Some New Translation Surfaces in 3-Minkowski Space

Yuan YUAN, Hui Li LIU*

Department of Mathematics, Northeastern University, Liaoning 110004, P. R. China

Abstract In this paper we study translation surfaces of some new types in 3-Minkowski space $E^3_1$ and give some classifications of such surfaces whose mean curvature and Gauss curvature satisfy certain conditions.

Keywords Minkowski space; translation surface; Weingarten surface.

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1. Introduction

For the study of the surfaces theory in 3-Euclidean space $E^3$ or 3-Minkowski space $E^3_1$, it is a very important and interesting problem to construct or classify the constant mean curvature or constant Gaussian curvature, or even more general, Weingarten surfaces. It is well-known that the translation surface is special and minimal one in 3-Euclidean space $E^3$ is Scherk surface. Here we consider translation surfaces in 3-Minkowski space. The second author gave some classification results for translation surfaces in [1] and [2]. However according to our recent work [3–6] we know that the results in [1] or [2] are only the Cases 1 and 2 of following 6 types of translation surfaces.

In 3-Minkowski space $E^3_1$, according to the spacelike direction, timelike direction and lightlike direction, the translation surfaces can be considered as the following six types

Type 1. Along spacelike direction and spacelike direction;
Type 2. Along spacelike direction and timelike direction;
Type 3. Along lightlike direction and lightlike direction;
Type 4. Along lightlike direction and spacelike direction;
Type 5. Along timelike direction and lightlike direction;
Type 6. Along timelike direction and timelike direction.

As we know that they are really different under Lorentz transformation in $E^3_1$. Using certain coordinate frames, we can express them in the different way [3, 6].

Let $E^3_1$ be the 3-Minkowski space with the inner product

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3.$$
Translation surface $S_a$ of Types 5 and 6 can be written as

$S_a : x(u, v) = \{X(u, v), Y(u, v), Z(u, v)\} = \{f(u + av) + g(v), u, v\}$.

(i) When $|a| = 1$, the surface $S_a$ is translation surface of Type 5.

(ii) When $|a| > 1$, the surface $S_a$ is translation surface of Type 6.

With $x_u = \frac{\partial x(u, v)}{\partial u}$, etc., the first fundamental form $I$ of the surface $S_a$ is given by

$$I = Edu^2 + 2Fdudv + Gdv^2,$$

$$E = \langle x_u, x_u \rangle = f_u^2 + 1,$$

$$F = \langle x_u, x_v \rangle = f_u(a_f v + g_v),$$

$$G = \langle x_v, x_v \rangle = (a_f v + g_v)^2 - 1.$$

For spacelike or timelike surface in $\mathbb{E}^3_1$, we have $EG - F^2 > 0$ or $EG - F^2 < 0$. The second fundamental form $II$ of $S_a$ is given by

$$II = Ldu^2 + 2Mdudv + Ndv^2,$$

$$L = \frac{1}{\sqrt{|EG - F^2|}} \det(x_u, x_v, x_{uu}) = \frac{f_{uu}}{\sqrt{|(a_f v + g_v)^2 - f_u^2 - 1|}},$$

$$M = \frac{1}{\sqrt{|EG - F^2|}} \det(x_u, x_v, x_{uv}) = \frac{af_{uv}}{\sqrt{|(a_f v + g_v)^2 - f_u^2 - 1|}},$$

$$N = \frac{1}{\sqrt{|EG - F^2|}} \det(x_u, x_v, x_{vv}) = \frac{a^2 f_{vv} + g_{vv}}{\sqrt{|(a_f v + g_v)^2 - f_u^2 - 1|}}.$$

The Gauss curvature $K$ and the mean curvature $H$ of $S_a$ are given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{f_{uu}((a^2 f_{vv} + g_{vv}) - a^2 f_{uv}^2)}{|(a_f v + g_v)^2 - f_u^2 - 1|}(a_f v + g_v)^2 - f_u^2 - 1),$$

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{(a^2 f_{vv} + g_{vv}) - 2a f_u f_{uv}(a_f v + g_v) + f_{uu}(a_f v + g_v)^2 - 1)}{2((a_f v + g_v)^2 - f_u^2 - 1)\sqrt{|(a_f v + g_v)^2 - f_u^2 - 1|}}.

