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A Liouville-Type Theorem for Higher-Order Parabolic Inequalities and Its Applications

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Abstract In this paper, we establish a Liouville-type theorem for a system of higher-order parabolic inequalities by using the method of test functions and an integral estimate. As an application, we observe the Fujita blow-up phenomena for the corresponding parabolic system, which in particular fills up the gap in the recent result of Pang et. al. (Existence and nonexistence of global solutions for a higher-order semilinear parabolic system, Indiana Univ. Math. J., 55(2006), 1113-1134). Moreover, the importance of this observation is that we do not impose any regularity assumption on the initial data.

Keywords Liouville theorem; parabolic inequalities; Fujita phenomena.

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1. Introduction

In this paper, we consider the nonexistence of nontrivial entire solution to the following high-order parabolic inequalities

$$|u_i|_t + (-\Delta)^m u_i \geq |u_{i+1}|^{p_i}, \quad i = 1, 2, \ldots, k, \ u_{k+1} := u_1, \ x \in R^N, \ t > 0, \tag{1.1}$$

where $m \geq 1$, $p_i > 1$, $N \geq 1$. This non-existence result is naturally called a Liouville theorem as the elliptic type problems. As an application, we observe the Fujita phenomena for the corresponding parabolic system

$$u_{it} + (-\Delta)^m u_i = u_{i+1}^{p_i}, \quad i = 1, 2, \ldots, k, \ u_{k+1} := u_1, \ x \in R^N, \ t > 0. \tag{1.2}$$

We remark that, over the past few years, the nonexistence of global solution to the inequalities attracted the interest of some authors. For instance, Kartsatos and Kurta [1] studied system

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(1.1) in the case of $k = m = 1$. Precisely, they proved that the following inequality

$$|u|_t - \Delta u \geq |u|^p, \quad x \in R^N,\ t > 0 \tag{1.3}$$

has no nontrivial solution in $R^N \times (0,\infty)$ if $1 < p \leq 1 + \frac{2}{N}$, and then, as an application, obtained the well-known results of Fujita [2] and Hayakawa [3]. Recently, Jiang and Zheng [4] studied the similar problem to inequality (1.3) but with double degenerate. For the system case, Mitidieri and Pohozaev [5] studied the following semilinear system

$$u_t - \Delta u \geq |v|^p,\ v_t - \Delta v \geq |u|^q, \quad x \in R^N,\ t > 0$$

with initial data $u_0, v_0 \in L^1_{\mathrm{loc}}(R^N)$ and $p, q > 1$. They obtained the nonexistence of nontrivial global solutions in the case of $\max\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\} \geq \frac{N}{2}$. Recently, Xiang et al. [6] studied the system (1.1) with $m = 1$, got a Liouville-type theorem, and observed the Fujita blow-up phenomena for the corresponding parabolic system. For the more studies in parabolic differential inequalities, we refer to [5, 7, 8]. In particular, [5] presented a general approach and concerned a large class of equations and inequalities.

On the other hand, the higher-order semilinear inequalities or equations with $m > 1$ appear in numerous problems in applications such as the flame propagation, the bi-stable phase transition and the higher-order diffusion. Over the past few years, the study of positive solution to the higher-order equations received considerable attention. The interested reader is referred to the monograph [9]. Here we only mention the works [10, 11]. In [10], Galaktionov and Pohozaev considered the Cauchy problem of the equation $u_t + (-\Delta)^m u = |u|^p$ with initial data $u_0 \in L^1(R^N) \bigcap L^\infty(R^N)$. By a comparison with self-similar solutions of majorizing order-preserving integral equations, they got the Fujita-type result for this equation. For the system case, Pang et al. [11] studied the system

$$\begin{aligned} &u_t + (-\Delta)^m u = |v|^p,\ v_t + (-\Delta)^m v = |u|^q, \quad x \in R^N,\ t > 0, \\ &u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad x \in R^N \end{aligned} \tag{1.4}$$

with $m > 1$, $p, q \geq 1$ and $pq > 1$ and showed that every solution with initial data having positive average value does not exist globally in time if $\min\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\} \geq \frac{N}{2m}$, while the global solutions with small initial data exist if $\max\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\} < \frac{N}{2m}$. For the more studies in the nonexistence of global solutions to parabolic problems, we refer to the classical book [12] and the interesting surveys [13, 14].

