatively nonexpansive mapping is a quasi-ϕ-strict asymptotically pseudo-contraction, but the converse may be not true.

Example 2.1 Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space X onto a nonempty closed convex subset C of X. Then, Π_C is a closed and quasi$-\phi-$strict asymptotically pseudo-contraction from X onto C with $F(\Pi_C) = C$ but not a relatively nonexpansive mapping. The class of relatively nonexpansive mappings [3] requires the strong restriction: $\widetilde{F(T)} = F(T)$.

Recall that a Banach space X has the property (K) if for any sequence $\{x_n\} \subset X$ and $x \in X$, if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. For more information concerning property (K) the reader is referred to [13] and references cited there.

A mapping $T : C \to C$ is said to be asymptotically regular on C if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0, \quad \forall x \in C.$$

The following lemmas are crucial for the proofs of the main results in this paper.

Lemma 2.1 ([10]) *Let C be a nonempty closed convex subset of a smooth Banach space X, $x_0 \in C$ and $x \in X$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0 \ \text{ for } \ y \in C.$$

Lemma 2.2 ([10]) *Let X be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of X and let $x \in X$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \ \text{ for all } \ y \in C.$$

Now we are in a position to prove the main results of this paper.

3. Main results

Theorem 3.1 *Let X be a reflexive, strictly convex, smooth Banach space such that X and X^* have the property(K). Assume C is a nonempty bounded closed convex subset of X. Let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-κ_i-strict asymptotically pseudo-contractions such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i $(i \in I)$ is asymptotically regular on C. Define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{cases} x_0 \in X \ \text{ chosen arbitrarily}, \\ C_{1,i} = C, C_1 = \bigcap\limits_{i\in I} C_{1,i}, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(x_n, T_i^n x_n) \leq \dfrac{2}{1-\kappa_i}\langle x_n - z, Jx_n - JT_i^n x_n\rangle + \dfrac{k_{n,i}^2 - 1}{1-\kappa_i} M\}, \\ C_{n+1} = \bigcap\limits_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \geq 0, \end{cases} \tag{3.1}$$

where I is an index set, $\kappa_i \in [0, 1)$, $\{k_{n,i}\}$ is an asymptotic sequence for T_i and $M = \sup\{\phi(x, y), x, y \in C\}$. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_F x_0$.

We remark that the sets $\{k_{n,i}\}$ and $\{\kappa_i\}$ in Theorem 3.1 are nets when I is an infinite uncountable set; otherwise, they are all sequences of real numbers.

Proof We split the proof into seven steps.

Step 1. Show that $\Pi_F x_0$ is well defined for every $x_0 \in X$.

To this end, we prove first that $F(T_i)$ is closed and convex for each $i \in I$. Let $\{p_n\}$ be a sequence in $F(T_i)$ with $p_n \to p$ as $n \to \infty$. We prove that $p \in F(T_i)$. From the definition of T_i, one has

$$\phi(p_n, T_i p) \le k_{1,i}^2 \phi(p_n, p) + \kappa_i \phi(p, T_i p),$$

where $1 \le k_{1,i} < \infty$. In view of (2.4), we obtain

$$\phi(p_n, p) + \phi(p, T_i p) + 2\langle p_n - p, Jp - JT_i p\rangle \le k_{1,i}^2 \phi(p_n, p) + \kappa_i \phi(p, T_i p),$$

i.e.,

$$\phi(p, T_i p) \le \frac{2}{1-\kappa_i}\langle p - p_n, Jp - JT_i p\rangle + \frac{k_{1,i}^2 - 1}{1-\kappa_i}\phi(p_n, p). \tag{3.2}$$

Take limits on the both sides of (3.2). Since $p_n \to p$ as $n \to \infty$, we have $\phi(p, T_i p) = 0$, which implies that $p = T_i p$. Hence $F(T_i)$ is closed, for all $i \in I$.