(2)

2. Main results

By a transformation

$$\begin{cases} y = u + av, \\
  z = v,
\end{cases}$$

and $\frac{\partial (y, z)}{\partial (u, v)} \neq 0$, from (1) and (2) we get

$$K = \frac{f_{yy} g_{zz}}{\varepsilon((a^2 f_y + g_z)^2 - f_y^2 - 1)^2},$$

$$H = \frac{g_{zz}(1 + f_y^2) + f_{yy}(a^4 - 1 + g_z^2)}{2\varepsilon((a^2 f_y + g_z)^2 - f_y^2 - 1)^{3/2}},$$

(3)

(4)

where $\varepsilon = \pm 1$. In the following, we will consider translation surfaces of Types 5 and 6 whose Gauss curvature $K$ and mean curvature $H$ satisfy certain conditions. They are usually called
Some new translation surfaces in 3-Minkowski space

Theorem 1 Let $S_a$ be a translation surface of Type 6 in $\mathbb{E}^3_1$. If $S_a$ is minimal, it is congruent to a plane or the functions $f$ and $g$ satisfy

\[
\begin{align*}
    f &= -\frac{1}{c} \log |\sec(-c(u + av) + c_1)| + c_2, \\
    g &= \frac{1}{c} \log |\sec(c\sqrt{a^4 - 1}v + c_1)| + c_2,
\end{align*}
\]

where $c, c_1, c_2$ are constants and $c \neq 0$.

Proof Let $S_a$ be a translation surface of Type 6 in $\mathbb{E}^3_1$. By a transformation in $\mathbb{E}^3_1$, the translation surface $S_a$ can be written as

\[x(u, v) = \{f(u + av) + g(v), u, v\}, \quad |a| > 1.\]

From (4), putting $H = 0$ gives

\[g_{zz}(1 + f_y^2) + f_{yy}(a^4 - 1 + g_z^2) = 0.\]

Hence

\[\frac{g_{zz}}{a^4 - 1 + g_z^2} = -\frac{f_{yy}}{1 + f_y^2} = c,\]

where $c$ is constant.

i) When $c = 0$, we have

\[g_{zz} = 0 \quad \text{and} \quad f_{yy} = 0.\]

Then the surface is a plane.

ii) When $c \neq 0$, we have

\[
\begin{align*}
    f &= -\frac{1}{c} \log |\sec(-c(u + av) + c_1)| + c_2, \\
    g &= \frac{1}{c} \log |\sec(c\sqrt{a^4 - 1}v + c_1)| + c_2,
\end{align*}
\]

where $c_1, c_2$ are constants. This completes the proof of Theorem (1). ☐

Theorem 2 Let $S_a$ be a translation surface of Type 6 with constant mean curvature $H \neq 0$ in $\mathbb{E}^3_1$. Then

(i) If $S_a$ is spacelike, it is congruent to the following surfaces or an open part of them in $\mathbb{E}^3_1$

(a) $X(u, v) = -\frac{\sqrt{a^4 - 1}}{2H} \sqrt{4H^2v^2 - 1} - a^2cv + c(u + av), c \in R$,

(b) $X(u, v) = -\frac{\sqrt{a^4 - 1}}{2H} \frac{1}{\sqrt{1 - a^4uv^2}} + \sqrt{4H^2(1 + av)^2 - 1} - a^2cv + c(u + av), c \in R$;

(ii) If $S_a$ is timelike, it is congruent to the following surfaces or an open part of them in $\mathbb{E}^3_1$

(c) $X(u, v) = -\frac{\sqrt{a^4 - 1}}{2H} \sqrt{4H^2v^2 - 1} - a^2cv + c(u + av), c \in R$,

(d) $X(u, v) = -\frac{\sqrt{a^4 - 1}}{2H} \frac{1}{\sqrt{1 - a^4uv^2}} + \sqrt{4H^2(1 + av)^2 - 1} - a^2cv + c(u + av), c \in R$.

Proof Let $S_a$ be a translation surface of Type 6 with constant mean curvature $H \neq 0$ in $\mathbb{E}^3_1$. 