Motivated by the above cited works, we establish a Liouville-type theorem for the higher-order parabolic inequalities system (1.1) by using the method of test functions and an integral estimate. As an application, we observe the Fujita blow-up phenomena for the corresponding parabolic system (1.2), which in particular fills up the gap in the recent result of Pang et. al. [11]. Moreover, the importance of this observation is that we do not impose any regularity assumption on the initial data, which is necessary in [5, 7, 10, 11], but suppose the existence of local L^p weak solutions (see Definition 2.1). Such an existence hypothesis is reasonable, at least, for the initial data considered in [5, 7, 10, 11].

The organization of this paper is as follows. In Section 2, we state some preliminaries and our main results. And then we prove these results in Section 3.

2. Preliminaries and results

Let us now state some preliminaries and the main results. For convenience, we set $p_0 = p_k$, $p_{k+1} = p_1$. We begin with the definition of a weak solution to system (1.1).

Definition 2.1 *We say $(u_1, u_2, \ldots, u_k)$ is a weak solution of system (1.1), if $u_i \in L^{p_{i-1}}_{\rm loc}(R^N \times (0,\infty))$ $(i = 1, 2, \ldots, k)$, and satisfy the integral inequalities*

$$\int_{R^N\times(0,\infty)} -|u_i|\varphi_t + u_i(-\Delta)^m\varphi \mathrm{d}x\mathrm{d}t \geq \int_{R^N\times(0,\infty)} |u_{i+1}|^{p_i}\varphi \mathrm{d}x\mathrm{d}t, \tag{2.1}$$

for any nonnegative function $\varphi \in C_0^\infty(R^N \times (0,\infty))$.

We shall say $(u_1, u_2, \ldots, u_k)$ is a weak solution of system (1.2) if (2.1) with inequalities replaced by equations holds for any $\varphi \in C_0^\infty(R^N \times (0,\infty))$.

The next definition concerns the boundedness of a weak solution from below.

Definition 2.2 *The weak solution $(u_1, u_2, \ldots, u_k)$ of system (1.1) or (1.2) is said to be bounded below by a positive constant on $R^N \times (0,\infty)$ if there exists a constant $\lambda > 0$ such that $u_i(x,t) \geq \lambda$, a.e. $(x,t) \in R^N \times (0,\infty)$, for some $i \in \{1, 2, \ldots, k\}$.*

Then we introduce some useful symbols. Let $(\alpha_1, \alpha_2, \ldots, \alpha_k)^{\rm T}$ be the unique solution of the algebraic system

$$\begin{pmatrix} -1 & p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & p_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & p_{k-1} \\ p_k & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_{k-1} \\ \alpha_k \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \\ 1 \end{pmatrix}.$$

One easily sees $p_i\alpha_{i+1} = \alpha_i + 1$ $(i = 1, 2, \ldots, k)$. A series of standard computations yield

$$\alpha_i = \frac{1 + p_i + p_ip_{i+1} + \cdots + p_ip_{i+1}...p_kp_1...p_{i-2}}{p_1p_2...p_k - 1}, \quad i = 1, 2, 3, \ldots, k.$$

For fixed R, we will set the time-space domains

$$\Sigma_1 := \left\{(x,t) \in R^N \times R_+ : 0 < |x|^{4m} + t^2 < R^4\right\}$$

and

$$\Sigma_2 := \left\{(x,t) \in R^N \times R_+ : 0 < |x|^{4m} + t^2 < 2R^4\right\},$$

respectively. Then $\Sigma_{1(\tau)} := \{(x,t) \in \Sigma_1 : t > \tau\}$.

Our main results read as follows.

Theorem 2.1 (Liouville-type Theorem) *Assume $(u_1, u_2, \ldots, u_k)$ is a weak solution of system (1.1). If $\max\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \geq \frac{N}{2m}$, then $u_i(x,t) = 0$, a.e. $(x,t) \in R^N \times (0,\infty)$ $(i = 1, 2, \ldots, k)$.*

Noticing that a nonnegative global weak solution of system (1.2) is a weak solution of system (1.1), we observe the following well-known Fujita-type result as an application of the above

Liouville-type theorem.