We next show that $F(T_i)$ is convex. To this end, for arbitrary p_1, $p_2 \in F(T_i)$, $t \in (0, 1)$, put $p_t = tp_1 + (1-t)p_2$. It suffices to show that $T_i p_t = p_t$. Indeed, in view of definitions of $\phi(x, y)$ and T_i, we have

$$\begin{aligned}
\phi(p_t, T_i^n p_t) =& \|p_t\|^2 - 2\langle p_t, JT_i^n p_t\rangle + \|T_i^n p_t\|^2 \\
=& \|p_t\|^2 - 2t\langle p_1, JT_i^n p_t\rangle - 2(1-t)\langle p_2, JT_i^n p_t\rangle + \|T_i^n p_t\|^2 \\
=& \|p_t\|^2 + t\phi(p_1, T_i^n p_t) + (1-t)\phi(p_2, T_i^n p_t) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
\le& \|p_t\|^2 + t[k_{n,i}^2 \phi(p_1, p_t) + \kappa_i \phi(p_t, T_i^n p_t)] + \\
& (1-t)[k_{n,i}^2 \phi(p_2, p_t) + \kappa_i \phi(p_t, T_i^n p_t)] - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
=& \|p_t\|^2 + \kappa_i \phi(p_t, T_i^n p_t) - t\|p_1\|^2 - (1-t)\|p_2\|^2 + \\
& k_{n,i}^2 t\|p_1\|^2 + k_{n,i}^2 (1-t)\|p_2\|^2 - k_{n,i}^2 \|p_t\|^2.
\end{aligned}$$

This implies that

$$(1-\kappa_i)\phi(p_t, T_i^n p_t) \le (k_{n,i}^2 - 1)(t\|p_1\|^2 + (1-t)\|p_2\|^2 - \|p_t\|^2).$$

Since $k_{n,i} \to 1$ and $\kappa_i \in [0, 1)$, we have $\phi(p_t, T_i^n p_t) \to 0$ as $n \to \infty$. Note that $0 \le (\|p_t\| - \|T_i^n p_t\|)^2 \le \phi(p_t, T_i^n p_t)$. Hence $\|T_i^n p_t\| \to \|p_t\|$ and consequently $\|J(T_i^n p_t)\| \to \|Jp_t\|$. This implies that $\{J(T_i^n p_t)\}$ is bounded. Since X is reflexive, X^* is also reflexive. So we can assume that

$$J(T_i^n p_t) \to f_0 \in X^* \tag{3.3}$$

weakly. On the other hand, in view of the reflexivity of X, one has $J(X) = X^*$, which means that for $f_0 \in X^*$, there exists $x \in X$, such that $J(x) = f_0$. Note that

$$\begin{aligned}
\phi(p_t, T_i^n p_t) &= \|p_t\|^2 - 2\langle p_t, J(T_i^n p_t)\rangle + \|T_i^n p_t\|^2 \\
&= \|p_t\|^2 - 2\langle p_t, J(T_i^n p_t)\rangle + \|J(T_i^n p_t)\|^2.
\end{aligned}$$

Using weakly lower semi-continuity of $\|\cdot\|^2$ and (3.3), we have

$$\begin{aligned}\liminf_{n\to\infty}\phi(p_t,T_i^np_t) &= \liminf_{n\to\infty}(\|p_t\|^2-2\langle p_t,J(T_i^np_t)\rangle+\|J(T_i^np_t)\|^2)\\ &\geq \|p_t\|^2-2\langle p_t,f_0\rangle+\|f_0\|^2=\|p_t\|^2-2\langle p_t,Jx\rangle+\|Jx\|^2\\ &=\phi(p_t,x).\end{aligned}$$

From $\phi(p_t,T_i^np_t)\to 0$ as $n\to\infty$, we have $\phi(p_t,x)=0$ and consequently $p_t=x$, which implies that $f_0=Jp_t$. Hence

$$J(T_i^np_t)\to Jp_t\in X^*$$

weakly. Since $\|J(T_i^np_t)\|\to\|Jp_t\|$ and X^* has the property (K), we have

$$\|J(T_i^np_t)-Jp_t\|\to 0. \tag{3.4}$$

Noting that $J^{-1}:X^*\to X$ is demi-continuous, we have

$$T_i^np_t\to p_t$$

weakly. Since $\|T_i^np_t\|\to\|p_t\|$, by using the property (K) of X, we have

$$\|T_i^np_t-p_t\|\to 0. \tag{3.5}$$

Since T_i is asymptotically regular, we have $T_i(T_i^np_t)=T_i^{n+1}p_t\to p_t$ as $n\to\infty$. Since T_i is closed, we see that $p_t=T_ip_t$. Hence $F(T_i)$ is closed and convex for each $i\in I$ and consequently $F=\bigcap_{i\in I}F(T_i)$ is closed and convex. By our assumption that $F=\bigcap_{i\in I}F(T_i)\neq\emptyset$, we have Π_Fx_0 is well defined for every $x_0\in X$.