We assume that \( f_{yy}g_{zz} \neq 0 \). Differentiating (4) with respect to \( y \) and \( z \), we obtain

\[
\begin{pmatrix}
\frac{g_{zzz}}{g_{zz}} \\
\frac{g_{zz}}{g_{zz}} \\
g_{zz}
\end{pmatrix}
= 3
\begin{pmatrix}
\frac{f_{yy}}{f_{yy} + 1}
\end{pmatrix}
= 3H.
\]

That is

\[
\begin{cases}
    f_{yy} = \left( \frac{H}{2} f_y^2 + c_1 f_y + c_2 \right) (f_y^2 + 1), \\
g_{zz} = \frac{k}{24} g_y^4 + k_1 g_z^3 + k_2 g_y^2 + k_3 g_z + k_4,
\end{cases}
\]

where \( k = \frac{3H}{a^4 - 1} \), \( c_1, c_2, k_1, k_2, k_3, k_4 \) are constants. Putting \( f_{yy} \) into (4) and considering the coefficient of \( f_y^4 \), we can get \( H = 0 \) or \( g(z) = \text{constant} \), which contradicts \( H \neq 0 \).

By a transformation in \( E_3 \) we can assume that \( f_{yy} = 0 \) and write \( f(y) = cy \). From (4) we have

\[
(c^2 + 1) g_{zz} = 2H((a^2 c + g_z)^2 - c^2 - 1)\frac{2}{a^2 + 1}
\]

or

\[
(c^2 + 1) g_{zz} = -2H(c^2 + 1 - (a^2 c + g_z)^2)\frac{2}{a^2 + 1}.
\]

Solving these equations, we obtain the following surfaces, respectively

\[
g(z) = -\frac{\sqrt{1 + c^2}}{2H} \sqrt{4H^2(z + c_1)^2 + 1 - a^2 c z + c_2}, \quad c_1, c_2, c \in R, \tag{7}
\]

which is spacelike and congruent to the surface (a) given by Theorem (2);

\[
g(z) = -\frac{\sqrt{1 + c^2}}{2H} \sqrt{4H^2(z + c_1)^2 + 1 - a^2 c z + c_2}, \quad c_1, c_2, c \in R, \tag{8}
\]

which is timelike and congruent to the surface (c) given by Theorem (2).

When \( g_{zz} = 0 \) we assume that \( g(z) = cz \). By (4) we have

\[
(a^4 + c^2 - 1) f_{yy} = 2H((a^2 f_y + c)^2 - f_y^2)\frac{2}{a^2 + 1}\tag{9}
\]

or

\[
(a^4 + c^2 - 1) f_{yy} = -2H(f_y^2 + 1 - (a^2 f_y + c)^2)\frac{2}{a^2 + 1}.\tag{10}
\]

Solving these equations, we obtain the following surfaces, respectively

\[
f(y) = -\frac{\sqrt{a^4 + c^2 - 1}}{2H \sqrt{a^4 - 1}} \sqrt{4H^2 \frac{(y + c_1)^2}{a^4 - 1} - \frac{a^2 c}{a^4 - 1} y + c_2}, \quad c_1, c_2, c \in R, \tag{11}
\]

which is spacelike and congruent to the surface (b) given by Theorem (2);

\[
f(y) = -\frac{\sqrt{a^4 + c^2 - 1}}{2H \sqrt{a^4 - 1}} \sqrt{4H^2 \frac{(y + c_1)^2}{a^4 - 1} + \frac{a^2 c}{a^4 - 1} y + c_2}, \quad c_1, c_2, c \in R, \tag{12}
\]

By (4) we have

\[
y = (f_y c_1 + c_2)^2 - (a^2 f_y + c_2)^2 + 1 - a^2 c f_y + c_1 - 1,
\]

and by (4) we can assume that \( f_y c_1 + c_2 = 0 \) we assume that \( y = 0 \).

Differentiating (4) with respect to \( y \) and considering the coefficient of \( f_y^4 \), we can get \( H = 0 \) or \( g(z) = \text{constant} \), which contradicts \( H \neq 0 \).
Some new translation surfaces in 3-Minkowski space

1127

which is timelike and congruent to the surface (d) given by Theorem (2). This completes the proof of Theorem (2). □

Theorem 3 Let \( S_a : x(u, v) = \{ f(u + av) + g(v), u, v \} \) be a translation surface of Type 5 or 6 with Gauss curvature \( K = 0 \) in \( \mathbb{E}^3_1 \). Then the functions \( f \) and \( g \) satisfy

\[
\begin{cases}
  f(u + av) = c_1(u + av) + c_2, & c_1, c_2 \in \mathbb{R}, \\
  g(v) \text{ is any function},
\end{cases}
\]

or

\[
\begin{cases}
  g = c_1 v + c_2, & c_1, c_2 \in \mathbb{R}, \\
  f(u + av) \text{ is any function}.
\end{cases}
\]