Corollary 2.1 *If* $\max\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \geq \frac{N}{2m}$, *then there exists no nontrivial nonnegative global weak solution to system (1.2).*

Remark The above result is consistent with that of [1, 5, 6, 10] if one takes special k, m and p_i. In particular, for $k = 2$, we improve the results of [11] in the sense of obtaining the precise Fujita critical curve. In fact, as mentioned in the Introduction, Pang et.al. [11] has shown that there exists nontrivial nonnegative global weak solution for $\max\{\alpha_1, \alpha_2\} < \frac{N}{2m}$, while there is no nontrivial nonnegative global weak solution for $\min\{\alpha_1, \alpha_2\} \geq \frac{N}{2m}$. Thus, Corollary 2.1 fills the gap and suggests that the critical Fujita curve is described by $\max\{\alpha_1, \alpha_2\} = \frac{N}{2m}$ and this curve belongs to the global non-existence case. Furthermore, unlike the usual parabolic problems, we have not taken the regularity of initial data into account.

Next, we establish a priori estimate for weak solutions of system (1.1), which is of independent interest. And then from this estimate, we obtain a corollary, which completes the above nonexistence results in some sense.

Theorem 2.2 (Universal Local L^{p_i} Estimates) *For any* $\tau > 0$, *there exists a positive constant* $C(\tau, N, p_i)$, *which depends only on* τ, N *and* p_i $(i = 1, 2, \ldots, k)$, *such that any weak solution* $(u_1, u_2, \ldots, u_k)$ *of system (1.1) satisfies*

$$\int_{\Sigma_{1(\tau)}} |u_{i+1}|^{p_i} \mathrm{d}x\mathrm{d}t \leq C(\tau, N, p_i) R^{2+\frac{N}{m}-2p_i\alpha_{i+1}}, \quad i = 1, 2, \ldots, k$$

for any fixed $R > 0$.

Remark We say that the above estimate is universal since the constant C is independent of the initial data at $t = 0$. This estimate is true for the nonnegative weak solution of system (1.2).

Corollary 2.1 says that the unique global weak solution is the trivial weak solution 0 under appropriate parameter restriction, while the following corollary shows that, though there could exist global weak solutions in the remainder parameter region, every component of the global weak solutions has a positive measure set in which it closes to 0.

Corollary 2.2 *There exist no weak solutions of systems (1.1) and (1.2) on* $R^N \times (0, \infty)$ *bounded below by a positive constant.*

3. Proof of main results

To give the proof of Theorems 2.1 and 2.2, we choose a suitable test function as [1, 6]. Firstly, we take functions χ, η satisfying

$\chi : [0, \infty) \to [0, 1]$ is smooth such that $\chi = 1$ on $[0, 1]$, $\chi = 0$ on $[2, \infty)$;

$\eta : [0, \infty) \to [0, 1]$ is smooth such that $\eta = 0$ on $[0, \varepsilon]$, $\eta = 1$ on $[2\varepsilon, \infty)$, and $\eta' \geq 0$,

for any fixed ε, and define

$$\xi(x, t) = \chi^s\Big(\frac{|x|^{4m} + t^2}{R^4}\Big), \quad s \gg 1 \tag{3.1}$$

for any fixed $R > 0$. We take further χ such that $\chi', \chi'', \ldots, \chi^{2m}$ are bounded.

Here and later, we denote by C the positive constants, which depend only on N, p_i $(i = 1, 2, \ldots, k)$. Then for such ξ, η, s, we have the following proposition, which plays an important role in the proof of Theorems 2.1 and 2.2.

Proposition 3.1 *There exists a positive constant C, which depends only on N and p_i $(i = 1, 2, \ldots, k)$, such that the weak solution $(u_1, u_2, \ldots, u_k)$ of system (1.1) satisfies*

$$\int_{\Sigma_2} |u_i|^{p_{i-1}} \xi\eta^2 \mathrm{d}x\mathrm{d}t \leq C\Big(\int_{\Sigma_2\setminus\Sigma_1} |u_i|^{p_{i-1}} \xi\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_1p_2\cdots p_k}} R^{(2+\frac{N}{m})\frac{(p_1p_2\cdots p_k-1)}{p_1p_2\cdots p_k}-2\frac{1+p_i+p_ip_{i+1}\cdots+p_ip_{i+1}\cdots p_kp_1\cdots p_{i-2}}{p_ip_{i+1}\cdots p_kp_1\cdots p_{i-2}}},$$

$i = 1, 2, ..., k$, where ξ, η are defined as the above.