Step 2. Show that C_n is closed and convex for all $n\geq 1$.

It suffices to show that for each $i\in I$, $C_{n,i}$ is closed and convex for every $n\geq 1$. This can be proved by induction on n. In fact, for $n=1$, $C_{1,i}=C$ is closed and convex. Assume that $C_{n,i}$ is closed and convex for some $n\geq 1$. For $z\in C_{n+1,i}\subset C_{n,i}$, one obtains that

$$\phi(x_n,T_i^nx_n)\leq\frac{2}{1-\kappa_i}\langle x_n-z,Jx_n-JT_i^nx_n\rangle+\frac{k_{n,i}^2-1}{1-\kappa_i}M.$$

It is easy to see that $C_{n+1,i}$ is closed and convex. Then, for all $n\geq 1$, $C_{n,i}$ is closed and convex. Consequently, $C_n=\bigcap_{i\in I}C_{n,i}$ is closed and convex for all $n\geq 1$.

Step 3. Show that $F\subset C_n$, for all $n\geq 1$.

It suffices to show that for each $i\in I, F\subset C_{n,i}$. It is obvious that $F\subset C=C_{1,i}$. Suppose that $F\subset C_{n,i}$ for some $n\geq 1$. For any $p'\in F$, one has $p'\in C_{n,i}$. By using the definition of T_i, we have

$$\phi(p',T_i^nx_n)\leq k_{n,i}^2\phi(p',x_n)+\kappa_i\phi(x_n,T_i^nx_n).$$

In view of (2.4), we can obtain

$$\begin{aligned}\phi(x_n,T_i^nx_n) &\leq\frac{2}{1-\kappa_i}\langle x_n-p',Jx_n-JT_i^nx_n\rangle+\frac{k_{n,i}^2-1}{1-\kappa_i}\phi(p',x_n)\\ &\leq\frac{2}{1-\kappa_i}\langle x_n-p',Jx_n-JT_i^nx_n\rangle+\frac{k_{n,i}^2-1}{1-\kappa_i}M\end{aligned}$$

which implies that $p' \in C_{n+1,i}$ and hence $F \subset C_{n,i}$ for all $n \geq 1$ and $i \in I$. Therefore, $F \subset \bigcap_{n=1}^{\infty} C_n = D \neq \emptyset$.

Step 4. Show that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.

From Lemma 2.2, one has

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0),$$

for each $w \in F \subset C_n$ and for all $n \geq 1$. Therefore, the sequence $\{\phi(x_n, x_0)\}$ is bounded. On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one has $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ for all $n \geq 1$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists.

Step 5. Show that $x_n \to p_0$ as $n \to \infty$, where $p_0 = \Pi_D x_0$.

By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. By Lemma 2.2, we have

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.7}$$

Letting $m, n \to \infty$ in (3.7), one has $\phi(x_m, x_n) \to 0$ as $n \to \infty$. Hence $|\|x_m\| - \|x_n\|| \to 0$. This implies that $\{\|x_n\|\}$ is cauchy and hence $\{x_n\}$ is bounded. Since X is reflexive, without loss of generality, we can assume that $x_n \to p_0$ weakly as $n \to \infty$. It is easy to show that $p_0 \in C_n$ for all $n \geq 1$. Hence $p_0 \in \bigcap_{n=1}^{\infty} C_n = D$. Noticing that $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \phi(p_0, x_0)$, and using weakly lower semi-continuity of $\phi(\cdot, x_0)$, we have

$$\phi(p_0, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(p_0, x_0),$$

which implies that $\phi(x_n, x_0) \to \phi(p_0, x_0)$ as $n \to \infty$. Hence $\|x_n\| \to \|p_0\|$. By the property (K) of X, we have $x_n \to p_0$. Hence $Jx_n \to Jp_0$ weakly. Noting that $x_n = \Pi_{C_n} x_0$ and $D \subset C_n$, from Lemma 2.1, we have

$$\langle x_n - y, Jx_0 - Jx_n \rangle \geq 0 \quad \text{for any} \quad y \in D.$$

Taking the limit as $n \to \infty$ yields

$$\langle p_0 - y, Jx_0 - Jp_0 \rangle \geq 0 \quad \text{for any} \quad y \in D,$$

which implies that $p_0 = \Pi_D x_0$.

