

Zero-Divisor Semigroups of Fan-Shaped Graph

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Abstract Let P_n be a path graph with n vertices, and let $F_n = P_n \cup \{c\}$, where c is adjacent to all vertices of P_n . The resulting graph is called a fan-shaped graph. The corresponding zero-divisor semigroups have been completely determined by Tang et al. for $n = 2, 3, 4$ and by Wu et al. for $n \geq 6$, respectively. In this paper, we study the case for $n = 5$, and give all the corresponding zero-divisor semigroups of F_n .

Keywords commutative zero-divisor semigroup; refinements of a star graph; fan-shaped graph; center.

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1. Introduction

Let S be a commutative semigroup with a zero element 0 . The zero-divisor graph of S is denoted by $\Gamma(S)$, a simple and undirected graph. The vertices of $\Gamma(S)$ are all the nonzero zero-divisors of S , and there is an edge between two distinct vertices a and b if and only if $ab = 0$. In 2005, DeMeyer and DeMeyer [1, Theorem 3] proved that a refinement of a star graph is the graph of a semigroup. Zero-divisor graphs of commutative semigroups have been studied in several papers, such as [2–8].

Let P_n be a path graph with n vertices. We use $F_n = P_n \cup \{c\}$ to denote a graph where c is adjacent to all vertices of P_n . It is easy to see that F_n is a refinement of a star graph, so F_n has corresponding semigroups. When $n = 2$, $F_2 = K_3$, and G has 4 corresponding mutually non-isomorphic semigroups [8]. If $n = 3$, then G is exactly $K_3 - e$, and G corresponds to 12 mutually non-isomorphic commutative semigroups [4]. If $n = 4$, Su and Tang in [3] proved that there are 47 mutually non-isomorphic semigroups corresponding to F_4 . If n is greater than or equal to 6, Wu et al. proved in [7] that F_n has $g(n)$ corresponding mutually non-isomorphic

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semigroups, where

$$g(n) = \begin{cases} \frac{1}{2}(2^n + 2^{\frac{n}{2}}), & \text{if } n \text{ is even,} \\ \frac{1}{2}(2^n + 2^{\frac{(n+1)}{2}}), & \text{if } n \text{ is odd.} \end{cases}$$

The main purpose of this paper is to determine all the non-isomorphic zero-divisor semigroups of F_5 .

For any semigroup S , let T be the set of all zero-divisors of S . Then T is an ideal of S and in particular, it is also a semigroup with the property that all of elements of T are zero-divisors of the semigroup. We call such semigroups zero-divisor semigroups. Obviously we have $\Gamma(S) \cong \Gamma(T)$. Throughout this paper, all semigroups considered are commutative zero-divisor semigroups. Let S be a semigroup with zero element 0 , where $0x = x$ for all $x \in S$. If $x \in S$, we denote $\text{ann}(x) = \{y \in S \mid xy = 0\}$. For a graph G , we use $V(G)$ to denote the vertex set of G . For any two distinct vertices a, b of G , if a is adjacent to b , we denote it as $a - b$.

2. Lemmas

We need the following two lemmas to prove our main theorem.

Lemma 2.1 *Let S be a commutative zero-divisor semigroup with $\Gamma(S) = F_5$. Then $c^2 = 0$.*

Proof If $c^2 \neq 0$, then by [8, Lemma 2.11], $S \setminus \{c\}$ is an ideal of S such that $\Gamma(S \setminus \{c\}) = F_5$. This is impossible by [1, Theorem 1]. \square

Lemma 2.2 *Let S be a zero-divisor semigroup with $\Gamma(S) = F_5$, where $V(P_5) = \{a_1, b_1, b_2, b_3, a_2\}$ with $a_1 - b_1 - b_2 - b_3 - a_2$. Then the following holds:*

- (1) $a_1b_3 = b_1a_2 = b_1b_3 = c$;
- (2) $a_1b_2, b_2a_2, a_1a_2 \in \{b_2, c\}$;
- (3) $b_1^2, b_3^2 \in \{c, 0\}$;
- (4) $a_1^2, a_2^2, b_2^2 \in \{b_2, c, 0\}$.

Proof (1) Since $a_1b_3 \in \text{ann}(b_1) \cap \text{ann}(b_2) \cap \text{ann}(a_2) = \{0, c\}$, $a_1b_3 = c$. Similarly, $a_2b_1 \in \text{ann}(b_3) \cap \text{ann}(b_2) \cap \text{ann}(a_1) = \{0, c\}$ and $b_1b_3 \in \text{ann}(a_1) \cap \text{ann}(b_2) \cap \text{ann}(a_2) = \{0, c\}$, therefore $a_2b_1 = b_1b_3 = c$.

(2) Since $a_1b_2 \in \text{ann}(b_1) \cap \text{ann}(b_3) = \{0, c, b_2\}$, $a_1b_2 \in \{c, b_2\}$. Similarly, $a_2b_2 \in \{c, b_2\}$ and $a_1a_2 \in \{c, b_2\}$.

(3) From (1) we have $a_1b_3^2 = 0$ and $a_2b_1^2 = 0$, so $b_1^2, b_3^2 \in \text{ann}(a_1) \cap \text{ann}(a_2) = \{0, c\}$.