Proof Without loss of generality, we only prove the case $i = 2$, since the estimates for other u_i $(i = 1, 3, 4, ..., k)$, can be similarly proved. Take the test function $\varphi(x, t) = \xi(x, t)\eta^2(t)$ in (2.1). Then, we have

$$\int_{\Sigma_2} |u_2|^{p_1} \xi\eta^2 \mathrm{d}x\mathrm{d}t \leq \int_{\Sigma_2} -|u_1|(\xi_t\eta^2 + 2\xi\eta\eta') + u_1\eta^2(-\Delta)^m\xi \mathrm{d}x\mathrm{d}t. \tag{3.2}$$

Recalling $0 \leq \xi, \eta \leq 1, \eta' \geq 0$ and using Hölder inequalities, we conclude from (3.2) that

$$\begin{aligned}
&\int_{\Sigma_2} |u_2|^{p_1} \xi\eta^2 \mathrm{d}x\mathrm{d}t \leq \int_{\Sigma_2} |u_1|\eta^2 \left(|\xi_t| + |(-\Delta)^m\xi|\right) \mathrm{d}x\mathrm{d}t\\
&\leq C\Big(\int_{\Sigma_2\setminus\Sigma_1} |u_1|^{p_k} \xi\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_k}} \Big[\int_{\Sigma_2} \Big(|\xi_t|\xi^{-\frac{1}{p_k}}\eta^{2-\frac{2}{p_k}}\Big)^{\frac{p_k}{p_k-1}} \mathrm{d}x\mathrm{d}t\Big]^{\frac{p_k-1}{p_k}} +\\
&\quad C\Big(\int_{\Sigma_2\setminus\Sigma_1} |u_1|^{p_k} \xi\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_k}} \Big[\int_{\Sigma_2} \Big(|(-\Delta)^m\xi|\xi^{-\frac{1}{p_k}}\eta^{2-\frac{2}{p_k}}\Big)^{\frac{p_k}{p_k-1}} \mathrm{d}x\mathrm{d}t\Big]^{\frac{p_k-1}{p_k}}\\
&\leq C\Big(\int_{\Sigma_2\setminus\Sigma_1} |u_1|^{p_k} \xi\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_k}} \times\\
&\quad \Big[\Big(\int_{\Sigma_2} |\xi_t|^{\frac{p_k}{p_k-1}}\xi^{-\frac{1}{p_k-1}}\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{p_k-1}{p_k}} + \Big(\int_{\Sigma_2} |(-\Delta)^m\xi|^{\frac{p_k}{p_k-1}}\xi^{-\frac{1}{p_k-1}}\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{p_k-1}{p_k}}\Big].
\end{aligned}$$

Consider the scaled variables $t = R^2\tau$, $x = R^{1/m}y$, and note that

$$\mathrm{d}x\mathrm{d}t = R^{2+N/m}\mathrm{d}\tau\mathrm{d}y,\ \xi_t = R^{-2}\xi_\tau,\ (-\Delta)_x^m\xi = R^{-2}(-\Delta)_y^m\xi.$$

By a series of computations, we get that there exist positive constants C_1, C_2 independent of R such that

$$\Big(\int_{\Sigma_2} |\xi_t|^{\frac{p_k}{p_k-1}}\xi^{-\frac{1}{p_k-1}}\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{p_k-1}{p_k}} \leq C_1 R^{(2+\frac{N}{m}-\frac{2p_k}{p_k-1})\frac{p_k-1}{p_k}},$$

$$\Big(\int_{\Sigma_2} |(-\Delta)^m\xi|^{\frac{p_k}{p_k-1}}\xi^{-\frac{1}{p_k-1}}\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{p_k-1}{p_k}} \leq C_2 R^{(2+\frac{N}{m}-\frac{2p_k}{p_k-1})\frac{p_k-1}{p_k}}.$$