Step 6. Show that $p_0 = T_i p_0$.

We prove first that $\{T_n^i x_n\}$ is bounded. Indeed, taking $p \in F \subset C_{n+1} \subset C_{n+1,i}$, we have

$$\phi(x_n, T_i^n x_n) \leq \frac{2}{1-\kappa_i} \langle x_n - p, Jx_n - JT_i^n x_n \rangle + \frac{k_{n,i}^2 - 1}{1-\kappa_i} M,$$

i.e.,

$$\|x_n\|^2 - 2\langle x_n, JT_i^n x_n \rangle + \|T_i^n x_n\|^2 \leq \frac{2}{1-\kappa_i} \|x_n - p\| (\|x_n\| + \|T_i^n x_n\|) + \frac{k_{n,i}^2 - 1}{1-\kappa_i} M.$$

It follows that

$$\|T_i^n x_n\|^2 \leq \frac{2}{1-\kappa_i} \|x_n - p\| \|x_n\| - \|x_n\|^2 + (\frac{2}{1-\kappa_i} \|x_n - p\| + 2\|x_n\|) \|T_i^n x_n\| + \frac{k_{n,i}^2 - 1}{1-\kappa_i} M.$$

Since $\{x_n\}$ is bounded, we obtain that $\{T_i^n x_n\}$ is bounded. From $x_{n+1} \in C_{n+1}$, one has

$$\phi(x_n, T_i^n x_n) \leq \frac{2}{1-\kappa_i}\langle x_n - x_{n+1}, Jx_n - JT_i^n x_n\rangle + \frac{k_{n,i}^2 - 1}{1-\kappa_i}M. \tag{3.8}$$

By Step 5, we obtain that $x_n \to p_0$ and hence $x_{n+1} - x_n \to 0$. Taking limits on the both sides of (3.8), from $\lim_{n\to\infty} k_{n,i} \to 1$, we obtain that $\phi(x_n, T_i^n x_n) \to 0$ as $n \to \infty$. Note that $0 \leq (\|x_n\| - \|T_i^n x_n\|)^2 \leq \phi(x_n, T_i^n x_n)$. Hence $\|T_i^n x_n\| \to \|p_0\|$ and consequently $\|J(T_i^n x_n)\| \to \|Jp_0\|$. This implies that $\{J(T_i^n x_n)\}$ is bounded. Since X is reflexive, X^* is also reflexive. So we can assume that

$$J(T_i^n x_n) \to f_0 \in X^* \tag{3.9}$$

weakly. On the other hand, in view of the reflexivity of X, one has $J(X) = X^*$, which means that for $f_0 \in X^*$, there exists $x \in X$, such that $Jx = f_0$. Note that

$$\begin{aligned}\phi(x_n, T_i^n x_n) &= \|x_n\|^2 - 2\langle x_n, J(T_i^n x_n)\rangle + \|T_i^n x_n||^2 \\ &= \|x_n\|^2 - 2\langle x_n, J(T_i^n x_n)\rangle + \|J(T_i^n x_n)||^2.\end{aligned} \tag{3.10}$$

Using weakly lower semi-continuity of $\|\cdot\|^2$ and (3.9), we have

$$\begin{aligned}\liminf_{n\to\infty} \phi(x_n, T_i^n x_n) &\geq \liminf_{n\to\infty}(\|p_0\|^2 - 2\langle p_0, f_0\rangle + \|f_0||^2) \\ &= \|p_0\|^2 - 2\langle p_0, Jx\rangle + \|Jx||^2 = \phi(p_0, x).\end{aligned}$$