Proof From (3), putting \( K = 0 \), we get

\[ f_{yy}g_{zz} = 0. \]

i) When \( f_{yy} = 0 \), we have

\[
\begin{cases}
  f = c_1 y + c_2 = c_1(u + av) + c_2, & c_1, c_2 \in \mathbb{R}, \\
  g(v) \text{ is any function}.
\end{cases}
\]

ii) When \( g_{zz} = 0 \), we get

\[
\begin{cases}
  g = c_1 z + c_2 = c_1 v + c_2, & c_1, c_2 \in \mathbb{R}, \\
  f(u + av) \text{ is any function}.
\end{cases}
\]

Theorem 4 There is no translation surface of Type 5 or 6 with constant Gauss curvature \( K \neq 0 \) in \( \mathbb{E}^3_1 \).

Proof Let \( S_a \) be a translation surface of Type 6 with constant Gauss curvature \( K \neq 0 \) in \( \mathbb{E}^3_1 \). From (3) we have \( f_{yy}g_{zz} \neq 0 \). Differentiating (3) with respect to \( y \) and \( z \), we obtain

\[ g_{zzz}((a^4 - 1)f_y + g_z) - 2a^2 g_{zz}^2 = 0. \]

If \( g_{zzz} = 0 \) and \( a \neq 0 \), then \( g_{zz} = 0 \), which contradicts the assumption \( K \neq 0 \). So when \( g_{zzz} \neq 0 \) we have

\[ (a^4 - 1)f_y = \frac{2a^2 g_{zz}^2}{g_{zzz}} - g_z = c, \]

that is

\[
\begin{cases}
  (a^4 - 1)f_y = c, \\
  \frac{2a^2 g_{zz}^2}{g_{zzz}} - g_z = c.
\end{cases}
\]

By (18) we get that \( f_{yy} = 0 \). That means \( K = 0 \). Therefore, there is no translation surface of Type 6 with constant Gauss curvature \( K \neq 0 \) in \( \mathbb{E}^3_1 \). The proof of translation surface of Type 5 is similar. This completes the proof of Theorem (4). □

With the same methods we can also obtain the following results. We omit the proofs.

Theorem 5 Let \( x(u, v) = \{ f(u + av) + g(v), u, v \} \) be a translation surface of Type 5 which is
minimal in $\mathbb{E}_1^3$. Then the surface is a plane or the functions $f$ and $g$ satisfy

$$
\begin{cases}
    f = \frac{1}{c} \log |\sec(c(u + av) + c_1)| + c_2, \\
    g = \frac{1}{c} \log |cv + c_1| + c_2,
\end{cases}
$$

(19)

where $c_1$, $c_2$, $c$ are constants and $c \neq 0$.

**Theorem 6** Let $S_a$ be a translation surface of Type 5 with constant mean curvature $H \neq 0$ in $\mathbb{E}_1^3$. Then

(i) If $S_a$ is spacelike, it is congruent to the following surfaces or an open part of them in $\mathbb{E}_1^3$

(a) $X(u, v) = -\sqrt{\frac{1}{2H^2}} \sqrt{4H^2v^2 - 1} + cu$, $c \in R$,

(b) $X(u, v) = -\frac{c}{2H^2} \frac{1}{u+v} + \frac{1-c^2}{2c} u + \frac{c^2 + 1}{2c} v$, $c \neq 0$ and $c \in R$;

(ii) If $S_a$ is timelike, it is congruent to the following surfaces or an open part of them in $\mathbb{E}_1^3$

(c) $X(u, v) = -\frac{\sqrt{\frac{1}{2H^2}}}{2H} \sqrt{4H^2v^2 + 1} + cu$, $c \in R$,

(d) $X(u, v) = \frac{c}{2H^2} \frac{1}{u+v} + \frac{1-c^2}{2c} u + \frac{c^2 + 1}{2c} v$, $c \neq 0$ and $c \in R$.

**Theorem 7** Let $S_a$ be a translation surface of Type 5 or 6 in $\mathbb{E}_1^3$ whose Gauss curvature $K$ and mean curvature $H$ satisfy $bH + cK = 0$ $(bc \neq 0)$. Then it is congruent to a plane or an open part of it.

**References**