From these two inequalities we obtain

$$\int_{\Sigma_2} |u_2|^{p_1} \xi\eta^2 \mathrm{d}x\mathrm{d}t \leq C\Big(\int_{\Sigma_2\setminus\Sigma_1} |u_1|^{p_k} \xi\eta^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_k}} R^{(2+\frac{N}{m}-\frac{2p_k}{p_k-1})\frac{p_k-1}{p_k}}. \tag{3.3}$$

Then after a series of similar computations, we can get

$$\int_{\Sigma_2}|u_3|^{p_2}\xi\eta^2\mathrm{d}x\mathrm{d}t\le C\Big(\int_{\Sigma_2\setminus\Sigma_1}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_1}}R^{\left(2+\frac{N}{m}-\frac{2p_1}{p_1-1}\right)\frac{p_1-1}{p_1}},\tag{3.4}$$

$$\cdots$$

$$\int_{\Sigma_2}|u_k|^{p_{k-1}}\xi\eta^2\mathrm{d}x\mathrm{d}t\le C\Big(\int_{\Sigma_2\setminus\Sigma_1}|u_{k-1}|^{p_{k-2}}\xi\eta^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_{k-2}}}R^{\left(2+\frac{N}{m}-\frac{2p_{k-2}}{p_{k-2}-1}\right)\frac{p_{k-2}-1}{p_{k-2}}},$$

$$\int_{\Sigma_2}|u_1|^{p_k}\xi\eta^2\mathrm{d}x\mathrm{d}t\le C\Big(\int_{\Sigma_2\setminus\Sigma_1}|u_k|^{p_{k-1}}\xi\eta^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_{k-1}}}R^{\left(2+\frac{N}{m}-\frac{2p_{k-1}}{p_{k-1}-1}\right)\frac{p_{k-1}-1}{p_{k-1}}}.$$

Combining these inequalities, we have

$$\begin{aligned}
\int_{\Sigma_2}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t&\le C\Big(\int_{\Sigma_2\setminus\Sigma_1}|u_1|^{p_k}\xi\eta^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_k}}R^{\left(2+\frac{N}{m}-\frac{2p_k}{p_k-1}\right)\frac{p_k-1}{p_k}}\\
&\le C\Big(\int_{\Sigma_2\setminus\Sigma_1}|u_k|^{p_{k-1}}\xi\eta^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_{k-1}p_k}}R^{\left(2+\frac{N}{m}-\frac{2p_{k-1}}{p_{k-1}-1}\right)\frac{p_{k-1}-1}{p_{k-1}p_k}}R^{\left(2+\frac{N}{m}-\frac{2p_k}{p_k-1}\right)\frac{p_k-1}{p_k}}\\
&\quad\cdots\\
&\le C\Big(\int_{\Sigma_2\setminus\Sigma_1}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_1p_2\cdots p_k}}R^{\left(2+\frac{N}{m}-\frac{2p_1}{p_1-1}\right)\frac{p_1-1}{p_1p_2\cdots p_k}}\times\\
&\quad\cdots R^{\left(2+\frac{N}{m}-\frac{2p_{k-1}}{p_{k-1}-1}\right)\frac{p_{k-1}-1}{p_{k-1}p_k}}R^{\left(2+\frac{N}{m}-\frac{2p_k}{p_k-1}\right)\frac{p_k-1}{p_k}}\\
&=C\Big(\int_{\Sigma_2\setminus\Sigma_1}|u_2|^{p_1}\xi^s\eta^2\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p_1p_2\cdots p_k}}R^{\left(2+\frac{N}{m}\right)\left(\frac{p_1-1}{p_1p_2\cdots p_k}+\ldots+\frac{p_{k-1}-1}{p_{k-1}p_k}+\frac{p_k-1}{p_k}\right)-2\left(\frac{1}{p_2\cdots p_k}+\ldots+\frac{1}{p_k}+1\right)},
\end{aligned}$$

which implies that the conclusion of Proposition 3.1 holds. □

Proof of Theorem 2.1 Without loss of generality, we take $\alpha_1=\max\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\ge\frac{N}{2m}$. Since $p_1\alpha_2=\alpha_1+1$, we see $2+\frac{N}{m}-2p_1\alpha_2=2(\frac{N}{2m}-\alpha_1)\le0$. Then according to (3.7) in the proof of Theorem 2.2 below, we have

$$\int_{\Sigma_2}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t\le C,\tag{3.5}$$

where C is independent of $R\ge1$. Thus, by monotonicity,

$$\int_{\Sigma_2\setminus\Sigma_1}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t\to0,\quad\text{as } R\to\infty.\tag{3.6}$$

