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Zero-Divisor Semigroups of Fan-Shaped Graph

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Abstract Let P_n be a path graph with n vertices, and let $F_n = P_n \cup \{c\}$, where c is adjacent to all vertices of P_n. The resulting graph is called a fan-shaped graph. The corresponding zero-divisor semigroups have been completely determined by Tang et al. for $n = 2, 3, 4$ and by Wu et al. for $n \geq 6$, respectively. In this paper, we study the case for $n = 5$, and give all the corresponding zero-divisor semigroups of F_n.

Keywords commutative zero-divisor semigroup; refinements of a star graph; fan-shaped graph; center.

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1. Introduction

Let S be a commutative semigroup with a zero element 0. The zero-divisor graph of S is denoted by $\Gamma(S)$, a simple and undirected graph. The vertices of $\Gamma(S)$ are all the nonzero zero-divisors of S, and there is an edge between two distinct vertices a and b if and only if $ab = 0$. In 2005, DeMeyer and DeMeyer [1, Theorem 3] proved that a refinement of a star graph is the graph of a semigroup. Zero-divisor graphs of commutative semigroups have been studied in several papers, such as [2–8].

Let P_n be a path graph with n vertices. We use $F_n = P_n \cup \{c\}$ to denote a graph where c is adjacent to all vertices of P_n. It is easy to see that F_n is a refinement of a star graph, so F_n has corresponding semigroups. When $n = 2$, $F_2 = K_3$, and G has 4 corresponding mutually non-isomorphic semigroups [8]. If $n = 3$, then G is exactly $K_3 - e$, and G corresponds to 12 mutually non-isomorphic commutative semigoups [4]. If $n = 4$, Su and Tang in [3] proved that there are 47 mutually non-isomorphic semigroups corresponding to F_4. If n is greater than or equal to 6, Wu et al. proved in [7] that F_n has $g(n)$ corresponding mutually non-isomorphic

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semigroups, where

$$g(n) = \begin{cases} \frac{1}{2}(2^n + 2^{\frac{n}{2}}), & \text{if } n \text{ is even,} \\ \frac{1}{2}(2^n + 2^{\frac{(n+1)}{2}}), & \text{if } n \text{ is odd.} \end{cases}$$

The main purpose of this paper is to determine all the non-isomorphic zero-divisor semigroups of F_5.

For any semigroup S, let T be the set of all zero-divisors of S. Then T is an ideal of S and in particular, it is also a semigroup with the property that all of elements of T are zero-divisors of the semigroup. We call such semigroups zero-divisor semigroups. Obviously we have $\Gamma(S) \cong \Gamma(T)$. Throughout this paper, all semigroups considered are commutative zero-divisor semigroups. Let S be a semigroup with zero element 0, where $0x = x$ for all $x \in S$. If $x \in S$, we denote $\mathrm{ann}(x) = \{y \in S | xy = 0\}$. For a graph G, we use $V(G)$ to denote the vertex set of G. For any two distinct vertices a, b of G, if a is adjacent to b, we denote it as $a - b$.

2. Lemmas

We need the following two lemmas to prove our main theorem.

Lemma 2.1 *Let S be a commutative zero-divisor semigroup with $\Gamma(S) = F_5$. Then $c^2 = 0$.*

Proof If $c^2 \neq 0$, then by [8, Lemma 2.11], $S \setminus \{c\}$ is an ideal of S such that $\Gamma(S \setminus \{c\}) = P_5$. This is impossible by [1, Theorem 1]. □

Lemma 2.2 *Let S be a zero-divisor semigroup with $\Gamma(S) = F_5$, where $V(P_5) = \{a_1, b_1, b_2, b_3, a_2\}$ with $a_1 - b_1 - b_2 - b_3 - a_2$. Then the following holds:*

(1) $a_1b_3 = b_1a_2 = b_1b_3 = c$;

(2) $a_1b_2, b_2a_2, a_1a_2 \in \{b_2, c\}$;

(3) $b_1^2, b_3^2 \in \{c, 0\}$;

(4) $a_1^2, a_2^2, b_2^2 \in \{b_2, c, 0\}$.

Proof (1) Since $a_1b_3 \in \mathrm{ann}(b_1) \cap \mathrm{ann}(b_2) \cap \mathrm{ann}(a_2) = \{0, c\}$, $a_1b_3 = c$. Similarly, $a_2b_1 \in \mathrm{ann}(b_3) \cap \mathrm{ann}(b_2) \cap \mathrm{ann}(a_1) = \{0, c\}$ and $b_1b_3 \in \mathrm{ann}(a_1) \cap \mathrm{ann}(b_2) \cap \mathrm{ann}(a_2) = \{0, c\}$, therefore $a_2b_1 = b_1b_3 = c$.

(2) Since $a_1b_2 \in \mathrm{ann}(b_1) \cap \mathrm{ann}(b_3) = \{0, c, b_2\}$, $a_1b_2 \in \{c, b_2\}$. Similarly, $a_2b_2 \in \{c, b_2\}$ and $a_1a_2 \in \{c, b_2\}$.

(3) From (1) we have $a_1b_3^2 = 0$ and $a_2b_1^2 = 0$, so $b_1^2, b_3^2 \in \mathrm{ann}(a_1) \cap \mathrm{ann}(a_2) = \{0, c\}$.

(4) From (1) we have $a_1^2b_3 = 0$ and $a_2^2b_1 = 0$, so $a_1^2, a_2^2 \in \mathrm{ann}(b_1) \cap \mathrm{ann}(b_3) = \{0, c, b_2\}$. Clearly, $b_2^2 \in \mathrm{ann}(b_1) \cap \mathrm{ann}(b_3) = \{0, c, b_2\}$. □

3. Main results

Now, we can prove our main theorem of this paper.

