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On Complete Totally Real Pseudo-Umbilical Submanifolds in a Complex Projective Space

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Abstract Let $\mathbf{M}^n$ be a totally real pseudo-umbilical submanifold in a complex projective space $\mathbf{CP}^{n+p}$. In this paper, we study the position of completeness of $\mathbf{M}^n$. By choosing a suitable frame field, we obtain a rigidity theorem such that $\mathbf{M}^n$ becomes totally umbilical submanifold and improve the related results.

Keywords complex projective space; totally real submanifolds; pseudo-umbilical submanifolds; complete.

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1. Introduction

Let CP^{n+p} be a $2(n+p)$-dimensional complex projective space endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4. Let M^n be an n-dimensional submanifold in CP^{n+p}. M^n is called totally real if each tangent space of M^n is mapped into the normal space by the complex structure J of CP^{n+p}. In [1], Du investigated the conditions under which totally real pseudo-umbilical submanifolds must be minimal for $p=0$. When M^n is a compact totally real minimal submanifold, the corresponding pinching theorem was obtained. In [2], when M^n is an n-dimensional compact totally real pseudo-umbilical submanifold with parallel mean curvature in CP^{n+p}, Zhang obtained a pinching theorem about the square of the length of the second fundamental form. In this paper, we study the complete totally real pseudo-umbilical submanifolds in CP^{n+p} for general complex codimension p, and obtain the following theorems.

Theorem *Let M^n be an n-dimensional complete totally real pseudo-umbilical submanifold with parallel mean curvature in CP^{n+p} $(p>0)$. Then either M^n is totally umbilical, or* $\inf\rho\le n(1+H^2)(n-\frac{5}{3})$, *where ρ is the scalar curvature of M^n.*

Corollary *Let M^n be an n-dimensional complete totally real pseudo-umbilical submanifold with*

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parallel mean curvature in CP^{n+p} $(p > 0)$. If the square of the length of second fundamental form $S < \frac{n}{3}(2 + 5H^2)$, where H is the mean curvature of M^n, then M^n is totally umbilical.

2. Basic formulas

Let M^n be an n-dimensional totally real submanifold in CP^{n+p}. Choose a local field of orthonormal frames

$$e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}, e_{1*} = Je_1, \ldots, e_{n^*} = Je_n, e_{(n+1)^*} = Je_{n+1}, \ldots, e_{(n+p)^*} = Je_{n+p}$$

in CP^{n+p}, in such a way that, restricted to M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n. For convenience, we use the following convention on the range of indices:

$$\begin{aligned}
&A, B, C, \ldots, = 1, \ldots, n+p, 1^*, \ldots, (n+p)^*;\\
&i, j, k, \ldots = 1, \ldots, n;\\
&\alpha, \beta, \gamma, \ldots = n+1, \ldots, n+p, 1^*, \ldots, (n+p)^*;\\
&\lambda, \mu \ldots = n+1, \ldots, n+p.
\end{aligned}$$

Let $\{\omega_A\}$ be the dual frames of $\{e_A\}$. Then the structure equations of CP^{n+p} are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D \tag{2.2}$$

where

$$\omega_{ij} = \omega_{i^*j^*}, \quad \omega_{i^*j} = \omega_{j^*i}, \quad \omega_{\lambda\mu} = \omega_{\lambda^*\mu^*},$$

$$\omega_{\lambda^*\mu} = \omega_{\mu^*\lambda}, \quad \omega_{i\mu} = \omega_{i^*\mu^*}, \quad \omega_{i^*\lambda} = \omega_{\lambda^*i}; \tag{2.3}$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}. \tag{2.4}$$

In (2.4), (J_{AB}) is the component of the linear transformation J, i.e.,

$$(J_{AB}) = \begin{pmatrix} 0 & I_{n+p} \\ -I_{n+p} & 0 \end{pmatrix} \tag{2.5}$$

where I_{n+p} denotes the identity matrix of degree $n+p$.

Restricting these forms to M, we have

$$\omega_\alpha = 0, \quad \omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \tag{2.6}$$

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \quad h^{i^*}_{jk} = h^{j^*}_{ik} = h^{k^*}_{ij}, \tag{2.7}$$

$$\begin{cases} d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \\ d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \end{cases} \tag{2.8}$$

$$R_{ijkl} = K_{ijkl} + \sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}), \tag{2.9}$$

$$d\omega_{\alpha\beta} = -\sum_{\gamma}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} + \frac{1}{2}\sum_{k,l}R_{\alpha\beta kl}\omega_k\wedge\omega_l, \tag{2.10}$$

$$R_{\alpha\beta ij} = K_{\alpha\beta ij} + \sum_{k}(h^{\alpha}_{ik}h^{\beta}_{kj} - h^{\alpha}_{jk}h^{\beta}_{ki}), \tag{2.11}$$

where h is the second fundamental form of M^n, and $R_{ijkl}, R_{\alpha\beta ij}$ are the components of the *Riemann* curvature tensor R and the normal curvature tensor $R^{\perp}$ with respect to $\{e_A\}$. The mean curvature vector ξ, the mean curvature H, the square of the length of the second fundamental form S and the scalar curvature ρ of M^n are defined as follows

$$\xi = \frac{1}{n}\sum_{\alpha}(\sum_{i}h^{\alpha}_{ii})e_{\alpha}, \quad H = \parallel \xi \parallel, \quad S = \parallel h \parallel^2, \tag{2.12}$$

$$\rho = n(n-1) + n^2H^2 - S. \tag{2.13}$$

Let h^{α}_{ijk} and h^{α}_{ijkl} be the covariant of h^{α}_{ij}. Then

$$h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = -K_{\alpha ijk} = 0, \tag{2.14}$$