It follows from $\phi(x_n, T_i^n x_n) \to 0$ that $\phi(p_0, x) = 0$ and consequently $p_0 = x$, which implies that $f_0 = Jp_0$. Hence

$$J(T_i^n x_n) \to Jp_0 \in X^*$$

weakly. Since $\|J(T_i^n x_n)\| \to \|Jp_0\|$ and X^* has the property (K), we have

$$\|J(T_i^n x_n) - Jp_0\| \to 0.$$

Noting that $J^{-1} : X^* \to X$ is demi-continuous, we have $T_i^n x_n \to p_0 \in X$ weakly. Since $\|T_i^n x_n\| \to \|p_0\|$ and X has the property(K), we obtain that $T_i^n x_n \to p_0$ as $n \to \infty$. By using the asymptotic regularity of T_i, we have $T_i^{n+1} x_n \to p_0$. Hence $T_i(T_i^n x_n) \to p_0$. From the closeness property of T_i, we have $T_i p_0 = p_0$. Therefore, $p_0 \in F = \bigcap_{i\in I} F(T_i)$.

Step 7. Show that $p_0 = \Pi_F x_0$.

It follows from $p_0 = \Pi_D x_0$ and $F \subset D$ that

$$\phi(p_0, x_0) \leq \phi(\Pi_F x_0, x_0) \leq \phi(p_0, x_0),$$

which implies that $\phi(\Pi_F x_0, x_0) = \phi(p_0, x_0)$. Hence, $p_0 = \Pi_F x_0$. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_F x_0$. This completes the proof. □

Remark 3.1 In Theorem 3.1, if we take $I = \{1, 2, \ldots, N\}$, $\mathbf{N}$ and R^+, respectively, then we obtain corresponding convergence theorems for a finite, an infinite countable and an infinite uncountable families of quasi-ϕ-strict asymptotically pseudo-contractions. Theorem 3.1 improves and extends Theorem 1.1 from a single closed and quasi-ϕ-strict pseudo-contraction to a family of closed and quasi-ϕ-strict asymptotically pseudo-contractions. Theorem 3.1 presents an affirmative answer to Question 1.1.

Remark 3.2 Theorem 3.1 improves and extends relative results of [3, 6, 7] in several aspects:

(i) Uniform convex and uniform smooth Banach spaces or Hilbert spaces are extended to reflexive, strictly convex, smooth Banach spaces with the property (K);

(ii) Relatively nonexpansive mappings or strict pseudo-contractions are extended to closed and quasi-ϕ-strict asymptotically pseudo-contractions;

(iii) Our Algorithm (3.1) is simpler than the ones used in [3, 6, 7].

From Theorem 3.1, we deduce the following corollaries immediately.

Corollary 3.1 *Let X be a reflexive, strictly convex, smooth Banach space such that X and X^* have the property (K). Assume C is a nonempty bounded closed convex subset of X. Let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-κ_i-strict asymptotically pseudo-contractions such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i $(i \in I)$ is asymptotically regular on C. Define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{cases} x_0 \in X \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, C_1 = \bigcap\limits_{i\in I} C_{1,i}, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(x_n, T_i^n x_n) \leq 2\langle x_n - z, Jx_n - JT_i^n x_n\rangle + (k_{n,i}^2 - 1)M\}, \\ C_{n+1} = \bigcap\limits_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \geq 0, \end{cases} \tag{3.11}$$

where $\{k_{n,i}\}$ is an asymptotic sequence for $\{T_i\}_{i\in I}$ and $M = \sup\{\phi(x,y),\ x, y \in C\}$. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_F x_0$.

Corollary 3.2 *Let X be a reflexive, strictly convex, smooth Banach space such that X and X^* have the property (K). Assume C is a nonempty bounded closed convex subset of X. Let $T : C \to C$ be a closed and quasi-ϕ-κ-strict asymptotically pseudo-contraction such that $F(T) \neq \emptyset$. Assume that every T is asymptotically regular on C. Define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{cases} x_0 \in X \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : \phi(x_n, T^n x_n) \leq \dfrac{2}{1-\kappa}\langle x_n - z, Jx_n - JT^n x_n\rangle + \dfrac{k_n^2 - 1}{1-\kappa} M\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \geq 0, \end{cases} \tag{3.12}$$

where $\kappa \in [0, 1)$, $\{k_n\}$ is an asymptotic sequence for T and $M = \sup\{\phi(x,y),\ x, y \in C\}$. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_{F(T)} x_0$.

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