Theorem 3.1 *The simple graph F_5 has 26 non-isomorphic corresponding zero-divisor semi-*

groups.

Proof Let $S = \{0, c, a_1, a_2, b_1, b_2, b_3\}$ be a zero-divisor semigroup with $\Gamma(S) = F_5$, where $V(F_5) = \{c, a_1, b_1, b_2, b_3, a_2\}$. And let the path P_5 be $a_1 - b_1 - b_2 - b_3 - a_2$. Then by Lemma 2.2(4), we have three cases to discuss according to the value of b_2^2. Notice that semigroups in different cases are not isomorphic.

Case 1 Suppose $b_2^2 = b_2$. If $a_1b_2 = c$, then $a_1b_2^2 = cb_2$, so $a_1b_2 = 0$, a contradiction. Hence, $a_1b_2 = b_2$. Similarly, $a_2b_2 = b_2$. If $a_1a_2 = c$, then $a_1a_2b_2 = cb_2$, this implies $a_1b_2 = 0$, a contradiction. So $a_1a_2 = b_2$. We can obtain that $a_1^2 = a_2^2 = b_2$ by the same way. If $b_1^2 = b_2^2 = c$, then we obtain the following multiplicative table (Table 1). If $b_1^2 = b_3^2 = 0$, then we obtain the following new table (Table 2). If $b_1^2 = 0$, $b_2^2 = c$, then we obtain a new table (Table 3). If $b_1^2 = c$, $b_2^2 = 0$, then the table is isomorphic to Table 3.

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	b_2	c	0
a_2		b_2	c	b_2	0	0
b_1			c	0	c	0
b_2				b_2	0	0
b_3					c	0
c						0

Table 1

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	b_2	c	0
a_2		b_2	c	b_2	0	0
b_1			0	0	c	0
b_2				b_2	0	0
b_3					0	0
c						0

Table 2

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	b_2	c	0
a_2		b_2	c	b_2	0	0
b_1			c	0	c	0
b_2				b_2	0	0
b_3					0	0
c						0

Table 3

Multiplicative tables (1-3) of zero-divisor semigroups of F_5

Case 2 Suppose $b_2^2 = c$. If $a_1b_2 = b_2$, then $b_2^2 = a_1b_2^2 = a_1c = 0$, a contradiction. So $a_1b_2 = c$. Similarly, $a_2b_2 = c$ and $a_1a_2 = c$. If $a_1^2 = b_2$, then $a_2b_2 = a_1^2a_2 = a_1(a_1a_2) = a_1c = 0$, a contradiction. So $a_1^2 \in \{0, c\}$ by Lemma 2.2(4). Similarly, $a_2^2 \in \{0, c\}$. In this case, we have following ten tables (Tables 4-13) according to the cardinal numbers of $\{a_i | a_i^2 = 0, i = 1, 2\}$ and $\{b_i | b_i^2 = 0, i = 1, 2\}$.

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 4

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 5

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 6

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 7

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 8

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 9

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					c	0
c						0

Table 10

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 11

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 12

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				c	0	0
b_3					0	0
c						0

Table 13

Multiplicative tables (4-13) of zero-divisor semigroups of F_5

Case 3 Suppose $b_2^2 = 0$. In this case, if $a_1b_2 = b_2$, then $a_1a_1b_2 = a_1b_2$, so $(a_1^2)b_2 = b_2$, so $a_1^2 \neq 0$ and $a_1^2 \neq c$, therefore $a_1^2 = b_2$. So $b_2^2 = a_1^2b_2 = a_1b_2 = b_2$, a contradiction. Hence, $a_1b_2 = c$. If $a_2b_2 = b_2$, then $a_1b_2 = a_2a_1b_2 = a_2c = 0$, a contradiction. So $a_2b_2 = c$. We have two subcases to discuss via the value of a_1a_2.

Subcase 3.1 Let $a_1a_2 = b_2$. Then $a_1^2a_2 = a_1b_2 = c$, so $a_1^2 = b_2$. Similarly, $a_2^2 = b_2$. In this case, we have three tables (Table 14–16) on S similar to Case 1.

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	c	c	0
a_2		b_2	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 14

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	c	c	0
a_2		b_2	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 15

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	b_2	b_2	0	c	c	0
a_2		b_2	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 16

Multiplicative tables (14-16) of zero-divisor semigroups of F_5

Subcase 3.2 Let $a_1a_2 = c$. Then $a_1^2a_2 = a_1c = 0$, so $a_1^2 \in \{0, c\}$. Similarly, $a_2^2 \in \{0, c\}$. In this case, we have ten tables (Table 17–26) on S similar to Case 2.

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 17

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 18

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 19

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 20

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		c	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 21

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 22

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					c	0
c						0

Table 23

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			c	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 24

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	c	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 25

$\cdot$	a_1	a_2	b_1	b_2	b_3	c
a_1	0	c	0	c	c	0
a_2		0	c	c	0	0
b_1			0	0	c	0
b_2				0	0	0
b_3					0	0
c						0

Table 26

Multiplicative tables (17-26) of zero-divisor semigroups of F_5

The final work is to verify that all of above twenty-six tables are associative tables. This work was done via the program written by Nong Minqiang. This completes the proof. $\Box$

From the above proof, we can see that the nilpotent semigroups corresponding to the fan-shaped graph are exactly the semigroups determined in Cases 2 and 3. Thus we have the following:

Corollary 3.2 *There exist* 23 *nilpotent semigroups corresponding to the simple graph* F_5.

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