Recalling Proposition 3.1 and using $\alpha_1\ge\frac{N}{2}$ again, we easily obtain $\int_{\Sigma_2}|u_2|^{p_1}\eta^2\mathrm{d}x\mathrm{d}t\to0$ as $R\to\infty$, which implies that $\int_{R^N\times(2\varepsilon,+\infty)}|u_2|^{p_1}\mathrm{d}x\mathrm{d}t=0$. We have $u_2(x,t)=0$, a.e. $(x,t)\in R^N\times(2\varepsilon,+\infty)$. Since ε is arbitrary, we see $u_2(x,t)=0$, a.e. $(x,t)\in R^N\times(0,+\infty)$. Using (3.4), we see $u_3(x,t)=0$, a.e. $(x,t)\in R^N\times(0,+\infty)$. Similarly, $u_4(x,t)=0,\ldots,u_k(x,t)=0,u_1(x,t)=0$, a.e. $(x,t)\in R^N\times(0,+\infty)$. □

Proof of Theorem 2.2 We only prove a priori estimate for the component u_2, since the others can be deduced by similar arguments. It follows from Proposition 3.1 that

$$\Big(\int_{\Sigma_2}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t\Big)^{1-\frac{1}{p_1p_2\cdots p_k}}\le CR^{\left(2+\frac{N}{m}\right)\frac{(p_1p_2\cdots p_k-1)}{p_1p_2\cdots p_k}-2\frac{1+p_i+p_ip_{i+1}\cdots+p_ip_{i+1}\cdots p_kp_1\cdots p_{i-2}}{p_ip_{i+1}\cdots p_kp_1\cdots p_{i-2}}}.$$

Namely,

$$\int_{\Sigma_2}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t\le CR^{2+\frac{N}{m}-2\frac{p_1+p_1p_2+\cdots+p_1p_2\cdots p_k}{p_1p_2\cdots p_k-1}}=CR^{2+\frac{N}{m}-2p_1\alpha_2}.\tag{3.7}$$

Since $\xi(x,t)\equiv 1$ in Σ_1, we conclude $\int_{\Sigma_1}|u_2|^{p_1}\eta^2\mathrm{d}x\mathrm{d}t\le\int_{\Sigma_2}|u_2|^{p_1}\xi\eta^2\mathrm{d}x\mathrm{d}t\le CR^{2+\frac{N}{m}-2p_1\alpha_2}$. Note that $\eta(t)=1$ on $[2\varepsilon,\infty)$. If we set $\varepsilon=\frac{\tau}{2}$, then we have

$$\int_{\Sigma_{1(\tau)}}|u_2|^{p_1}\mathrm{d}x\mathrm{d}t\le C(\tau,N,p_i)R^{2+\frac{N}{m}-2p_1\alpha_2},$$

as desired. □

Proof of Corollary 2.2 We prove this result by a contradiction. Without loss of generality, we suppose on the contrary that there exists a positive constant λ such that $u_2(x,t)\ge\lambda$, a.e. $(x,t)\in R^N\times(0,\infty)$. Then we could obtain from Theorem 2.2 that

$$\lambda^{p_1}|\Sigma_1(1)|\le\int_{\Sigma_1}|u_2|^{p_1}\mathrm{d}x\mathrm{d}t\le CR^{2+\frac{N}{m}-2p_1\alpha_2},$$

for any $R>0$. Therefore $R^{2+\frac{N}{m}}\le CR^{2+\frac{N}{m}-2p_1\alpha_2}$ for any fixed $R>0$, which leads to a contradiction as $R\to\infty$ by C being R independent. □

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