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_{m}(h^{\alpha}_{im}R_{mjkl} + h^{\alpha}_{mj}R_{mikl}) - \sum_{\beta}h^{\beta}_{ij}R_{\alpha\beta kl}. \tag{2.15}$$

Lemma 1 ([3]) *Let $A_1, A_2, \ldots, A_m$ be symmetric $(n\times n)$-matrices, $m \geq 2$. Then*

$$-2\sum_{\alpha,\beta}[\mathrm{tr}(A^2_{\alpha}A^2_{\beta}) - \mathrm{tr}(A_{\alpha}A_{\beta})^2] - \sum_{\alpha,\beta}[\mathrm{tr}(A_{\alpha}A_{\beta})]^2 \geq -\frac{3}{2}S^2.$$

Lemma 2 ([4]) *Let M^n be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below. If F is a C^2-function bounded from above on M^n, then for any $\varepsilon > 0$, there is a point $x \in M^n$ such that $\sup F - \varepsilon < F(x)$, $|\mathrm{grad}\, F| < \varepsilon$, $\Delta F < \varepsilon$.*

Lemma 3 *Let M^n be a totally real pseudo-umbilical submanifold with parallel mean curvature in CP^{n+p} $(p > 0)$. Then the Ricci curvature R_{ii} satisfies $R_{ii} \geq n - 1 + nH^2 - S$.*

Proof From the theorem of [2], we can choose $\xi = He_{n+1}$. With the fact that M^n is pseudo-umbilical, we can get

$$\mathrm{tr}\, H_{\alpha} = \begin{cases} nH, & \alpha = n+1 \\ 0, & \alpha \neq n+1 \end{cases}, \qquad h^{n+1}_{ij} = H\delta_{ij}.$$

From the definition of Ricci curvature and (2.9)

$$\begin{aligned} R_{ii} &= \sum_{j(\neq i)}R_{ijij} = \sum_{j(\neq i)}K_{ijij} + \sum_{\alpha}\sum_{j(\neq i)}[h^{\alpha}_{ii}h^{\alpha}_{jj} - (h^{\alpha}_{ij})^2] \\ &= n - 1 + \sum_{j,\alpha}h^{\alpha}_{ii}h^{\alpha}_{jj} - \sum_{\alpha}(h^{\alpha}_{ii})^2 - \sum_{\alpha}\sum_{j(\neq i)}(h^{\alpha}_{ij})^2 \\ &= n - 1 + nH^2 - \sum_{j,\alpha}(h^{\alpha}_{ij})^2 \geq n - 1 + nH^2 - S. \quad \square \end{aligned}$$

3. Proof of Theorem

Let $\xi = He_{n+1}$. From the fact that M^n is pseudo-umbilical, it follows

$$\operatorname{tr} H_\alpha = \begin{cases} nH, & \alpha = n+1 \\ 0, & \alpha \neq n+1 \end{cases}, \qquad h_{ij}^{n+1} = H\delta_{ij}. \tag{3.1}$$

Set $\tau = \sum_{\substack{i,j \\ \alpha > n+1}} (h_{ij}^\alpha)^2 = S - nH^2$. Since M^n has parallel mean curvature vector, we can get $\sum_k h_{kkij}^\alpha = 0$, and the mean curvature H is a constant.

By using (2.9), (2.11), (2.14), (2.15) and (3.1), it is not difficult to get

$$\begin{aligned}
\frac{1}{2}\Delta\tau &= \sum_{\substack{i,j,k \\ \alpha\neq n+1}} (h_{ijk}^\alpha)^2 + \sum_{\substack{i,j,k \\ \alpha\neq n+1}} h_{ij}^\alpha \Delta h_{ij}^\alpha \\
&= \sum_{\substack{i,j,k \\ \alpha\neq n+1}} (h_{ijk}^\alpha)^2 + \sum_{\substack{i,j,k,m \\ \alpha\neq n+1}} h_{ij}^\alpha (h_{im}^\alpha R_{mkjk} + h_{km}^\alpha R_{mijk}) - \sum_{\substack{i,j,k,\beta \\ \alpha\neq n+1}} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} \\
&= \sum_{\substack{i,j,k \\ \alpha\neq n+1}} (h_{ijk}^\alpha)^2 + \sum_m \operatorname{tr} H_{m^*}^2 + n(1+H^2)\tau + \\
&\quad \sum_{\alpha,\beta\neq n+1} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta\neq n+1} [\operatorname{tr}(H_\alpha H_\beta)]^2.
\end{aligned}$$