(4) From (1) we have $a_1^2b_3 = 0$ and $a_2^2b_1 = 0$, so $a_1^2, a_2^2 \in \text{ann}(b_1) \cap \text{ann}(b_3) = \{0, c, b_2\}$. Clearly, $b_2^2 \in \text{ann}(b_1) \cap \text{ann}(b_3) = \{0, c, b_2\}$. \square

3. Main results

Now, we can prove our main theorem of this paper.

Theorem 3.1 *The simple graph F_5 has 26 non-isomorphic corresponding zero-divisor semi-*

groups.

Proof Let $S = \{0, c, a_1, a_2, b_1, b_2, b_3\}$ be a zero-divisor semigroup with $\Gamma(S) = F_5$, where $V(F_5) = \{c, a_1, b_1, b_2, b_3, a_2\}$. And let the path P_5 be $a_1 - b_1 - b_2 - b_3 - a_2$. Then by Lemma 2.2(4), we have three cases to discuss according to the value of b_2^2 . Notice that semigroups in different cases are not isomorphic.

Case 1 Suppose $b_2^2 = b_2$. If $a_1b_2 = c$, then $a_1b_2^2 = cb_2$, so $a_1b_2 = 0$, a contradiction. Hence, $a_1b_2 = b_2$. Similarly, $a_2b_2 = b_2$. If $a_1a_2 = c$, then $a_1a_2b_2 = cb_2$, this implies $a_1b_2 = 0$, a contradiction. So $a_1a_2 = b_2$. We can obtain that $a_1^2 = a_2^2 = b_2$ by the same way. If $b_1^2 = b_2^2 = c$, then we obtain the following multiplicative table (Table 1). If $b_1^2 = b_3^2 = 0$, then we obtain the following new table (Table 2). If $b_1^2 = 0, b_2^2 = c$, then we obtain a new table (Table 3). If $b_1^2 = c, b_2^2 = 0$, then the table is isomorphic to Table 3.

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	b_2	c	0
a_2		b_2	c	b_2	0	0
b_1			c	0	c	0
b_2				b_2	0	0
b_3					c	0
c						0

Table 1

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	b_2	c	0
a_2		b_2	c	b_2	0	0
b_1			0	0	c	0
b_2				b_2	0	0
b_3					0	0
c						0

Table 2

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	b_2	c	0
a_2		b_2	c	b_2	0	0
b_1			c	0	c	0
b_2				b_2	0	0
b_3					0	0
c						0

Table 3

Multiplicative tables (1-3) of zero-divisor semigroups of F_5

Case 2 Suppose $b_2^2 = c$. If $a_1b_2 = b_2$, then $b_2^2 = a_1b_2^2 = a_1c = 0$, a contradiction. So $a_1b_2 = c$. Similarly, $a_2b_2 = c$ and $a_1a_2 = c$. If $a_1^2 = b_2$, then $a_2b_2 = a_1^2a_2 = a_1(a_1a_2) = a_1c = 0$, a contradiction. So $a_1^2 \in \{0, c\}$ by Lemma 2.2(4). Similarly, $a_2^2 \in \{0, c\}$. In this case, we have following ten tables (Tables 4-13) according to the cardinal numbers of $\{a_i|a_i^2 = 0, i = 1, 2\}$ and $\{b_i|b_i^2 = 0, i = 1, 2\}$.

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 4

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 5

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 6

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 7

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 8

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 9

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 10

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 11

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 12

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 13

Multiplicative tables (4-13) of zero-divisor semigroups of F_5

Case 3 Suppose $b_2^2 = 0$. In this case, if $a_1b_2 = b_2$, then $a_1a_1b_2 = a_1b_2$, so $(a_1^2)b_2 = b_2$, so $a_1^2 \neq 0$ and $a_1^2 \neq c$, therefore $a_1^2 = b_2$. So $b_2^2 = a_1^2b_2 = a_1b_2 = b_2$, a contradiction. Hence, $a_1b_2 = c$. If $a_2b_2 = b_2$, then $a_1b_2 = a_2a_1b_2 = a_2c = 0$, a contradiction. So $a_2b_2 = c$. We have two subcases to discuss via the value of a_1a_2 .

Subcase 3.1 Let $a_1a_2 = b_2$. Then $a_1^2a_2 = a_1b_2 = c$, so $a_1^2 = b_2$. Similarly, $a_2^2 = b_2$. In this case, we have three tables (Table 14–16) on S similar to Case 1.

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	c	c	0
a_2		b_2	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 14

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	c	c	0
a_2		b_2	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 15

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	c	c	0
a_2		b_2	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 16

Multiplicative tables (14-16) of zero-divisor semigroups of F_5

Subcase 3.2 Let $a_1a_2 = c$. Then $a_1^2a_2 = a_1c = 0$, so $a_1^2 \in \{0, c\}$. Similarly, $a_2^2 \in \{0, c\}$. In this case, we have ten tables (Table 17–26) on S similar to Case 2.

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 17

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 18

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 19

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 20

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 21

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 22

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 23

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 24

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 25

\cdot	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 26

Multiplicative tables (17-26) of zero-divisor semigroups of F_5

The final work is to verify that all of above twenty-six tables are associative tables. This work was done via the program written by Nong Minqiang. This completes the proof. \square

From the above proof, we can see that the nilpotent semigroups corresponding to the fan-shaped graph are exactly the semigroups determined in Cases 2 and 3. Thus we have the following:

Corollary 3.2 *There exist 23 nilpotent semigroups corresponding to the simple graph F_5 .*

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