From Lemma 1 we can get

$$\begin{aligned}
\frac{1}{2}\Delta\tau &\geq \sum_{\substack{i,j,k \\ \alpha\neq n+1}} (h_{ijk}^\alpha)^2 + \sum_m \operatorname{tr} H_{m^*}^2 + \tau[n(1+H^2) - \frac{3}{2}\tau] \\
&\geq \tau[n(1+H^2) - \frac{3}{2}\tau].
\end{aligned} \tag{3.2}$$

(1) If $\inf \rho \leq n(1+H^2)(n-\frac{5}{3})$, the theorem is correct.

(2) If $\inf \rho > n(1+H^2)(n-\frac{5}{3})$, i.e., $\rho > n(1+H^2)(n-\frac{5}{3})$, by (2.13) we have

$$S = n(n-1) + n^2H^2 - \rho \leq \frac{2}{3}n(1+H^2) + nH^2,$$

then

$$\tau = S - nH^2 \leq \frac{2}{3}n(1+H^2).$$

Since H is a constant, τ is bounded from above. By Lemma 3, we have

$$R_{ii} \geq n - 1 + nH^2 - S \geq n - 1 - \frac{2}{3}n(1+H^2),$$

which means that Ricci curvature is bounded from below. Now we consider the following smooth function on M^n defined by $F = (f+a)^{\frac{1}{2}}$, $f = \tau$, where $a > 0$ is a real number. Obviously, F is bounded from above, so we can apply Lemma 2 to F. For any $\varepsilon > 0$, there is a point $x \in M$ such that

$$F(x) > \sup F - \varepsilon, \quad |\operatorname{grad} F| < \varepsilon, \quad \Delta F < \varepsilon, \tag{3.3}$$

$$dF = \frac{1}{2}(f+a)^{-\frac{1}{2}} df,$$

$$\Delta F = \frac{1}{2}[-\frac{1}{2}(f+a)^{-\frac{3}{2}} \|df\|^2 + (f+a)^{-\frac{1}{2}}\Delta f]$$

$$= \frac{1}{2}[-2\|dF\|^2 + \Delta f](f+a)^{-\frac{1}{2}}$$
$$= \frac{1}{2F}(-2\|dF\|^2 + \Delta f).$$

Therefore, $F\Delta F = -\|dF\|^2 + \frac{1}{2}\Delta f$, i.e.,

$$\frac{1}{2}\Delta f = F\Delta F + \|dF\|^2. \tag{3.4}$$

By (3.3) and (3.4), we have

$$\frac{1}{2}\Delta f < \varepsilon^2 + \varepsilon F = \varepsilon(\varepsilon + F). \tag{3.5}$$

Choose a sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$, as $m \to \infty$. For any ε_m, there exists a point $x_m \in M$ satisfying (3.3). Therefore $\varepsilon_m[\varepsilon_m + F(x_m)] \to 0$, $m \to \infty$. On the other hand, by (3.3)

$$F(x_m) > \sup\, F - \varepsilon_m. \tag{3.6}$$

Because F is bounded from above, $\{F(x_m)\}$ is a bounded sequence. Thus $F(x_m) \to F_0$. By the definition of supremum and (3.6), we have $F_0 = \sup\, F$. Hence the definition of F gives rise to $f(x_m) \to f_0 = \sup\, f$. By (3.2), (3.5), we have

$$f[n(1+H^2) - \frac{3}{2}f] \le \frac{1}{2}\Delta f < \varepsilon^2 + \varepsilon F,$$
$$f(x_m)[n(1+H^2) - \frac{3}{2}f(x_m)] < \varepsilon_m^2 + \varepsilon_m F(x_m) \le \varepsilon_m^2 + \varepsilon_m F_0.$$

Let $m \to \infty$. We get $\varepsilon_m \to 0$, $f(x_m) \to f_0$. Then

$$f_0[n(1+H^2) - \frac{3}{2}f_0] \le 0.$$

a) If $f_0 = 0$, $f = \tau = 0$. So $S = nH^2$. This means that M^n is totally umbilical;

b) If $f_0 > 0$,

$$n(1+H^2) - \frac{3}{2}f_0 \le 0, \quad f_0 \ge \frac{2}{3}n(1+H^2).$$

So $\sup\, f \ge \frac{2}{3}n(1+H^2)$, by (2.13), we have $\inf \rho \le n(n-\frac{5}{3})(1+H^2)$. □

Proof of Corollary From the proof of Theorem, we can know that, when $\tau < \frac{2}{3}n(1+H^2)$, i.e., $S < \frac{n}{3}(2+5H^2)$, we have $f_0 = 0$. Then $f = \tau = 0$. So $S = nH^2$, i.e., M^n is totally umbilical. □

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